MA40254 Differential and geometric analysis : Exercises 3

Hand in answers by 1:15pm on Wednesday 25 October for the Seminar of Thursday 26 October
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Show that $M := \{(x, y) \in \mathbb{R}^2 : y = x^2\}$ is a 1-dimensional submanifold of $\mathbb{R}^2$.

[**Solution**: *M is the graph of the smooth function $h\colon \mathbb{R} \to \mathbb{R}$ with $h(x) = x^2$. Thus there is a parametrization $\varphi\colon \mathbb{R} \to M$ with $\varphi(x) = (x, x^2)$. This is a diffeomorphism because $F\colon \mathbb{R}^2 \to \mathbb{R}$ with $F(x, y) = x$ is smooth and $F|_M = \varphi^{-1}$. Alternatively, we can apply the Regular Value Theorem: $f\colon \mathbb{R}^2 \to \mathbb{R}$ with $f(x, y) = y - x^2$ is smooth and $Df_{(x,y)}$ is represented by the matrix $[-2x\ 1]$. This is nonzero for all (x, y) so 0 is a regular value and hence $M = f^{-1}(0)$ is a 1-dimensional submanifold of $\mathbb{R}^2$. This is related to the first approach, because close to the origin, the proof of the regular value theorem gives a parametrization using the graph of h.*]

1. Let $U \subset \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}^n$ be a twice differentiable function such that Df_x is invertible at $x \in U$. Let K be the operator norm of $(Df_x)^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ and N the supremum of the operator norm of $D(Df)_z : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ for $z \in U$ (where $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is itself equipped with the operator norm).

Suppose that $0 < \delta < 1/(2KN)$ and the open ball $B_\delta(x)$ is contained in U.

(i) Let $\tilde{f} := (Df_x)^{-1} \circ f$. Show that the image $\tilde{f}(B_\delta(x))$ contains $B_{\delta/4}(\tilde{f}(x))$.

[**Hint**: *Apply the Mean Value Inequality to Df and use Lemma 1.22.*]

(ii) Show that the image $f(B_\delta(x))$ contains the ball $B_{\delta/(4K)}(f(x))$.

[**Hint**: *What can you say about the image of $B_{\delta/(4K)}(f(x))$ under $(Df_x)^{-1}$?*]

2. Consider parametrisations of φ of $S^n := \{y \in \mathbb{R}^{n+1} : \|y\| = 1\}$ that are "graphs over a coordinate plane" in the following sense: $\varphi : B^n \to U$, for $B^n := \{x \in \mathbb{R}^n : \|x\| < 1\}$ and $U \subset S^n$ some open subset, and all but one of the $n+1$ components of $\varphi(x)$ is equal to a component of x (e.g., φ could be of the form $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, g(x_1, \ldots, x_n), x_i, \ldots x_n))$. How many parametrisations of this form are required for the images U to cover S^n?

[**Hint**: *If $e_i \in \mathbb{R}^{n+1}$ is one of the $n+1$ basis vectors, how many parametrisations of the given type have e_i (or $-e_i$) contained in their image?*]

3. Define two parametrisations $\varphi_+, \varphi_- : \mathbb{R}^n \to S^n$ as follows. For $x \in \mathbb{R}^n$, consider the line $L_\pm$ in $\mathbb{R}^{n+1}$ that contains $(x, 0)$ and $(0, \pm 1)$, and let $\varphi_\pm(x)$ be the intersection point (other than $(0, \pm 1)$) of $L_\pm$ with S^n.

Show that

$$\varphi_\pm(x) = \frac{1}{1 + \|x\|^2}\left(2x,\ \pm(\|x\|^2 - 1)\right).$$

What are the images $U_\pm$ of $\varphi_\pm$?

[**Hint**: *Parametrize $L_\pm$ by $t \in \mathbb{R}$ and find $y \in L_\pm$ with $\|||y||^2 = 1\|$. This should give a quadratic equation for t—one solution gives the point $(0, \pm 1)$ and the other is $\varphi_\pm(x)$.*]

4. Let k be a positive integer, and let $M = \{(x, y, z) \in \mathbb{R}^3 : x^k + y^k + z^k = 1\}$.

(i) Show that M is a submanifold of $\mathbb{R}^3$.

[**Hint**: *Show that 1 is a regular value of $(x, y, z) \mapsto x^k + y^k + z^k$.*]

(ii) Show that if k is even then M is diffeomorphic to S^2.

[**Hint**: *To see the difference between when k is even or odd, it may be helpful to draw a sketch of the set $\{(x,y) \in \mathbb{R}^2 : x^k + y^k = 1\}$ for $k = 1,2,3,4$. Now you need to find a smooth function $f : M \to S^2$ with a smooth inverse $g : S^2 \to M$. To do this, find explicitly smooth maps $F\colon \mathbb{R}^3 \setminus \{0\} \to S^2$ and $G\colon \mathbb{R}^3 \setminus \{0\} \to M$ such that the restriction of F to M and the restriction of G to S^2 are inverses.*]

5. Fix $k > 0$, and define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x,y,z) = \frac{x^2 + y^2}{(x^2 + y^2 + z^2 + k)^2}.$$

What are the regular values of f? For each regular value q of f, describe $f^{-1}(q)$.
[**Hint**: *First identify the points where $\frac{\partial f}{\partial z} = 0$, then identify which of those have $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ too.*]

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1. (i) By the chain rule $D\tilde{f}_z = (Df_x)^{-1} \circ Df_z$ for all z. Hence

$$\|D\tilde{f}_z - \mathrm{Id}_{\mathbb{R}^n}\| = \|(Df_x)^{-1}\|\|Df_z - Df_x\| \leq K\|Df_z - Df_x\| \leq KN\|z - x\|$$

by the Mean Value Inequality for Df, which is $< 1/2$ for $z \in B_\delta(x)$. Lemma 1.22 thus ensures that the image $\tilde{f}(B_\delta(x))$ contains $B_{\delta/4}(\tilde{f}(x))$.

