MA40254 Differential and geometric analysis : Exercises 2

Hand in answers by 1:15pm on Wednesday 18 October for the Seminar of Thursday 19 October
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** For $U \subseteq \mathbb{R}^n$ open, let $f : U \to \mathbb{R}^n$ be a local diffeomorphism. Show $f(U)$ is open.

[**Hint**: *Apply the Inverse Function Theorem at each $x \in U$: this implies the existence of a neighbourhood $U' \subseteq U$ of x such that $f(U')$ is open in $\mathbb{R}^n$. Thus $f(U)$ contains an open neighbourhood of each of its elements.*]

1. Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be C^1. Suppose Df_x is surjective for every $x \in U$. Show that $f(U)$ be open in $\mathbb{R}^m$?

[**Hint**: *Reduce this problem to the solution of question 1 for each $x \in U$ by restricting f to $\{x + v : v \in W\}$ for a suitable m-dimensional subspace $W \leq \mathbb{R}^n$.*]

2. Let U be an open subset of $\mathbb{R}^2 \setminus \{0\}$, and let $f : U \to \mathbb{R}_{>0} \times \mathbb{R}$, $(x, y) \mapsto (r, \theta)$ be a smooth function such that $x = r(x, y) \cos\theta(x, y)$ and $y = r(x, y) \sin\theta(x, y)$ for any $(x, y) \in U$. Compute $Df_{(x,y)}$ for $(x, y) \in U$.

[**Hint**: *What is the relationship of f to $g : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$, $(r, \theta) \mapsto (r \cos\theta, r \sin\theta)$ and what does this imply about Df and Dg?*]

3. Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x, y, z) := (x^2 + y^2 + z,\ xz^2 - yz)$. For which $(x, y, z) \in \mathbb{R}^3$ is $Df_{(x,y,z)} : \mathbb{R}^3 \to \mathbb{R}^2$ surjective?

[**Hint**: *When are the two rows of $Df_{(x,y,z)}$ linearly independent? One approach is to compute the three 2×2 minors. You should find that there are two cases depending on whether $z = 0$ or not.*]

4. Let $GL_n(\mathbb{R}) \subset M_{n,n}(\mathbb{R})$ be the subset of invertible matrices in the vector space of real $n \times n$ matrices. Let $\mathrm{inv} : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ be the (continuous) function $A \mapsto A^{-1}$.

(i) Explain why $GL_n(\mathbb{R})$ is open in $M_{n,n}(\mathbb{R})$.

(ii) Show that the derivative of inv at the identity matrix $I \in GL_n(\mathbb{R})$ is

$$D\mathrm{inv}_I = -\,\mathrm{Id}_{M_{n,n}(\mathbb{R})}\,.$$

(iii) Identify the derivative $D\mathrm{inv}_A : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R})$ at $A \in GL_n(\mathbb{R})$, and deduce that inv is smooth.

[**Hint**: *(i) The determinant is a continuous function. (ii) Note that $(I+X)((I+X)^{-1} - I + X) = X^2$. (iii) Define L_A and $R_A : Mat_{n,n}(\mathbb{R}) \to Mat_{n,n}(\mathbb{R})$ by $X \mapsto AX$ and $X \mapsto XA$. What can you say about $L_A \circ \mathrm{inv} \circ R_A$? Now apply the chain rule.*]

5. (i) Let the function $\chi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\chi(t) := \begin{cases} 0 & \text{for } t \leq 0 \\ e^{-1/t} & \text{for } t > 0 \end{cases}$$

Show that there is a polynomial p_n of degree $\leq 2n$ such that for $t > 0$, the nth derivative of $\chi(t)$ is $p_n(1/t)\chi(t)$. Hence or otherwise, prove that χ is a smooth function on $\mathbb{R}$. [You may assume results about "exponentials dominating polynomials".]

[**Hint**: *Use induction: you don't need to find an explicit expression for* p_n.]

(ii) For any $x \in \mathbb{R}^n$ and any $r > 0$, show that there is a smooth function $\rho\colon \mathbb{R}^n \to \mathbb{R}$ such that $\{y \in \mathbb{R}^n : \rho(y) \neq 0\}$ is the open ball $B_r(x)$. [This is called a "bump function".]

[**Hint**: *Precompose* χ *with a suitable function of* $\|y - x\|$.]

DMJC 10 October