MA40254 Differential and geometric analysis : Exercises 4

Hand in answers by 1:15pm on Wednesday 1 November for the Seminar of Thursday 2 November
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Compute the tangent space T_pM to the 1-dimensional submanifold $M := \{(x, y) \in \mathbb{R}^2 : y = x^2\}$ of $\mathbb{R}^2$ at the point $p = (t, t^2)$.

[**Solution**: *As in Exercises 3, there are two approaches. The first is to use the parametrization* $\varphi\colon \mathbb{R} \to M$ *with* $\varphi(x) = (x, x^2)$*. Then* $D\varphi_p$ *has matrix* $\left[\begin{smallmatrix}1\\2t\end{smallmatrix}\right]$*, and* $T_pM = \operatorname{im} D\varphi_p = \{(\lambda, 2\lambda t) \in \mathbb{R}^2 : \lambda \in \mathbb{R}\}$*. Alternatively, since* $M = f^{-1}(0)$ *is the inverse image of a regular value of* $f(x, y) = x^2 - y$*, the exercise below implies* $T_pM = \ker Df_p = \{(a, b) \in \mathbb{R}^2 : -2ta + b = 0\}$*, since* Df_p *is represented by the matrix* $[-2t\ 1]$*.*]

1. (i) Let $M \subseteq \mathbb{R}^s$ be a submanifold. Let $P \subseteq \mathbb{R}^s$ be an open subset that contains M, and let $f : P \to \mathbb{R}^m$ be a smooth function. Suppose that the restriction of f to M is constant. Show that $T_pM \subseteq \ker Df_p \subseteq \mathbb{R}^s$ for any $p \in M$.

(ii) Let $P \subseteq \mathbb{R}^s$ be an open subset, $f : P \to \mathbb{R}^m$ a smooth function, $q \in \mathbb{R}^m$ a regular value of f, and $M := f^{-1}(q)$. Show that $T_pM = \ker Df_p \subseteq \mathbb{R}^s$ for any $p \in M$.

[**Hint**: *Let* φ *be a parametrisation of* M*, and consider the derivative of* $f \circ \varphi$*. The chain rule and the rank-nullity theorem may be helpful!*]

2. For points $x, y \in \mathbb{R}^2$ with $x \neq y$, let $S(x, y) = \{tx + (1-t)y : t \in (0, 1)\} \subset \mathbb{R}^2$. For which $x, y, x', y' \in \mathbb{R}^2$ is $S(x, y) \cup S(x', y')$ a submanifold of $\mathbb{R}^2$?

[**Hint**: *One way that the union of the line segments can fail to be a submanifold is if they intersect in a single point. Consider a neighbourhood of such a point, with the point itself removed from it. How many pieces does it have?*]

3. Let $O(n) = \{A \in GL_n(\mathbb{R}) : A^T = A^{-1}\}$. Show that $O(n)$ is a submanifold of $M_{n,n}(\mathbb{R})$. What is $T_IO(n) \subseteq M_{n,n}(\mathbb{R})$ (the tangent space of $O(n)$ at the identity matrix $I \in O(n)$)?

[**Hint**: *Describe* $O(n)$ *using the function* $Mat_{n,n}(\mathbb{R}) \to \{\text{symmetric matrices}\}$, $A \mapsto A^T A$*.*]

4. Let V be a real vector space of dimension n, and $\mathrm{Alt}^2(V)$ the space of alternating 2-forms on V, that is bilinear maps $\omega : V \times V \to \mathbb{R}$ such that $\omega(v, v) = 0$ for any $v \in V$. What is the dimension of $\mathrm{Alt}^2(V)$?

[**Hint**: *Use a basis to identify* $\mathrm{Alt}^2(V)$ *with a subspace of the space of* $n \times n$ *matrices.*]

5. For $v_1, \ldots, v_n \in \mathbb{R}^n$, let $\mathrm{Det}(v_1, \ldots, v_n) \in \mathbb{R}$ denote the determinant of the $n \times n$ matrix with columns $v_1, \ldots, v_n$.

(i) Show that Det spans $\mathrm{Alt}^n(\mathbb{R}^n)$.

[**Hint**: *You could argue in terms of the characterisation of the determinant function from Algebra 1B, or make use of the result on the dimension of* $\mathrm{Alt}^n(\mathbb{R}^n)$*.*]

(ii) For any $u, v \in \mathbb{R}^3$, show that there is a unique $u \times v \in \mathbb{R}^3$ such that for any $w \in \mathbb{R}^3$,

$$\mathrm{Det}(u, v, w) = (u \times v).w.$$

Here the right hand side is the Euclidean inner product of the vectors $u \times v$ and w.

