MA40254 Differential and geometric analysis : Exercises 3

Hand in answers by 1:15pm on Wednesday 25 October for the Seminar of Thursday 26 October
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Show that $M := \{(x,y) \in \mathbb{R}^2 : y = x^2\}$ is a 1-dimensional submanifold of $\mathbb{R}^2$.

1. Let $U \subset \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}^n$ be a twice differentiable function such that Df_x is invertible at $x \in U$. Let K be the operator norm of $(Df_x)^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ and N the supremum of the operator norm of $D(Df)_z : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ for $z \in U$ (where $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is itself equipped with the operator norm).

Suppose that $0 < \delta < 1/(2KN)$ and the open ball $B_\delta(x)$ is contained in U.

(i) Let $\tilde{f} := (Df_x)^{-1} \circ f$. Show that the image $\tilde{f}(B_\delta(x))$ contains $B_{\delta/4}(\tilde{f}(x))$.

(ii) Show that the image $f(B_\delta(x))$ contains the ball $B_{\delta/(4K)}(f(x))$.

2. Consider parametrisations of φ of $S^n := \{y \in \mathbb{R}^{n+1} : \|y\| = 1\}$ that are "graphs over a coordinate plane" in the following sense: $\varphi : B^n \to U$, for $B^n := \{x \in \mathbb{R}^n : \|x\| < 1\}$ and $U \subset S^n$ some open subset, and all but one of the $n+1$ components of $\varphi(x)$ is equal to a component of x (e.g., φ could be of the form $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, g(x_1, \ldots, x_n), x_i, \ldots x_n))$. How many parametrisations of this form are required for the images U to cover S^n?

3. Define two parametrisations $\varphi_+, \varphi_- : \mathbb{R}^n \to S^n$ as follows. For $x \in \mathbb{R}^n$, consider the line $L_\pm$ in $\mathbb{R}^{n+1}$ that contains $(x, 0)$ and $(0, \pm 1)$, and let $\varphi_\pm(x)$ be the intersection point (other than $(0, \pm 1)$) of $L_\pm$ with S^n.

Show that

$$\varphi_\pm(x) = \frac{1}{1 + \|x\|^2}\left(2x,\ \pm(\|x\|^2 - 1)\right).$$

What are the images $U_\pm$ of $\varphi_\pm$?

4. Let k be a positive integer, and let $M = \{(x,y,z) \in \mathbb{R}^3 : x^k + y^k + z^k = 1\}$.

(i) Show that M is a submanifold of $\mathbb{R}^3$.

(ii) Show that if k is even then M is diffeomorphic to S^2.

5. Fix $k > 0$, and define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x,y,z) = \frac{x^2 + y^2}{(x^2 + y^2 + z^2 + k)^2}.$$

What are the regular values of f? For each regular value q of f, describe $f^{-1}(q)$.

DMJC 17 October