# MA40254 Differential and geometric analysis : Exercises 2

*Hand in answers by 1:15pm on Wednesday 18 October for the Seminar of Thursday 19 October*
*Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709*

**0** (Warmup)**.** For $U \subseteq \mathbb{R}^n$ open, let $f : U \to \mathbb{R}^n$ be a local diffeomorphism. Show $f(U)$ is open.

**1.** Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be $C^1$. Suppose $Df_x$ is surjective for every $x \in U$. Show that $f(U)$ be open in $\mathbb{R}^m$?

**2.** Let $U$ be an open subset of $\mathbb{R}^2 \setminus \{0\}$, and let $f : U \to \mathbb{R}_{>0} \times \mathbb{R}$, $(x, y) \mapsto (r, \theta)$ be a smooth function such that $x = r(x, y)\cos\theta(x, y)$ and $y = r(x, y)\sin\theta(x, y)$ for any $(x, y) \in U$. Compute $Df_{(x,y)}$ for $(x, y) \in U$.

**3.** Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x, y, z) := (x^2 + y^2 + z,\ xz^2 - yz)$. For which $(x, y, z) \in \mathbb{R}^3$ is $Df_{(x,y,z)} \colon \mathbb{R}^3 \to \mathbb{R}^2$ surjective?

**4.** Let $GL_n(\mathbb{R}) \subset M_{n,n}(\mathbb{R})$ be the subset of invertible matrices in the vector space of real $n \times n$ matrices. Let $\mathrm{inv} : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ be the (continuous) function $A \mapsto A^{-1}$.

(i) Explain why $GL_n(\mathbb{R})$ is open in $M_{n,n}(\mathbb{R})$.

(ii) Show that the derivative of inv at the identity matrix $I \in GL_n(\mathbb{R})$ is

$$D\mathrm{inv}_I = -\operatorname{Id}_{M_{n,n}(\mathbb{R})}\,.$$

(iii) Identify the derivative $D\mathrm{inv}_A : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R})$ at $A \in GL_n(\mathbb{R})$, and deduce that inv is smooth.

**5.** (i) Let the function $\chi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\chi(t) := \begin{cases} 0 & \text{for } t \le 0 \\ e^{-1/t} & \text{for } t > 0 \end{cases}$$

Show that there is a polynomial $p_n$ of degree $\le 2n$ such that for $t > 0$, the $n$th derivative of $\chi(t)$ is $p_n(1/t)\chi(t)$. Hence or otherwise, prove that $\chi$ is a smooth function on $\mathbb{R}$. [You may assume results about "exponentials dominating polynomials".]

(ii) For any $x \in \mathbb{R}^n$ and any $r > 0$, show that there is a smooth function $\rho\colon \mathbb{R}^n \to \mathbb{R}$ such that $\{y \in \mathbb{R}^n : \rho(y) \neq 0\}$ is the open ball $B_r(x)$. [This is called a "bump function".]

*DMJC 10 October*