Introduction to Crystals ($\otimes$ rule)

f.d. reps of reductive Lie alg

regard crystals as combinatorial obj!

basic data. set B partial functions

$$e_i : B \to B, \ i \in I$$

convenient to draw them as a directed graph w. edges labelled by I

$b \xrightarrow{i} c$ means $e_i(b) = c$

e_i : injective. nilpotent. (??)

Motivation: - understand branching rules
- generalize combinatorics of tableaux.

$\mathfrak{h} \subset \mathfrak{g}$ restrict to $V_G(\lambda)\downarrow = \bigoplus U^\lambda_\mu \otimes V_H(\mu)$

interested in $\dim \cancel{V} V^\lambda_\mu$

- wt multiplicities $T \subset G$, $\mathfrak{h} \subset \mathfrak{g}$
- tensor product mult. $\Delta : G \to G \times G$

$\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$

(plethysm, Schur functors)

Any branching rule is given by altanating sum over Weyl gp $\sum_{a \in W} (-1)^{\ell(a)} \cdots$

<u>Schur polynomials</u> :

$$s_\lambda = \frac{\det \left(x_j^{\lambda_i + n - i}\right)}{\det \left(x_j^{n-i}\right)} \qquad \text{symmetric.}$$

$$s_\lambda = \sum_T x^{wt(T)} \qquad T: \text{ semi standard tabeaux}$$

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 3 & & \\ \cline{1-1} \end{array} \rightsquigarrow x_1 x_2 x_3^2$$

- Wt mult. : weyl character formula
- tensor product multiplicity : Brauer/Kling K.

$GL(n)$, Littelwood - Rich coeficients

<u>Additional data</u> : $f_i : B \to B$, $i \in I$

$$f_i(c) = b \Leftrightarrow e_i(b) = c$$

$\varepsilon_i : B \to \mathbb{N}$ $\quad \varepsilon_i(b) = \max\{k \mid e_i^k(b) \neq 0\}$

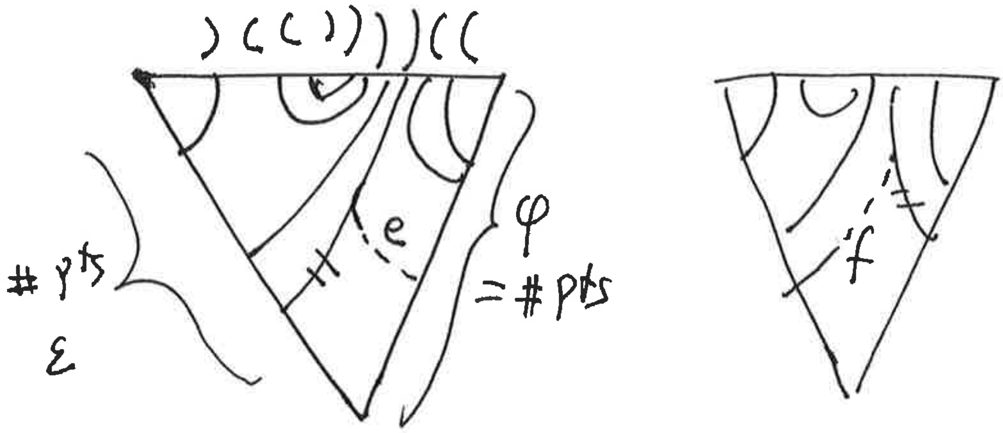
$\varphi_i : B \to \mathbb{N}$, $\quad \varphi_i(b) = \max\{k \mid f_i^k(b) \neq 0\}$

$wt(b) \; \text{"}{:=}\text{"} \; \varepsilon(b) - \phi(B)$ converts from roots to wts

~~?~~ $wt(e_i(b)) = wt(b) - \alpha_i$

$wt(f_i(b)) = wt(b) + \alpha_i$

Tropical △s.



no-crossing.

