MA40254 Differential and geometric analysis : Exercises 5

Hand in answers by 1:15pm on Wednesday 8 November for the Seminar of Thursday 9 November
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Let V be a real vector space of dimension n, $\alpha_1, \alpha_2 \ldots \alpha_k, \beta_j \in \mathcal{M}^1(V)$ and $\lambda, \mu \in \mathbb{R}$. Show that $\alpha_1\alpha_2\cdots(\lambda\alpha_j + \mu\beta_j)\alpha_{j+1}\cdots\alpha_k = \lambda\,\alpha_1\alpha_2\cdots\alpha_j\alpha_{j+1}\cdots\alpha_k + \mu\,\alpha_1\alpha_2\cdots\beta_j\alpha_{j+1}\cdots\alpha_k$ in $\mathcal{M}^k(V)$ and deduce that

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge (\lambda\alpha_j + \mu\beta_j) \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_k$$
$$= \lambda\,\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_j \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_k + \mu\,\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \beta_j \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_k.$$

[**Solution**: *Evaluating the left hand side of the first identity on $v_1, v_2, \ldots v_k \in V$ (using the definition) gives*

$$\alpha_1(v_1)\alpha_2(v_2)\cdots(\lambda\alpha_j + \mu\beta_j)(v_j)\alpha_{j+1}(v_{j+1})\cdots\alpha_k(v_k) \in \mathbb{R}$$

Now $(\lambda\alpha_j + \mu\beta_j)(v_j) = \lambda\,\alpha_j(v_j) + \mu\,\beta_j(v_j)$ (pointwise operations) so the result follows by the distributive law. To obtain the second identity, apply alt *to both sides of the first identity, and use that* $\mathrm{alt}\colon \mathcal{M}^k(V) \to \mathrm{Alt}^k(V)$ *is linear:* $\mathrm{alt}(\lambda\alpha + \mu\beta) = \sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)\sigma\cdot(\lambda\alpha + \mu\beta)$ *and it is easy to see that $\sigma\cdot(\lambda\alpha + \mu\beta) = \lambda\,\sigma\cdot\alpha + \mu\,\sigma\cdot\beta$ (evaluate both sides on $v_1, v_2, \ldots v_k$).*]

1. Let V be a real vector space of dimension n.

(i) Let $v_1, \ldots, v_k \in V$ and $\alpha_1, \ldots, \alpha_k \in V^*$. Let $A \in M_{k,k}(\mathbb{R})$ be the matrix with $A_{ij} = \alpha_i(v_j)$. Show that $(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \ldots, v_k) = \det A$.

[**Hint**: *Compare the result of evaluating the full antisymmetrisation of $\alpha_1\cdots\alpha_k \in \mathcal{M}^k(V)$ on $v_1, \ldots, v_k$ with the sum formula for the determinant of a matrix.*]

(ii) Let $\phi : V \to V$ be a linear operator. Show that for any $\alpha \in \mathrm{Alt}^n(V)$,

$$\phi^*\alpha = (\det\phi)\alpha \in \mathrm{Alt}^n(V)$$

[**Hint**: *Recall that* $\det\phi$ *means the determinant of the matrix A representing ϕ with respect to any basis $e_1, e_2, \ldots e_n$ of V. Use the dual basis $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n$ to write the entries A_{ij} in the form of part (i), and note that $\varepsilon_1 \wedge \varepsilon_2 \wedge \ldots \wedge \varepsilon_n$ is a basis for* $\mathrm{Alt}^n(V)$.]

2. Let $e_1, \ldots, e_5 \in \mathbb{R}^5$ be the standard basis, and let $\varepsilon_1, \ldots, \varepsilon_5 \in (\mathbb{R}^5)^*$ be the dual basis. Let

$$\alpha = 3\varepsilon_1 \wedge \varepsilon_3 + \varepsilon_2 \wedge (7\varepsilon_3 - 2\varepsilon_5) \in \mathrm{Alt}^2(\mathbb{R}^5).$$

(i) Evaluate $\alpha(e_1 + 2e_3, e_3 + e_4) \in \mathbb{R}$.