(ii) Since $(Df_x)^{-1}$ is linear with $\|(Df_x)^{-1}\|_{op} = K$, the image of $B_{\delta/(4K)}(f(x))$ under $(Df_x)^{-1}$ is contained in $B_{\delta/4}(\tilde{f}(x))$, and hence the image of $B_{\delta/4}(\tilde{f}(x))$ under Df_x contains $B_{\delta/(4K)}(f(x))$. Hence $f(B_\delta(x)) = (Df_x)(\tilde{f}(B_\delta(x)))$ contains $B_{\delta/(4K)}(f(x))$.

2. One can cover S^n by $2n+2$ parametrisations of this form, two for each of the $n+1$ coordinates. For example, writing the last coordinate as a function of the other n we get

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \sqrt{1 - x_1^2 - \cdots x_n^2})$$

and

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, -\sqrt{1 - x_1^2 - \cdots x_n^2}).$$

Now, each of the $2n+2$ vectors $\pm e_1, \ldots, \pm e_{n+1}$ is contained in the image of only one parametrisation of this form, so S^n cannot be covered by less than $2n+2$ such parametrisations.

3. Points on the two lines in question can be written as $(1-t)(0, \pm 1) + t(x, 0)$ with $t \in \mathbb{R}$. The intersection points with S^2 are given by

$$1 = \|(1-t)(0, \pm 1) + t(x, 0)\|^2 = \|(1-t)(0, \pm 1)\|^2 + \|tx\|^2 = 1 - 2t + t^2(\|x\|^2 + 1),$$

with solutions $t = 0$ and $t = \frac{2}{1+\|x\|^2}$. The former corresponds to $(0, 0, \pm 1)$, while the latter gives the desired expression for $\varphi_\pm(x)$.

$U_\pm$ is $S^n \setminus \{(0, \pm 1)\}$. (Thus we have covered S^n by two parametrisations.)

4. (i) The derivative of the function $\mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto x^k + y^k + z^k$ is represented by

$$\begin{pmatrix} kx^{k-1} & ky^{k-1} & kz^{k-1} \end{pmatrix},$$

which vanishes only at the origin. In particular, 1 is a regular value of this function, so its preimage M is a submanifold.

(ii) Define $F : \mathbb{R}^3 \setminus \{0\} \to S^2$ and $G : \mathbb{R}^3 \setminus \{0\} \to M$ by

$$F(x, y, z) = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}, \quad G(x, y, z) = \frac{(x, y, z)}{(x^k + y^k + z^k)^{1/k}}$$

These are both well-defined smooth functions (k even ensures that $x^k + y^k + z^k$ never vanishes), so the restrictions $f = F|_M : M \to S^2$ and $g = G|_{S^2} : S^2 \to M$ are smooth too, and they are inverse to each other.

5. We first identify the points $(x,y,z) \in \mathbb{R}^3$ where $Df_{(x,y,z)} : \mathbb{R}^3 \to \mathbb{R}$ fails to be surjective. Since the codomain is $\mathbb{R}$, that just means checking when all the partial derivatives are zero. Now $\partial f/\partial z = \frac{-4(x^2+y^2)z}{(x^2+y^2+z^2+k)^3}$ vanishes only when either $z = 0$ or $x = y = 0$.

For fixed z, we have $f(x,y,z) = g(R)$ where $g(R) := \frac{R}{(R+z^2+k)^2}$ and $R = x^2+y^2$. Hence

$$\frac{\partial f}{\partial x} = 2xg'(R), \quad \frac{\partial f}{\partial y} = 2yg'(R),$$

so both vanish if and only if $x = y = 0$ or

$$0 = g'(R) = \frac{-R+z^2+k}{(R+z^2+k)^3}.$$

Hence the set of points where $Df_{(x,y,z)}$ fails to be surjective is the union of the line $\{x = y = 0\}$ and the circle $\{z = 0,\ x^2+y^2 = k\}$. The image of this set is $\{0, \frac{1}{4k}\}$, so the set of regular values is $\mathbb{R} \setminus \{0, \frac{1}{4k}\}$. For $q > 0$

$$f(x,y,z) = q \iff x^2+y^2+z^2+k = \tfrac{1}{\sqrt{q}}\sqrt{x^2+y^2} \iff (\sqrt{x^2+y^2} - \tfrac{1}{2\sqrt{q}})^2 + z^2 = \tfrac{1}{4q} - k.$$

If $0 < q < \frac{1}{4k}$, then this is an equation defining a torus in $\mathbb{R}^3$, while if $q < 0$ or $q > \frac{1}{4k}$ then $f^{-1}(q)$ is empty.