# MA40254 Differential and geometric analysis : Exercises 1

*Hand in answers by 1:15pm on Wednesday 11 October for the Seminar of Thursday 12 October*
*Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709*

**1.** Let $U$ be the open interval $(-1,1) \subset \mathbb{R}$, and let $f : U \to \mathbb{R},\ x \mapsto x^2$.

(i) For $x \in U$, what are the domain and codomain of $Df_x$, the derivative of $f$ at $x$?

(ii) For which $x \in U$ is $Df_x$ injective?

(iii) What are the domain and codomain of the derivative function $Df$?

(iv) Is $Df$ injective?

[**Hint**: *$Df_x$ is a linear map represented by a $1 \times 1$ matrix, whose only entry is $\frac{df}{dx}$.*]

**2.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y) \mapsto (x^3 - y^2, xy)$. For which $(x,y) \in \mathbb{R}^2$ is $Df_{(x,y)}$ an isomorphism?

[**Hint**: *First find the $2 \times 2$ matrix that represents $Df_{(x,y)}$.*]

**3.** Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces, and let $\mathcal{L}(V,W)$ be the vector space of linear maps $V \to W$. For $\phi \in \mathcal{L}(V,W)$, define its operator norm by

$$\|\phi\|_{op} := \sup_v \|\phi(v)\|_W,$$

where the supremum is taken over $v \in V$ such that $\|v\|_V = 1$. Show that $\|\cdot\|_{op}$ is a norm on $\mathcal{L}(V,W)$.

[**Hint**: *Use the defining properties of the norm on $W$ to derive the properties required for the operator norm.*]

**4.** Which of the following functions are smooth?

(i) $f : S \to \mathbb{R},\ x \mapsto \sqrt[3]{x^3 - 2}$, where $S := \mathbb{Q} \subset \mathbb{R}$.

(ii) $g : S \to \mathbb{R},\ (x,y) \mapsto \begin{cases} \sqrt{y} \text{ if } x \geq 0 \\ -\sqrt{y} \text{ if } x \leq 0 \end{cases}$, where $S := \{(x,y) : y = x^2\} \subset \mathbb{R}^2$.

(iii) $h : S \to \mathbb{R},\ (x,y) \mapsto \sqrt{y}$, where $S := \{(x,y) : y = x^2\} \subset \mathbb{R}^2$.

[**Hint**: *(i) What is the biggest open subset $U \subset \mathbb{R}$ such that the function $x \mapsto \sqrt[3]{x^3 - 2}$ is differentiable on $U$?*

*(ii) Find a simple function $G : \mathbb{R}^2 \to \mathbb{R}$ such that the restriction of $G$ to $S$ equals $g$.*

*(iii) Suppose $U \subseteq \mathbb{R}^2$ is an open subset containing the origin, and $H : U \to \mathbb{R}$ is a smooth function whose restriction to $U \cap S$ equals $h$. For a suitable interval $I \subseteq \mathbb{R}$, what can you say about the composition of $I \to \mathbb{R}^2,\ t \mapsto (t, t^2)$ with $H$?* ]

**5.** Let $U \subseteq \mathbb{R}^n$ open, and let $f : U \to \mathbb{R}^m$ be a smooth function. If $Df_x$ is injective for every $x \in U$, must $f$ be injective?

[**Hint**: *We will see later that if $Df_x$ at $x$ in $U$ then $f$ is injective on a neighbourhood of $x$—but does this imply $f$ is injective on all of $U$? Think of simple examples (angles on a circle maybe?)*]

**6.** (i) Compute the derivative of the matrix multiplication map

$$m : M_{m,n}(\mathbb{R}) \times M_{n,p}(\mathbb{R}) \to M_{m,p}(\mathbb{R}),\ (A, B) \mapsto AB$$

(ii) Compute the derivative of $s : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R}),\ A \mapsto A^2$.

[**Hint**: *(i) This is the product rule for matrix multiplication, so we expect $Dm_{(A,B)}(X, Y) = XB + AY$: prove that this expectation is correct, using the operator norm on matrices (viewed as linear maps), which satisfies $\|XY\| \leq \|X\|\|Y\|$.*

*(ii) Write $s$ as the composition of the diagonal map $M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R}) \times M_{n,n}(\mathbb{R}),\ A \mapsto (A, A)$ and $m : M_{n,n}(\mathbb{R}) \times M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R})$.* ]

**1.** (i) $Df_x$ is the linear map $v \mapsto 2xv$ with domain $\mathbb{R}$ and codomain $\mathbb{R}$.

(ii) $v \mapsto 2xv$ is injective if and only if $x \neq 0$.

