1.

APK

Models of Crystals

1) Semistandard tableaux – for (irreps of) $SL(V)$

$\dim V = n$ type A_{n-1}

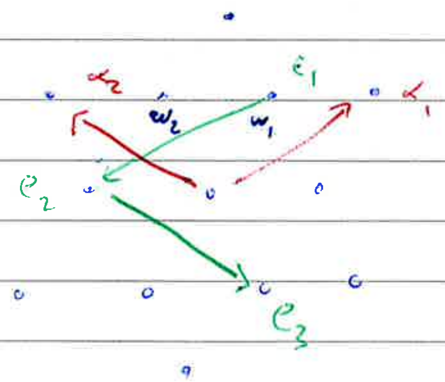
for tautological rep V

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots n-1 \xrightarrow{n-1} n$$

vertices $i \longleftrightarrow$ weights e_i of V in $\Lambda = \mathbb{Z}^n/(1,\dots,1)$

arrows j — i.e. $\sum e_i = 0$

i.e. partial f'ns $F_j \longleftrightarrow$ simple roots $\alpha_j = e_j - e_{j+1}$



irreps $V_\lambda \longleftrightarrow$ partition λ with $< n$ parts

crystal $= \{$s-s tableau filled with weights $1, \dots, n$ of shape $\lambda\}$ e.g. $\lambda = 3, 2$ $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}$ 'highest' weight of crystal V_λ

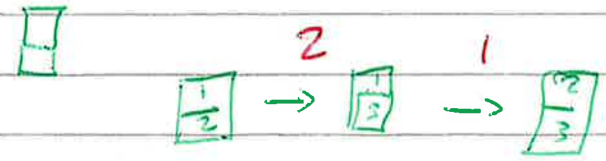
non decr. →

incr. ↓

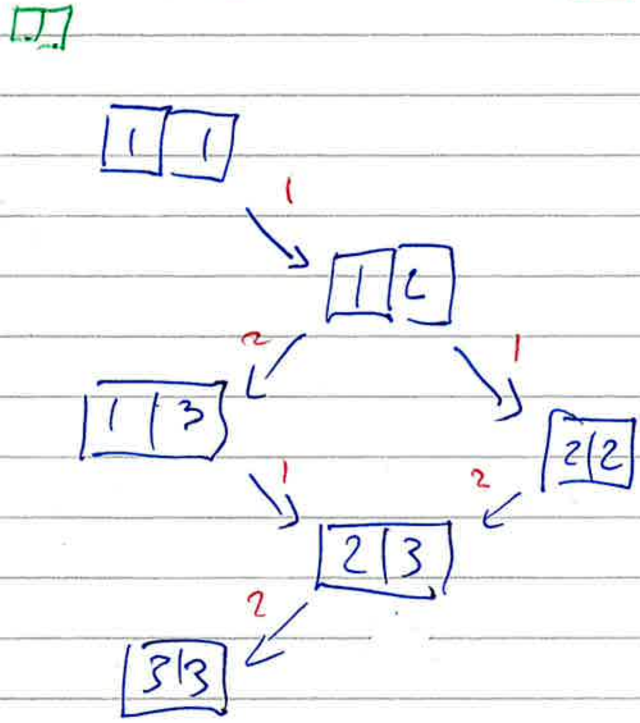
F_j = change last unmatched wt j to $j+1$

e.g for $SL(3)$

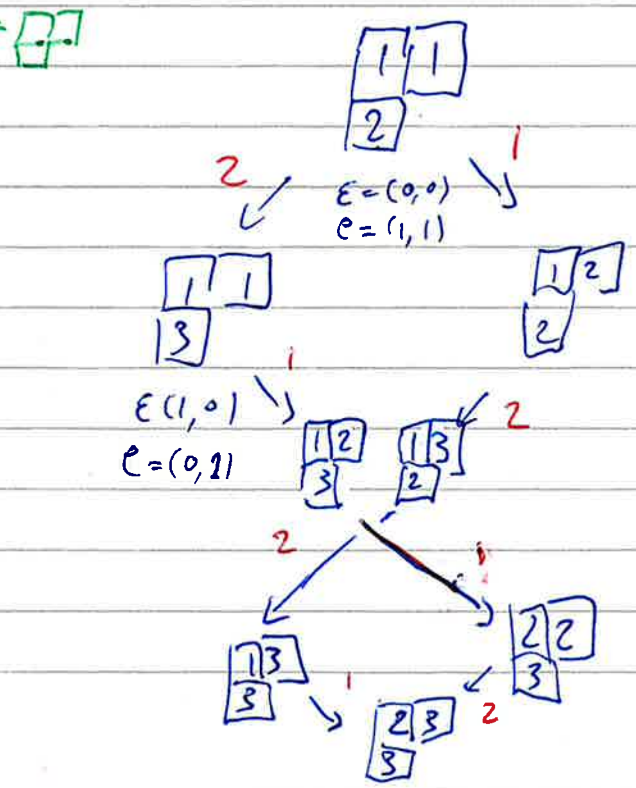
$\lambda =$



$\lambda =$



$\lambda =$



str. fns ε, e

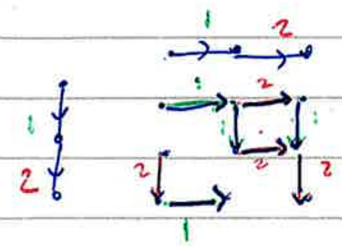
wt = $e - \varepsilon$ in

basis of <u>fundamental</u>

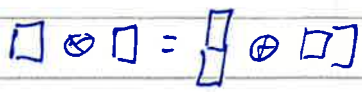
<u>weights</u>

2) Tensor product rule (Littlewood - Richardson for A_{n-1})

