MA40254 Differential and geometric analysis : Exercises 2

Hand in answers by 1:15pm on Wednesday 18 October for the Seminar of Thursday 19 October
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** For $U \subseteq \mathbb{R}^n$ open, let $f : U \to \mathbb{R}^n$ be a local diffeomorphism. Show $f(U)$ is open.

[**Solution**: *Apply the Inverse Function Theorem at each $x \in U$: this implies the existence of a neighbourhood $U' \subseteq U$ of x such that $f(U')$ is open in $\mathbb{R}^n$. Thus $f(U)$ contains an open neighbourhood of each of its elements.*]

1. Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be C^1. Suppose Df_x is surjective for every $x \in U$. Show that $f(U)$ be open in $\mathbb{R}^m$?

[**Hint**: *Reduce this problem to the solution of question 1 for each $x \in U$ by restricting f to $\{x + v : v \in W\}$ for a suitable m-dimensional subspace $W \leq \mathbb{R}^n$.*]

2. Let U be an open subset of $\mathbb{R}^2 \setminus \{0\}$, and let $f : U \to \mathbb{R}_{>0} \times \mathbb{R}$, $(x, y) \mapsto (r, \theta)$ be a smooth function such that $x = r(x, y)\cos\theta(x, y)$ and $y = r(x, y)\sin\theta(x, y)$ for any $(x, y) \in U$. Compute $Df_{(x,y)}$ for $(x, y) \in U$.

[**Hint**: *What is the relationship of f to $g : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$, $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$ and what does this imply about Df and Dg?*]

3. Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x, y, z) := (x^2 + y^2 + z,\ xz^2 - yz)$. For which $(x, y, z) \in \mathbb{R}^3$ is $Df_{(x,y,z)} : \mathbb{R}^3 \to \mathbb{R}^2$ surjective?

[**Hint**: *When are the two rows of $Df_{(x,y,z)}$ linearly independent? One approach is to compute the three 2×2 minors. You should find that there are two cases depending on whether $z = 0$ or not.*]

4. Let $GL_n(\mathbb{R}) \subset M_{n,n}(\mathbb{R})$ be the subset of invertible matrices in the vector space of real $n \times n$ matrices. Let $\mathrm{inv} : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ be the (continuous) function $A \mapsto A^{-1}$.

(i) Explain why $GL_n(\mathbb{R})$ is open in $M_{n,n}(\mathbb{R})$.

(ii) Show that the derivative of inv at the identity matrix $I \in GL_n(\mathbb{R})$ is

$$D\mathrm{inv}_I = -\,\mathrm{Id}_{M_{n,n}(\mathbb{R})}\,.$$

(iii) Identify the derivative $D\mathrm{inv}_A : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R})$ at $A \in GL_n(\mathbb{R})$, and deduce that inv is smooth.

[**Hint**: *(i) The determinant is a continuous function. (ii) Note that $(I+X)((I+X)^{-1} - I + X) = X^2$. (iii) Define L_A and $R_A : Mat_{n,n}(\mathbb{R}) \to Mat_{n,n}(\mathbb{R})$ by $X \mapsto AX$ and $X \mapsto XA$. What can you say about $L_A \circ \mathrm{inv} \circ R_A$? Now apply the chain rule.*]

5. (i) Let the function $\chi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\chi(t) := \begin{cases} 0 & \text{for } t \leq 0 \\ e^{-1/t} & \text{for } t > 0 \end{cases}$$

Show that there is a polynomial p_n of degree $\leq 2n$ such that for $t > 0$, the nth derivative of $\chi(t)$ is $p_n(1/t)\chi(t)$. Hence or otherwise, prove that χ is a smooth function on $\mathbb{R}$. [You may assume results about “exponentials dominating polynomials”.]

[**Hint**: *Use induction: you don't need to find an explicit expression for p_n.*]

(ii) For any $x \in \mathbb{R}^n$ and any $r > 0$, show that there is a smooth function $\rho : \mathbb{R}^n \to \mathbb{R}$ such that $\{y \in \mathbb{R}^n : \rho(y) \neq 0\}$ is the open ball $B_r(x)$. [This is called a “bump function”.]

