MA40254 Differential and geometric analysis : Exercises 5

Hand in answers by 1:15pm on Wednesday 8 November for the Seminar of Thursday 9 November
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Let V be a real vector space of dimension n, $\alpha_1, \alpha_2 \dots \alpha_k, \beta_j \in \mathcal{M}^1(V)$ and $\lambda, \mu \in \mathbb{R}$. Show that $\alpha_1\alpha_2 \cdots (\lambda\alpha_j + \mu\beta_j)\alpha_{j+1} \cdots \alpha_k = \lambda\, \alpha_1\alpha_2 \cdots \alpha_j\alpha_{j+1} \cdots \alpha_k + \mu\, \alpha_1\alpha_2 \cdots \beta_j\alpha_{j+1} \cdots \alpha_k$ in $\mathcal{M}^k(V)$ and deduce that

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge (\lambda\alpha_j + \mu\beta_j) \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_k$$
$$= \lambda\, \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_j \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_k + \mu\, \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \beta_j \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_k.$$

1. Let V be a real vector space of dimension n.

(i) Let $v_1, \dots, v_k \in V$ and $\alpha_1, \dots, \alpha_k \in V^*$. Let $A \in M_{k,k}(\mathbb{R})$ be the matrix with $A_{ij} = \alpha_i(v_j)$. Show that $(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \dots, v_k) = \det A$.

(ii) Let $\phi : V \to V$ be a linear operator. Show that for any $\alpha \in \mathrm{Alt}^n(V)$,

$$\phi^*\alpha = (\det\phi)\alpha \in \mathrm{Alt}^n(V)$$

2. Let $e_1, \dots, e_5 \in \mathbb{R}^5$ be the standard basis, and let $\varepsilon_1, \dots, \varepsilon_5 \in (\mathbb{R}^5)^*$ be the dual basis. Let

$$\alpha = 3\varepsilon_1 \wedge \varepsilon_3 + \varepsilon_2 \wedge (7\varepsilon_3 - 2\varepsilon_5) \in \mathrm{Alt}^2(\mathbb{R}^5).$$

(i) Evaluate $\alpha(e_1 + 2e_3, e_3 + e_4) \in \mathbb{R}$.

(ii) Express $\alpha \wedge (2\varepsilon_1 + \varepsilon_2 - 3\varepsilon_4) \in \mathrm{Alt}^3(\mathbb{R}^5)$ in terms of the standard basis $\varepsilon_1\wedge\varepsilon_2\wedge\varepsilon_3$, $\varepsilon_1\wedge\varepsilon_2\wedge\varepsilon_4$, $\dots$, $\varepsilon_3\wedge\varepsilon_4\wedge\varepsilon_5$.

3 (Less essential)**.** For a real inner product space V and $v \in V$, define $v^\flat \in V^*$ to be the linear map $V \to \mathbb{R}$, $w \mapsto v.w$. For $\alpha \in \mathrm{Alt}^{k+1}(V)$ and $v \in V$, define the 'contraction' $v \lrcorner \alpha \in \mathrm{Alt}^k(V)$ by

$$(v \lrcorner \alpha)(w_1, \dots, w_k) = \alpha(v, w_1, \dots, w_k)$$

for all $w_1, \dots, w_k \in V$. Show that the cross product on $\mathbb{R}^3$ is related to the wedge product on $(\mathbb{R}^3)^*$ by

$$(u \times v) \lrcorner \mathrm{Det} = u^\flat \wedge v^\flat \in \mathrm{Alt}^2(\mathbb{R}^3)$$

for any $u, v \in \mathbb{R}^3$.

4. Let $\phi : V \to W$ be a linear map between real vector spaces. Show that

(i) $\phi^* \mathrm{alt}(\alpha) = \mathrm{alt}(\phi^*\alpha) \in \mathrm{Alt}^k(V)$ for any $\alpha \in \mathcal{M}^k(W)$.

(ii) $\phi^*(\alpha_1 \wedge \cdots \wedge \alpha_k) = (\phi^*\alpha_1) \wedge \cdots \wedge (\phi^*\alpha_k) \in \mathrm{Alt}^k(V)$ for any $\alpha_1, \dots, \alpha_k \in \mathrm{Alt}^1(W)$.

5. Let V and W be vector spaces with bases v_1, v_2, v_3 and w_1, w_2, w_3, w_4 respectively. Let $\phi : V \to W$ be the linear map represented with respect to these bases by

$$\begin{pmatrix} 2 & 0 & -3 \\ 1 & 6 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 5 \end{pmatrix}$$

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in V^*$ and $\delta_1, \delta_2, \delta_3, \delta_4 \in W^*$ denote the dual bases to the given bases. Compute

$$\phi^*(3\delta_1 \wedge \delta_3 + \delta_2 \wedge \delta_4) \in \mathrm{Alt}^2(V)$$

in terms of the standard basis $\varepsilon_i \wedge \varepsilon_j : i < j$ for $\mathrm{Alt}^2(V)$.

DMJC 31 October