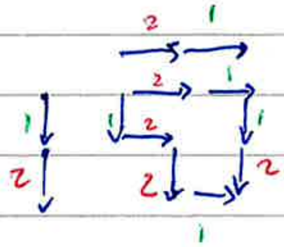
$V \otimes V$

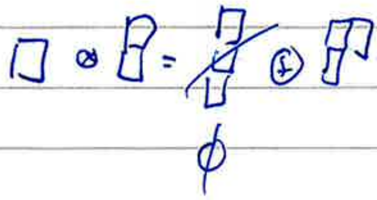


$\Lambda^2 V \oplus S^2 V$



$V \otimes V^*$





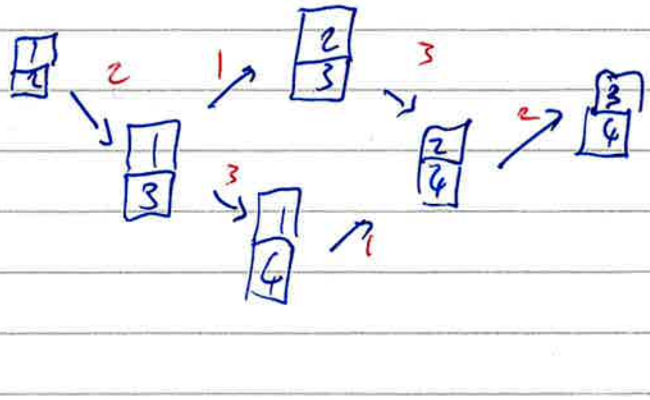
2) Preprojective modules

Another 'model' for k i.e. $\Lambda^k(V)$ of

$SL(V)$ 'fundamental' rep'ns

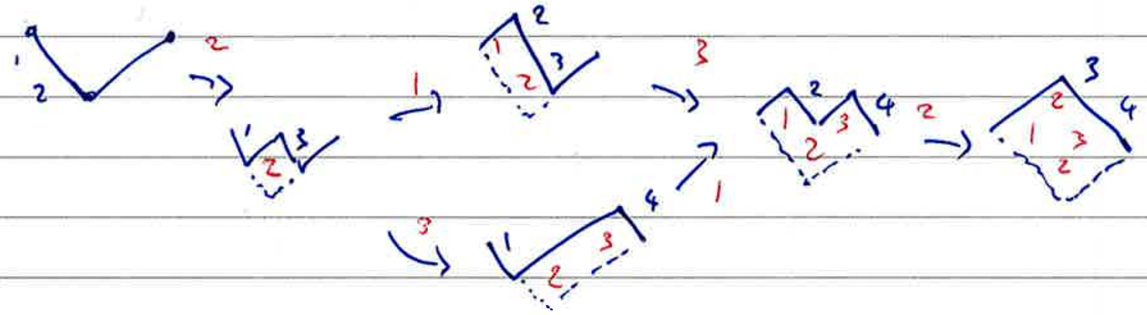
eg. $SL(4)$

$k=2$



crystal $\longleftrightarrow$ {k subsets J of $\{1, \dots, n\}$ $\longleftarrow$ ..

$\rightsquigarrow$ down-right paths $(0,k) \to (n-k, 0)$ in $\mathbb{Z}^2$



<u>Obs</u>

no numbers inside

Submodules of the projective-injective module Q_k for preproj. algebra $\Pi(A_3)$ $\xleftrightarrow{1:1}$ Young subdiagrams of $k \times n-k$ box, F_j add box in column j

<u>Def</u> Q a Dynkin quiver of type $\Delta \in \{A, D, E\}$

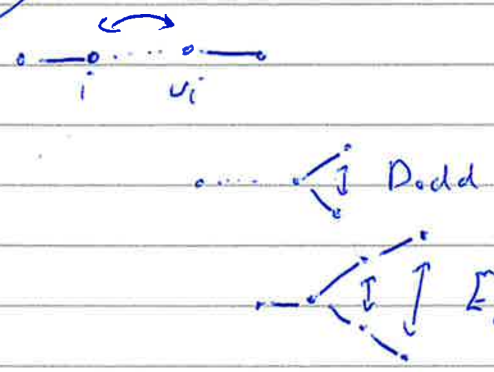
$$\underset{i}{\bullet} \xrightarrow{a} \underset{j}{\bullet} \qquad Q$$

$$e_i \underset{y_a}{\overset{x_a}{\rightleftarrows}} e_j \qquad \bar{Q}$$

$$\Pi(\Delta) = \mathbb{C}\bar{Q} / (\mu_i) \qquad \mu_i = \sum_{\overset{a}{\to} i} x_a y_a - \sum_{i \overset{a}{\to}} y_a x_a$$

index. projective $P_i = \Pi e_i$ 'paths from i'

index injective $Q_i = P_{\nu i}$ | ν = Nakayama



or $\nu = id$

type A_{n-1}

$Q_k = P_{n-k}$

'=' $k \times n-k$ box standing on its corner.