[**Hint**: *Precompose χ with a suitable function of $\|y - x\|$.*]

MA40254 Differential and geometric analysis : Solutions 2

1. For any $x \in U$, the hypothesis implies there is a subspace $W \leq \mathbb{R}^n$, with $\mathbb{R}^n = W \oplus \ker Df_x$, such that the restriction of Df_x to W gives a linear isomorphism $W \to \mathbb{R}^m$. Now consider the translation map $T_x : W \to \mathbb{R}^n$, $z \mapsto x + z$. Then $\tilde{U} := T_x^{-1}(U)$ is an open subset of W, containing the origin. Let $\tilde{f} := f \circ T_x : \tilde{U} \to \mathbb{R}^m$. This is a smooth function, with $D\tilde{f}_0 = Df_x|_W$ by the chain rule. Since that is an isomorphism, the image $\tilde{f}(\tilde{U})$ contains an open neighbourhood of $\tilde{f}(0) = f(x)$ by the Inverse Function Theorem. Since $\tilde{f}(\tilde{U}) \subseteq f(U)$, therefore $f(U)$ too contains an open neighbourhood of $f(x)$. Applying this argument for each $x \in U$ shows that $f(U)$ is open.

2. Let $g : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$, $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$. Then $g \circ f = \mathrm{Id}_U$, so $Df_{(x,y)}$ is the inverse of $Dg_{f(x,y)}$. The matrix representing $Dg_{(r,\theta)}$ is $\left(\begin{smallmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{smallmatrix}\right)$, with inverse

$$\frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{pmatrix}.$$

3. $Df_{(x,y,z)} : \mathbb{R}^3 \to \mathbb{R}^2$ is represented by the matrix

$$\begin{pmatrix} 2x & 2y & 1 \\ z^2 & -z & 2xz - y \end{pmatrix}.$$

$Df_{(x,y,z)}$ is not surjective if and only if this matrix has less than full rank—equivalently its rows are linearly dependent, or its three minors are all zero. The first minor is

$$2x(-z) - 2yz^2 = -2z(x + yz),$$

so vanishes if and only if $x = -yz$ or $z = 0$. When $x = -yz$, the vanishing of the other two minors

$$2x(2xz - y) - z^2, \qquad 2y(2xz - y) + z,$$

reduces to $y = \pm\sqrt{\frac{z}{4z^2+2}}$. On the other hand, if $z = 0$ then the vanishing of the other two minors reduces to $x = 0$ or $y = 0$. The case $x = 0$ is already covered by $x = -yz$ as before. On the other hand, if $y = 0$ then the same formula for y as before is still valid. All in all, we can say that the derivative fails to be surjective precisely when

$$y = z = 0, \text{ or}$$
$$y = \pm\sqrt{\frac{z}{4z^2+2}} \text{ and } x = -yz.$$

4. (i) $GL_n(\mathbb{R})$ is the complement to the zero set of $\det : M_{n,n}(\mathbb{R}) \to \mathbb{R}$. The determinant function is continuous, so its zero set is closed.

(ii) We claim that $D\mathrm{inv}_I = -\mathrm{Id}_{M_{n,n}(\mathbb{R})}$. This is equivalent to saying that

$$\frac{\|(I+X)^{-1} - I + X\|}{\|X\|} \to 0$$

as $X \to 0$ in $M_{n,n}(\mathbb{R})$. We can use any norms we like, but lets use the operator norm. Then, since $(I+X)((I+X)^{-1} - I + X) = I + (X^2 - I) = X^2$,

$$\|(I+X)^{-1} - I + X\| \leq \|(I+X)^{-1}\| \|X^2\| \leq \|(I+X)^{-1}\| \|X\|^2.$$

Because inv is continuous, certainly $\|(I+X)^{-1}\| < 2\|I\| = 2$ for $\|X\|$ small, so

$$\frac{\|(I+X)^{-1} - I + X\|}{\|X\|} < 2\|X\| \to 0$$

as $X \to 0$, thus proving that $D\mathrm{inv}_I(X) = -X$.