(ii) Express $\alpha \wedge (2\varepsilon_1 + \varepsilon_2 - 3\varepsilon_4) \in \mathrm{Alt}^3(\mathbb{R}^5)$ in terms of the standard basis $\varepsilon_1\wedge\varepsilon_2\wedge\varepsilon_3$, $\varepsilon_1\wedge\varepsilon_2\wedge\varepsilon_4$, $\ldots$, $\varepsilon_3\wedge\varepsilon_4\wedge\varepsilon_5$.

[**Hint**: *When expanding, bear in mind that $\varepsilon_i \wedge \varepsilon_j = -\varepsilon_j \wedge \varepsilon_i$, and $\varepsilon_i \wedge \varepsilon_i = 0$.*]

3 (Less essential)**.** For a real inner product space V and $v \in V$, define $v^\flat \in V^*$ to be the linear map $V \to \mathbb{R}$, $w \mapsto v.w$. For $\alpha \in \mathrm{Alt}^{k+1}(V)$ and $v \in V$, define the 'contraction' $v \lrcorner \alpha \in \mathrm{Alt}^k(V)$ by

$$(v \lrcorner \alpha)(w_1, \ldots, w_k) = \alpha(v, w_1, \ldots, w_k)$$

for all $w_1, \ldots, w_k \in V$. Show that the cross product on $\mathbb{R}^3$ is related to the wedge product on $(\mathbb{R}^3)^*$ by

$$(u \times v) \lrcorner \mathrm{Det} = u^\flat \wedge v^\flat \in \mathrm{Alt}^2(\mathbb{R}^3)$$

for any $u, v \in \mathbb{R}^3$.

[**Hint**: *Express* u *and* v *in terms of the standard basis, i.e., write* $u = u_1e_1 + u_2e_2 + u_3e_3$ *etc., and find the components of both sides of the equation with respect to the standard basis* $\varepsilon_1 \wedge \varepsilon_2, \varepsilon_1 \wedge \varepsilon_3, \varepsilon_2 \wedge \varepsilon_3$ *of* $\mathrm{Alt}^2\mathbb{R}^3$.]

4. Let $\phi : V \to W$ be a linear map between real vector spaces. Show that

(i) $\phi^* \operatorname{alt}(\alpha) = \operatorname{alt}(\phi^*\alpha) \in \mathrm{Alt}^k(V)$ for any $\alpha \in \mathcal{M}^k(W)$.

[**Hint**: *First check that* $\sigma \cdot (\phi^*\alpha) = \phi^*(\sigma \cdot \alpha) \in \mathcal{M}^k(V)$ *for any* $\alpha \in \mathcal{M}^k(W)$ *and* $\sigma \in S_k$.]

(ii) $\phi^*(\alpha_1 \wedge \cdots \wedge \alpha_k) = (\phi^*\alpha_1) \wedge \cdots \wedge (\phi^*\alpha_k) \in \mathrm{Alt}^k(V)$ for any $\alpha_1, \ldots, \alpha_k \in \mathrm{Alt}^1(W)$.

[**Hint**: *First show* $\phi^*(\alpha_1 \cdots \alpha_k) = (\phi^*\alpha_1) \cdots (\phi^*\alpha_k)$ *by evaluating both sides on* $v_1, \ldots v_k$.]

5. Let V and W be vector spaces with bases v_1, v_2, v_3 and w_1, w_2, w_3, w_4 respectively. Let $\phi : V \to W$ be the linear map represented with respect to these bases by

$$\begin{pmatrix} 2 & 0 & -3 \\ 1 & 6 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 5 \end{pmatrix}$$

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in V^*$ and $\delta_1, \delta_2, \delta_3, \delta_4 \in W^*$ denote the dual bases to the given bases. Compute

$$\phi^*(3\delta_1 \wedge \delta_3 + \delta_2 \wedge \delta_4) \in \mathrm{Alt}^2(V)$$

in terms of the standard basis $\varepsilon_i \wedge \varepsilon_j : i < j$ for $\mathrm{Alt}^2(V)$.

[**Hint**: *What matrix represents* $\phi^* : W^* \to V^*$ *with respect to the bases* δ_i *and* ε_j*? There is a reason why the dual map* ϕ^* *is sometimes called the "transpose" of* ϕ*! You should find, for instance, that* $\phi^*\delta_1 = 2\varepsilon_1 - 3\varepsilon_3$.]

DMJC 31 October