Xu Saping

Geometric construction of crystals for $U_q(g)$

(I)

$U_q^+ \otimes K \otimes U_q^-$

$\mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n^+$ $\quad U(\mathfrak{g}) \quad$ $U_q(\mathfrak{g})$

Simple Lie algebra $\quad$ enveloping algebra $\quad$ quantised enveloping algebra

$\lim_{q\to 1} U_q(\mathfrak{g}) = U(\mathfrak{g})$ 'quantum group'

h_i was previously called $\alpha_i^\vee$

- e_i, f_i, h_i
- $[h_i, h_j] = 0$
- $[h_i, e_j] = \langle h_i, \alpha_j \rangle e_j$
- Serre relations
- $[e_i, f_j] = \delta_{ij} h_i$

- $e_i, f_j, K_\lambda \quad (\lambda \in P)$ wt lattice
- $K_\lambda K_\mu = K_{\lambda+\mu}$
- $K_\lambda e_j = q^{(\lambda,\alpha_j)} e_j K_\lambda$
- $K_\lambda f_j = q^{-(\lambda,\alpha_j)} f_j K_\lambda$
- $[e_i, f_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$

$\left(\begin{matrix} A\text{-}D\text{-}E \\ q_i = q \ \forall i \end{matrix}\right)$

algebra over $\mathbb{Q}(q)$

- $i \neq j$ $\quad$ q-Serre relations

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix} f_i^n f_j f_i^{1-a_{ij}-n} = 0 \qquad \sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} e_i^n e_j e^{1-a_{ij}-n} = 0$$

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad [n]_q! = [n]_q \cdots [1] \qquad \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q! \, [n-m]_q!}$$

We shall construct $B(\infty)$ which is a crystal for U_q^- ($\overset{?}{=} U_q(n^-)$) as a $U_q(g)$ module

(I) Kashiwara's operators

Let $A = \{ f/g \in \mathbb{Q}(q) \mid g(0) \neq 0 \}$

Let M be a $U_q(g)$ module. (f-d integral module)

Let (L, B) be as follows:

① L is a free A-submodule of M s.t. $\longrightarrow$ ($L \cong A^n$)

$$M \cong \mathbb{Q}(q) \otimes_A L$$

② B is a basis of the $\mathbb{Q}$-vector space L/qL

Fact: $M = \bigoplus_\gamma M_\gamma$ $\leftarrow$ weight space decomposition

$$= \bigoplus_{0 \leq n \leq \langle h_i, \nu \rangle} f_i^n (\ker e_i \cap M_\nu)$$

over ν and n $\rightarrow$

"divided power": $f_i^{(n)} = f^n / [n]_q!$

Def (Kashiwara's operators)

$$M \longrightarrow M$$

$\tilde{f}_i : \quad f_i^{(n)} u \longmapsto f_i^{(n+1)} u \qquad u \in \ker e_i \cap M_\nu$

$\tilde{e}_i : \quad f_i^{(n)} u \longmapsto f_i^{(n-1)} u \qquad \text{and } 0 \leq n \leq \langle h_i, \nu \rangle$

Def (L, B) is called a crystal basis for M if

- $\tilde{e}_i L \subset L$, $\tilde{f}_i L \subset L$

$\Rightarrow$ $\tilde{e}_i$ and $\tilde{f}_i$ act on L/qL

- $\tilde{e}_i B \subset B\{0\}$, $\tilde{f}_i B \subset B \cup \{0\}$

(could be 3) $\rightarrow$ - $L = \bigoplus_{\nu \in P} L_\nu \quad B = \bigsqcup_{\gamma \in P} B_\gamma$

where $L_\gamma = L \cap M_\gamma$, $B_\gamma = B \cap \left(L_\gamma / qL_\gamma \right)$

- $b, b' \in B$, $b' = \tilde{f}_i b \Leftrightarrow b = \tilde{e}_i b'$

Example $\lambda \in P_+$, $V(\lambda)$ = simple $U_q(g)$-module of highest weight λ

u_λ = highest weight wt vector

$L(\lambda) = \mathcal{A}$ - submodule spanned/generated by $\tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} u_\lambda$ and 'by things of the form'

$B(\lambda) = \{ \underbrace{\tilde{f}_{i_1}, \cdots, \tilde{f}_{i_\ell} u_\lambda \neq 0}_{\text{i.e. } \neq 0 \text{ in the quotient}} \in L/qL \}$

Then $(L(\lambda), B(\lambda))$ = a crystal basis for $V(\lambda)$

In particular, $B(\lambda)$ is a crystal.

End Example

The highest wt $U_q(g)$ - module U_q^- is

- generated by 1
- satisfying $e_i \cdot 1 = 0 \ \forall i$

<u>Fact</u>: $U_q^- = \bigoplus_{n \geq 0} f_i^{(n)} \ker e_i \qquad \forall i$

<u>Define</u>: $\tilde{e}_i, \tilde{f}_i$ by

$\tilde{f}_i : f_i^{(n)} u \longmapsto f_i^{(n+1)} u \qquad (u \in \ker e_i)$

$\tilde{e}_i : f_i^{(n)} u \longmapsto f_i^{(n-1)} u$

Example $\mathcal{L}(\infty)$ = A-submodule of U_q^- generated by $\tilde{f}_{i_1} \cdots \tilde{f}_{i_p} \cdot 1$

$$\mathcal{B}(\infty) = \Big\{ \underbrace{\tilde{f}_{i_1}, \dots, \tilde{f}_{i_p} \cdot 1 \neq 0 \text{ in } \frac{\mathcal{L}(\infty)}{q\mathcal{L}(\infty)}}_{\text{i.e. } \neq 0 \text{ in the quotient}} \Big\}$$

$(\mathcal{L}(\infty), \underbrace{\mathcal{B}(\infty)}_{\text{a crystal}})$ is a crystal basis for U_q^-

$\mathcal{B}(\infty)$: characterised by some properties

$i \in I$: $\mathcal{B}_i$ is the crystal defined below

$$\mathcal{B}_i = \{ b_i(n) : n \in \mathbb{Z} \} \sqcup \{b_0\}$$

$$\mathrm{wt}(b_i(n)) = n\alpha_i$$

$$\varphi_i(b_i(n)) = n, \quad \varepsilon_i(b_i(n)) = -n$$

$$\varphi_j(b_i(n)) = -\infty, \quad \varepsilon_j(b_i(n)) = -\infty \quad (j \neq i)$$

$$\tilde{e}_i(b_i(n)) = b_i(n+1)$$

$$\tilde{f}_i(b_i(n)) = b_i(n-1)$$

$\tilde{e}_j(b_i(n)) = \tilde{f}_j(b_i(n)) = 0 \quad (i \neq j)$

$b_0 \in B$ with wt 0

Assume the following hold

① $\mathrm{wt}(B) \subset Q_-$

② b_0 is the unique elt of B with wt 0.

③ $\varepsilon_i(b_0) = 0 \quad \forall i$

④ $\varepsilon_i(b) \in \mathbb{Z} \quad \forall b, i$

⑤ For $\forall i$, $\exists$ 'strict' embedding $\Psi_i : B \to B \otimes B_i$

⑥ $\Psi_i(B) = B \times \{\tilde{f}_i^n b_i, n \geq 0\} \quad (b_i = b_i(0))$

⑦ For any $b \in B$ s.t. $b \neq b_0$ $\exists i$ s.t.

Then $B \cong B(\infty)$ $\quad \Psi_i(b) = b' \otimes \tilde{f}_i^n b_i$ with $n > 0$