for any dominant weight $\lambda = \sum \lambda_i \omega_i$ have injective $Q(\lambda) = \bigoplus_i \lambda_i Q_i$

Note: $\underline{\dim}\, Q_i = \omega_i + \omega_{\nu i}$ in the basis of simple roots $\{\alpha_j\}$ diff of highest and lowest weight.

Def For $M = \{M_i, i \in Q_0, x_a, y_a, a \in Q_1\}$ a Π module and $\beta \in \mathbb{N}^{Q_0}$

$$Gr_\beta(M) \subseteq \prod_{i \in Q_0} Gr(\beta_i, M_i)$$ of submodules

of M e.g. $x_a(N_{ta}) \subseteq N_{ha}$ $\forall a$?
$y_a(N_{ha}) \subseteq N_{ta}$

Thm (Lusztig) For $\mathfrak{g}_\Delta$ a simple Lie alg of type Δ and $L(\lambda)$ irrep of h.w. λ then the mult of the μ wt space $L(\lambda)_\mu$ = # components $Gr_{\lambda-\mu}(Q(\lambda))$

Def For any Π module $M \leq Q(\lambda)$

let $\varepsilon_i(M) = \dim \mathrm{Hom}(M, S_i)$ 'ith top'

These have generic (minimal) values on components

$\longrightarrow$ $e_i(M) = \dim \mathrm{Hom}(S_i, Q(\lambda)/M)$ 'put i on top'

Expectation These are the structure functions for $\coprod_\beta \{\text{cts of } Gr_\beta(Q(\lambda))\}$ as the crystal of $L(\lambda)$

Variation

$\coprod_\beta \{\text{cpnts of } Rep_\beta(\Pi(\Delta))\}$ is the crystal $B_\Delta(\infty)$ of $U(n^-)$ $\mathfrak{g}_\Delta = n^+ \oplus h \oplus n^-$

in particular for wt $\mu = -\sum \beta_i \alpha_i$

$\dim U(n^-)_\mu = \#$ cpnt's $Rep_\beta(\Pi)$

str. functions. ε_i = generic values of $\dim Hom(M, S_i)$ as before

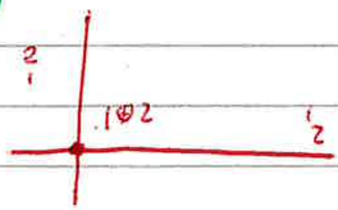
but $\varphi_i := (\alpha_i^\vee, wt) + \varepsilon_i$ doesn't measure forward length of root strings.

$\boxed{\text{e.g.}}$ $\Delta = A_2$  $xy = 0$, $yx = 0$

indecs Π modules are $\begin{smallmatrix}1\\2\end{smallmatrix}$, $\begin{smallmatrix}2\\1\end{smallmatrix}$, 1, 2

generic in cpnt of $Rep(\Pi)$ iff don't have both simples $1, 2$ as summands

e.g. $\beta = (1,1)$



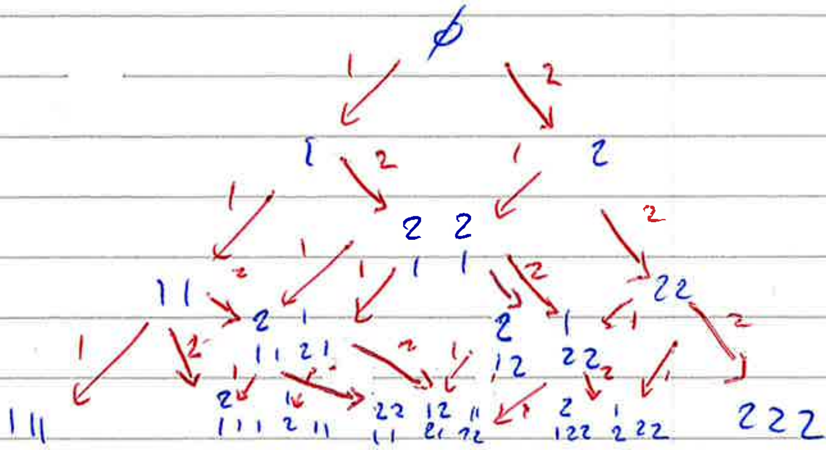


Fig. $B(\infty)$