• No pts on the side

$\begin{cases} l \\ r \end{cases} \Rightarrow \begin{cases} e = 0 \\ f = 0 \end{cases}$

Read as words in {(,)}

Bijections

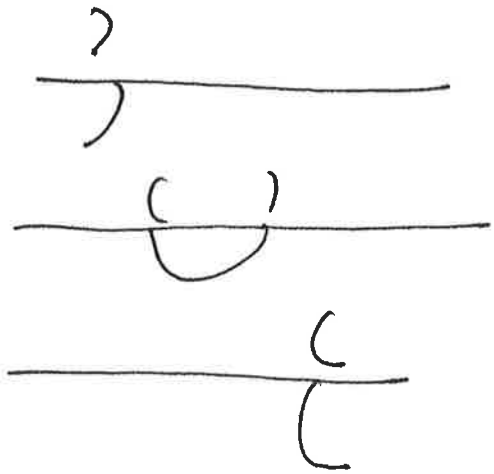
$\{\text{triangles}\} \overset{1\text{-}1}{\longleftrightarrow} \{\text{words}\}$

crystals — free monoid.

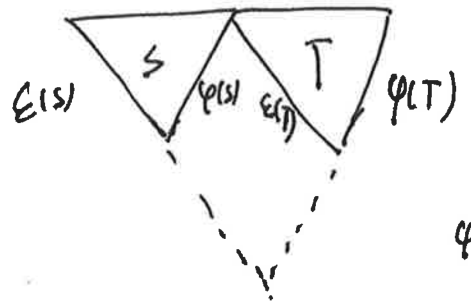
↑ — ↑

monoid str.? — crystal str.?

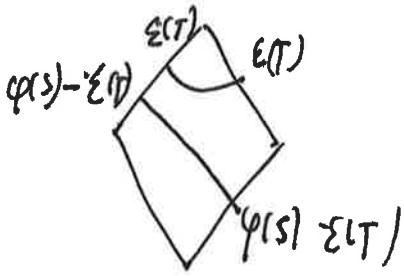
3 types of arcs



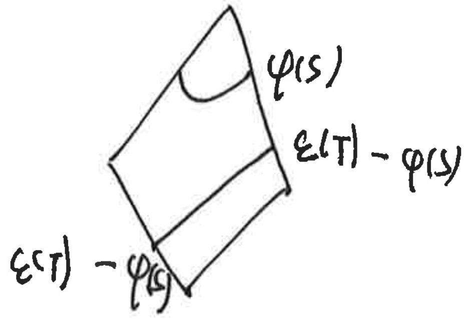
multiply $\triangle$s :



$\cdot\ \varphi(S) \geq \varepsilon(T)$



$U^3 =$



$\varepsilon(T) \geq \varphi(S)$

$$e(S \otimes T) = \begin{cases} e(S) \otimes T & \text{if } \varepsilon(T) > \varphi(S) \\ S \otimes e(T) & \text{if } \varphi(S) \geq \varepsilon(T) \end{cases}$$

(S, T are triangles)

$$\left(e_i(S \otimes T) = \begin{cases} e_i(S) \otimes T & \text{if } \varepsilon_i(T) > \varphi_i(S) \\ S \otimes e_i(T) & \text{if } \varphi_i(S) \geq \varepsilon_i(T) \end{cases} \right)$$

Lusztig involution
semistandard tableaux

h. wt $B \otimes B'$: $b \otimes b'$

h. wt $\varphi_i(b) \geq \varepsilon_i(b')$ $\forall i$.

$B' \otimes B$: $b' \otimes b$

$B \otimes B' \cong B' \otimes B$

Lusztig involution.

$B' \otimes B \longrightarrow$

$b \otimes b' \longmapsto \xi(\xi(b') \otimes \xi(b))$

θ : involution on simple roots

$\theta(\alpha_i) = -w_0 \alpha_i$ w_0 longest ~~wt~~ elements

$e_i \cdot \xi = \xi \cdot f_{\theta(\alpha_i)}$

$f_i \cdot \xi = \xi \, e_{\theta(\alpha_i)}$

$\xi(b), b$: on the same component.

On semistandard tab. ξ = evacuation (Schützenberger)