MA40254 Differential and geometric analysis : Exercises 1

Hand in answers by 1:15pm on Wednesday 11 October for the Seminar of Thursday 12 October
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

1. Let U be the open interval $(-1, 1) \subset \mathbb{R}$, and let $f : U \to \mathbb{R},\ x \mapsto x^2$.

(i) For $x \in U$, what are the domain and codomain of Df_x, the derivative of f at x?

(ii) For which $x \in U$ is Df_x injective?

(iii) What are the domain and codomain of the derivative function Df?

(iv) Is Df injective?

[**Hint**: *Df_x is a linear map represented by a 1×1 matrix, whose only entry is $\frac{df}{dx}$.*]

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (x^3 - y^2, xy)$. For which $(x, y) \in \mathbb{R}^2$ is $Df_{(x,y)}$ an isomorphism?

[**Hint**: *First find the 2×2 matrix that represents $Df_{(x,y)}$.*]

3. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces, and let $\mathcal{L}(V, W)$ be the vector space of linear maps $V \to W$. For $\phi \in \mathcal{L}(V, W)$, define its operator norm by

$$\|\phi\|_{op} := \sup_v \|\phi(v)\|_W,$$

where the supremum is taken over $v \in V$ such that $\|v\|_V = 1$. Show that $\|\cdot\|_{op}$ is a norm on $\mathcal{L}(V, W)$.

[**Hint**: *Use the defining properties of the norm on W to derive the properties required for the operator norm.*]

4. Which of the following functions are smooth?

(i) $f : S \to \mathbb{R},\ x \mapsto \sqrt[3]{x^3 - 2}$, where $S := \mathbb{Q} \subset \mathbb{R}$.

(ii) $g : S \to \mathbb{R},\ (x, y) \mapsto \begin{cases} \sqrt{y} & \text{if } x \geq 0 \\ -\sqrt{y} & \text{if } x \leq 0 \end{cases}$, where $S := \{(x, y) : y = x^2\} \subset \mathbb{R}^2$.

(iii) $h : S \to \mathbb{R},\ (x, y) \mapsto \sqrt{y}$, where $S := \{(x, y) : y = x^2\} \subset \mathbb{R}^2$.

[**Hint**: *(i) What is the biggest open subset $U \subset \mathbb{R}$ such that the function $x \mapsto \sqrt[3]{x^3 - 2}$ is differentiable on U?*

(ii) Find a simple function $G : \mathbb{R}^2 \to \mathbb{R}$ such that the restriction of G to S equals g.

(iii) Suppose $U \subseteq \mathbb{R}^2$ is an open subset containing the origin, and $H : U \to \mathbb{R}$ is a smooth function whose restriction to $U \cap S$ equals h. For a suitable interval $I \subseteq \mathbb{R}$, what can you say about the composition of $I \to \mathbb{R}^2,\ t \mapsto (t, t^2)$ with H?]

5. Let $U \subseteq \mathbb{R}^n$ open, and let $f : U \to \mathbb{R}^m$ be a smooth function. If Df_x is injective for every $x \in U$, must f be injective?

[**Hint**: *We will see later that if Df_x at x in U then f is injective on a neighbourhood of x—but does this imply f is injective on all of U? Think of simple examples (angles on a circle maybe?)*]

6. (i) Compute the derivative of the matrix multiplication map

$$m : M_{m,n}(\mathbb{R}) \times M_{n,p}(\mathbb{R}) \to M_{m,p}(\mathbb{R}),\ (A, B) \mapsto AB$$

(ii) Compute the derivative of $s : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R}),\ A \mapsto A^2$.

[**Hint**: *(i) This is the product rule for matrix multiplication, so we expect $Dm_{(A,B)}(X, Y) = XB + AY$: prove that this expectation is correct, using the operator norm on matrices (viewed as linear maps), which satisfies $\|XY\| \leq \|X\|\|Y\|$.*

(ii) Write s as the composition of the diagonal map $M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R}) \times M_{n,n}(\mathbb{R}),\ A \mapsto (A, A)$ and $m : M_{n,n}(\mathbb{R}) \times M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R})$.]

DMJC 3 October