(iii) $Df$ is a function $U \to \mathcal{L}(\mathbb{R}, \mathbb{R})$, where $\mathcal{L}(\mathbb{R}, \mathbb{R})$ is the space of linear maps from $\mathbb{R}$ to itself. ($\mathcal{L}(\mathbb{R}, \mathbb{R})$ is naturally isomorphic to $\mathbb{R}$.)

(iv) The linear maps $Df_x : v \mapsto 2xv$ and $Df_y : v \mapsto 2yv$ are equal if and only if $2x = 2y$ if and only if $x = y$, so $Df$ is injective.

**2.** $Df_{(x,y)}$ is represented by the matrix of partial derivatives

$$\begin{pmatrix} 3x^2 & -2y \\ y & x \end{pmatrix}.$$

This is invertible if and only if the determinant $3x^3 + 2y^2$ is non-zero, i.e., $x = -\sqrt[3]{2y^2/3}$.

**3.** There are three things to check.

(i) For $\phi \in \mathcal{L}(V, W)$, we have $\|\phi\|_{op} \geq 0$ with equality if and only if $\phi = 0$.

Since $\|w\|_W$ is non-negative for all $w \in W$, the $\sup_v \|\phi(v)\|_W$ is non-negative too. If the latter is 0, then $\phi(v) = 0$ for all $v \in V$ such that $\|v\|_V = 1$, and by linearity $\phi$ must be 0.

(ii) $\|\lambda\phi\|_{op} = |\lambda|\|\phi\|_{op}$

$\sup_v \|\lambda\phi(v)\|_W = \sup_v |\lambda|\|\phi(v)\|_W = |\lambda| \sup_v \|\phi(v)\|_W$

(iii) $\|\phi + \psi\|_{op} \leq \|\phi\|_{op} + \|\psi\|_{op}$.

$\sup_v \|\phi(v) + \psi(v)\|_W \leq \sup_v(\|\phi(v)\|_W + \|\psi(v)\|_W) \leq \sup_v \|\phi(v)\|_W + \sup_v \|\psi(v)\|_W$

**4.** (i) Let $U := \mathbb{R} \setminus \{\sqrt[3]{2}\}$. Then $F : U \to \mathbb{R},\ x \mapsto \sqrt[3]{x^3 - 2}$ is a smooth function on an open set whose restriction to $S$ equals $f$. Thus $f$ is smooth.

(ii) Note that $g(x, y) = x$ for any $(x, y) \in S$. Thus if we set $G : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x$ then $G$ is a smooth function whose restriction to $S$ equals $g$. Hence $g$ is smooth.

(iii) Suppose $h$ is smooth. Then there is an open subset $U \subseteq \mathbb{R}^2$ containing the origin, and a smooth function $H : U \to \mathbb{R}$ whose restriction to $U \cap S$ equals $h$.

Let $\varphi : \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (t, t^2)$, and let $I := \varphi^{-1}(U)$. Then the composition $H \circ \gamma : I \to \mathbb{R}$ is smooth by the chain rule. But in fact $H \circ \varphi(t) = |t|$, which is not smooth at $t = 0 \in I$.

**5.** No: take $U := \mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}^2$, $\theta \mapsto (\cos\theta, \sin\theta)$. Then $Df_\theta(v) = (-v\sin\theta, v\cos\theta)$ which is injective for all $\theta$; but $f(2\pi) = f(0)$. [An alternative counterexample is the map $x \mapsto x^2$ from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$.]

**6.** (i) We check that $Dm_{(A,B)}(X,Y) = XB + AY$:

$$\frac{\|(m(A+X, B+Y) - m(A,B) - (XB+AY)\|}{\|(X,Y)\|} = \frac{\|XY\|}{\|(X,Y)\|}.$$

We are free to choose which norms we use: convenient choices are the operator norm on matrices (viewed as linear maps), because this satisfies $\|XY\| \le \|X\|\|Y\|$, and, on pairs of matrices, the norm $\|(X,Y)\| = \max\{\|X\|, \|Y\|\}$. Then the ratio above is $\le \min\{\|X\|, \|Y\|\}$, which tends to 0 as $(X,Y) \to 0$.

(ii) Let $\Delta : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R}) \times M_{n,n}(\mathbb{R})$ denote the diagonal map $A \mapsto (A,A)$. Then $s = m \circ \Delta$, so by the chain rule $Ds_A(X) = Dm_{(A,A)}(X,X) = AX + XA$.