MA40254 Differential and geometric analysis : Exercises 4

Hand in answers by 1:15pm on Wednesday 1 November for the Seminar of Thursday 2 November
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Compute the tangent space T_pM to the 1-dimensional submanifold $M := \{(x, y) \in \mathbb{R}^2 : y = x^2\}$ of $\mathbb{R}^2$ at the point $p = (t, t^2)$.

[**Solution**: *As in Exercises 3, there are two approaches. The first is to use the parametrization* $\varphi\colon \mathbb{R} \to M$ *with* $\varphi(x) = (x, x^2)$*. Then* $D\varphi_p$ *has matrix* $\left[\begin{smallmatrix}1\\2t\end{smallmatrix}\right]$*, and* $T_pM = \operatorname{im} D\varphi_p = \{(\lambda, 2\lambda t) \in \mathbb{R}^2 : \lambda \in \mathbb{R}\}$*. Alternatively, since* $M = f^{-1}(0)$ *is the inverse image of a regular value of* $f(x, y) = x^2 - y$*, the exercise below implies* $T_pM = \ker Df_p = \{(a, b) \in \mathbb{R}^2 : -2ta + b = 0\}$*, since* Df_p *is represented by the matrix* $[-2t\ 1]$*.*]

1. (i) Let $M \subseteq \mathbb{R}^s$ be a submanifold. Let $P \subseteq \mathbb{R}^s$ be an open subset that contains M, and let $f : P \to \mathbb{R}^m$ be a smooth function. Suppose that the restriction of f to M is constant. Show that $T_pM \subseteq \ker Df_p \subseteq \mathbb{R}^s$ for any $p \in M$.

(ii) Let $P \subseteq \mathbb{R}^s$ be an open subset, $f : P \to \mathbb{R}^m$ a smooth function, $q \in \mathbb{R}^m$ a regular value of f, and $M := f^{-1}(q)$. Show that $T_pM = \ker Df_p \subseteq \mathbb{R}^s$ for any $p \in M$.

[**Hint**: *Let* φ *be a parametrisation of* M*, and consider the derivative of* $f \circ \varphi$*. The chain rule and the rank-nullity theorem may be helpful!*]

2. For points $x, y \in \mathbb{R}^2$ with $x \neq y$, let $S(x, y) = \{tx + (1-t)y : t \in (0, 1)\} \subset \mathbb{R}^2$. For which $x, y, x', y' \in \mathbb{R}^2$ is $S(x, y) \cup S(x', y')$ a submanifold of $\mathbb{R}^2$?

[**Hint**: *One way that the union of the line segments can fail to be a submanifold is if they intersect in a single point. Consider a neighbourhood of such a point, with the point itself removed from it. How many pieces does it have?*]

3. Let $O(n) = \{A \in GL_n(\mathbb{R}) : A^T = A^{-1}\}$. Show that $O(n)$ is a submanifold of $M_{n,n}(\mathbb{R})$. What is $T_IO(n) \subseteq M_{n,n}(\mathbb{R})$ (the tangent space of $O(n)$ at the identity matrix $I \in O(n)$)?

[**Hint**: *Describe* $O(n)$ *using the function* $Mat_{n,n}(\mathbb{R}) \to \{\textit{symmetric matrices}\}$, $A \mapsto A^TA$*.*]

4. Let V be a real vector space of dimension n, and $\mathrm{Alt}^2(V)$ the space of alternating 2-forms on V, that is bilinear maps $\omega : V \times V \to \mathbb{R}$ such that $\omega(v, v) = 0$ for any $v \in V$. What is the dimension of $\mathrm{Alt}^2(V)$?

[**Hint**: *Use a basis to identify* $\mathrm{Alt}^2(V)$ *with a subspace of the space of* $n \times n$ *matrices.*]

5. For $v_1, \ldots, v_n \in \mathbb{R}^n$, let $\mathrm{Det}(v_1, \ldots, v_n) \in \mathbb{R}$ denote the determinant of the $n \times n$ matrix with columns $v_1, \ldots, v_n$.

(i) Show that Det spans $\mathrm{Alt}^n(\mathbb{R}^n)$.

[**Hint**: *You could argue in terms of the characterisation of the determinant function from Algebra 1B, or make use of the result on the dimension of* $\mathrm{Alt}^n(\mathbb{R}^n)$*.*]

(ii) For any $u, v \in \mathbb{R}^3$, show that there is a unique $u \times v \in \mathbb{R}^3$ such that for any $w \in \mathbb{R}^3$,

$$\mathrm{Det}(u, v, w) = (u \times v).w.$$

Here the right hand side is the Euclidean inner product of the vectors $u \times v$ and w.

[**Hint**: *For fixed* u, v*, the left hand side is linear in* w*. Now recall the Riesz representation theorem for inner product spaces (Alg 2A).*]

DMJC 24 October