(iii) Given $A \in GL_n(\mathbb{R})$, define linear maps

$$L_A : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R}), X \mapsto AX,$$
$$R_A : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R}), X \mapsto XA.$$

The fact that $B^{-1} = (AB)^{-1}A$ for any $B \in GL_n(\mathbb{R})$ means that $\mathrm{inv} = R_A \circ \mathrm{inv} \circ L_A$. By the chain rule (and using that L_A and R_A are linear maps)

$$D\mathrm{inv}_I = R_A \circ D\mathrm{inv}_A \circ L_A.$$

Hence

$$D\mathrm{inv}_A = (R_A)^{-1} \circ (-\mathrm{Id}_{M_{n,n}(\mathbb{R})}) \circ (L_A)^{-1} = R_{-A^{-1}} \circ L_{A^{-1}},$$

i.e.,

$$D\mathrm{inv}_A(X) = -A^{-1}XA^{-1}.$$

An alternative proof is to observe that $m \circ (\mathrm{Id}, \mathrm{inv}) : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R}); A \mapsto AA^{-1} = I$ is constant, so

$$0 = Dm_{A,A^{-1}} \circ (D\,\mathrm{Id}_A, D\mathrm{inv}_A)(X) = Dm_{A,A^{-1}}(X, D\mathrm{inv}_A(X)) = XA^{-1} + AD\mathrm{inv}_A(X)$$

(by the product rule) and again we get $D\mathrm{inv}_A(X) = -A^{-1}XA^{-1}$.

In other words $D\mathrm{inv} = RL \circ \mathrm{inv}$ where $RL(B) = R_{-B} \circ L_B$ is a product of the linear maps $B \mapsto R_{-B}$ and $B \mapsto L_B$, and hence is differentiable by the product rule with derivative $D(RL)_B(Z) = R_{-Z} \circ L_B + R_{-B} \circ L_Z$. Since inv is differentiable, $D\mathrm{inv}$ is differentiable by the chain rule with $D(D\mathrm{inv})_A(X) = R_{A^{-1}XA^{-1}} \circ L_{A^{-1}} + R_{-A^{-1}} \circ L_{-A^{-1}XA^{-1}}$. Applying this to Y we get

$$D^2\mathrm{inv}_A(X,Y) = A^{-1}Y(A^{-1}XA^{-1}) + (A^{-1}XA^{-1})YA^{-1}.$$

In general, induction on k, using linearity, the chain rule, product rule and $D\mathrm{inv}$, shows that

$$D^k\mathrm{inv}_A(X_1, \ldots X_k) = (-1)^k \sum_{\sigma \in S_k} A^{-1}X_{\sigma(1)}A^{-1}X_{\sigma(2)} \cdots A^{-1}X_{\sigma(k)}A^{-1}.$$

Alternatively, observe that $A^{-1} = \mathrm{adj}(A)/\det(A)$ where $\mathrm{adj}(A)$ and $\det(A)$ are polynomial in the entries of A. Hence $\mathrm{adj}(A)$ and $1/\det(A)$ have partial derivatives of all orders (by induction in the latter case, since $\det(A)$ is nonvanishing), and now the iterated product rule shows A^{-1} is smooth.

5. (i) We use induction, as the claim clearly holds for $n = 0$ with $p_n(x) = 1$. Suppose the claim holds for $n = k$. Then for $t > 0$

$$\chi^{(k+1)}(t) = \frac{d}{dt}\left(p_k\left(1/t\right)e^{-1/t}\right) = \left(-\frac{p_k'(1/t)}{t^2} + \frac{p_k\left(1/t\right)}{t^2}\right)e^{-1/t},$$

so if we set $p_{k+1}(x) := x^2(-p_k'(x) + p_k(x))$ we are done.

For $t < 0$ it is clear that the nth derivative of $\chi(t)$ is 0. We prove $\chi^{(n)}(t)$ is differentiable at $t = 0$ with derivative $\chi^{(n+1)}(0) = 0$ by induction on n: $(\chi^{(n)}(0+\varepsilon) - \chi^{(n)}(0))/\varepsilon = \chi^{(n)}(\varepsilon)/\varepsilon$ which is $p_n(1/\varepsilon)e^{-1/\varepsilon}/\varepsilon$ for $\varepsilon > 0$ and zero for $\varepsilon < 0$. Thus it has limit 0 as $\varepsilon \to 0$ since exponentials dominate polynomials.

(ii) Define $\rho : \mathbb{R}^n \to \mathbb{R}$ by

$$\rho(y) := \chi\left(r^2 - \|y - x\|^2\right).$$

This is a smooth function (by the chain rule) that is non-zero precisely on $B_r(x)$.