[**Hint**: *For fixed* u, v*, the left hand side is linear in* w*. Now recall the Riesz representation theorem for inner product spaces (Alg 2A).*]

MA40254 Differential and geometric analysis : Solutions 4

1. (i) Let $\varphi : U' \to U \subseteq M$ be a parametrisation with $p \in U$, say $p = \varphi(x)$. Then $f \circ \varphi : U' \to \mathbb{R}^m$ is constant, so the chain rule gives

$$Df_p \circ D\varphi_x = D(f \circ \varphi)_x = 0.$$

Thus T_pM, the image of $D\varphi_x : \mathbb{R}^n \to \mathbb{R}^s$, is contained in the kernel of $Df_p : \mathbb{R}^s \to \mathbb{R}^m$.

(ii) The dimension of $\ker Df_p$ is $s - m$ by the Rank-Nullity theorem. On the other hand, we know that $f^{-1}(q)$ is a submanifold of dimension $s - m$, and that T_pM has the same dimension as M. Since T_pM and $\ker Df_p$ have equal dimension, equality must hold in $T_pM \subseteq \ker Df_p$.

2. For $S(x, y) \cup S(x', y')$ to be a submanifold of $\mathbb{R}^2$, one needs $S(x, y)$ to be disjoint from the closure $\overline{S(x', y')} \subset \mathbb{R}^2$ and vice versa, or that x, y, x', y' are all colinear.

If $S(x, y)$ and $S(x', y')$ intersect in a single point z, then for any open neighbourhood $U \subseteq S(x, y) \cup S(x', y')$ of z, $U \setminus \{z\}$ has 4 connected components. Thus z has no neighbourhood U diffeomorphic to an interval.

Similarly, if $x \in S(x', y')$ then for any neighbourhood U of x, $U \setminus \{x\}$ has 3 connected components.

3. For any $A \in M_{n,n}(\mathbb{R})$, the matrix $A^T A$ is symmetric. So if we let $S \subseteq M_{n,n}(\mathbb{R})$ denote the subspace of symmetric matrices, then $f : M_{n,n}(\mathbb{R}) \to S,\ A \mapsto A^T A$ is a well-defined function, and $O(n)$ is $f^{-1}(I)$. Clearly f is smooth, so to show that $O(n)$ is a submanifold, we need only check that $I \in S$ is a regular value of f.

For any $A \in M_{n,n}(\mathbb{R})$, the derivative

$$Df_A : M_{n,n}(\mathbb{R}) \to S$$

maps

$$X \mapsto A^T X + X^T A.$$

We want to prove that this is surjective if $A \in O(n)$. So suppose that $Y \in S$. Then setting $X = \frac{1}{2} AY$ gives $Df_A(X) = Y$. Thus Df_A is indeed surjective whenever $A \in O(n)$.

Now $T_I O(n)$ equals the kernel of $Df_I : M_{n,n}(\mathbb{R}) \to S,\ X \mapsto X + X^T$, i.e., $T_I O(n) \subseteq M_{n,n}(\mathbb{R})$ is the subspace of anti-symmetric matrices.

4. Recall that, if we choose a basis $e_1, \ldots, e_n$ for V, then any bilinear form $\omega : V \times V \to \mathbb{R}$ can be represented by a matrix $A \in M_{n,n}(\mathbb{R})$, namely

$$A_{ij} := \omega(e_i, e_j).$$

Conversely, any $A \in M_{n,n}(\mathbb{R})$ defines a unique bilinear form. The condition that ω is alternating is equivalent to A being anti-symmetric (i.e., $A = -A^T$), so $\mathrm{Alt}^2(V)$ is isomorphic to the subspace of anti-symmetric $n \times n$ matrices, which has dimension $\binom{n}{2}$.

5. (i) This is equivalent to the theorem from Algebra 1B that any function $M_{n,n}(\mathbb{R}) \to \mathbb{R}$ that is multilinear and alternating as a function of the columns is a scalar multiple of det.

Alternatively, use the fact that $\dim \mathrm{Alt}^n(\mathbb{R}^n) = \binom{n}{n} = 1$. To deduce that Det is a basis, it therefore suffices to check that $\mathrm{Det} \neq 0$. If $e_1, \ldots, e_n \in \mathbb{R}^n$ is the standard basis, then $\mathrm{Det}(e_1, \ldots, e_n) = \det I = 1$, so Det is indeed a non-zero element of $\mathrm{Alt}^n(\mathbb{R}^n)$.

(ii) Recall that the determinant is linear as a function of each column, e.g., if we fix u and v then $w \mapsto \mathrm{Det}(u, v, w)$ is a linear map $\mathbb{R}^3 \to \mathbb{R}$. The Riesz representation theorem implies that for any linear function $\mathbb{R}^3 \to \mathbb{R}$ there is a unique vector $z \in \mathbb{R}^3$ such that $w \mapsto z.w$ equals the given functional. So in particular there is a z such that $z.w = \mathrm{Det}(u, v, w)$ for all $w \in \mathbb{R}^3$, and we can define $u \times v$ to be this z. (Of course $u \times v$ turns out to have a familiar expression in terms of the components of u and v.)