# I Crystals

Recall For any precrystal $B$ we have the following maps: (called root operators)

We will construct other crystal from these

$$f_i : B \sqcup \{\circ\} \longrightarrow B \sqcup \{\circ\}$$
$$b \longmapsto b' \text{ if } \exists\, b \xrightarrow{i} b', \ \circ \text{ else}$$
$$\circ \longmapsto \circ$$

$$e_i : B \sqcup \{\circ\} \longrightarrow B \sqcup \{\circ\}$$
$$b \longmapsto b' \text{ if } \exists\, b' \xrightarrow{i} b, \ \circ \text{ else}$$
$$\circ \longmapsto \circ$$

Let $\mathfrak{g}$ be a c.s-s f.d Lie alg with Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and roots $\Delta = \Delta^+ \sqcup \Delta^-$

**Def** A $\mathfrak{g}$-crystal is a precrystal $B$ together with a morphism of precrystals

$$wt : B \longrightarrow \Lambda$$

such that

$$\underset{\mathfrak{h}^*}{\varphi_i(b) - \varepsilon_i(b) = \langle wt(b)}, \underset{\mathfrak{h}}{\alpha_i^\vee} \rangle$$

where

$$\varphi_i(b) = \max\{n \geq 0 \mid f_i^n b \neq \circ\}$$

and

$$\varepsilon_i(b) = \max\{n \geq 0 \mid e_i^n b \neq \circ\}$$

## Silly Example

Let $V$ be an irreducible rep of $\mathfrak{g}$ with highest weight $\alpha$

vertices : weights of $V$

colours : simple roots (Indexing set)

edges : $\{ \lambda \xrightarrow{i} \lambda - \alpha_i \}_{i, \lambda}$

The inclusion into $\Lambda$ makes this a crystal.

This illustrates how having a crystal isn't 'enough'. What could we require? We consider the following.

For every crystal $B$ we have a Character

$$\mathrm{Char}\, B := \sum_{v \in B} e^{wt(v)} \in \mathbb{Z}[\Lambda_w]$$

Intergral group ring on the abelian group $\Lambda_w$

Want: for every irreducible rep $V$ a crystal $B_V$ s.t

$$\mathrm{Char}\, B_V = \mathrm{Char}\, V := \sum_{\beta \in h^*} \dim V_\beta \, e^\beta \in \mathbb{Z}[\Lambda_w]$$

This is not true for the silly example. (For the reasons David explained last week!)

Is true for Littelmann Path Model.

# II Path Root operators

$$h^*_{\mathbb{R}} := \langle \Delta \rangle_{\mathbb{R}}$$

Recall $\dim(h^*_{\mathbb{R}}) = \dim h^*$ and the Killing form is real, pos-semi definite on $h^*_{\mathbb{R}}$.

**Def** (Say what a path is)

(i) Let $\Pi$ be piece wise linear paths, $\pi$, s.t $\pi(0) = 0$ and $\pi(1) \in \Lambda_W$.

Weyl chamber

Not needed till IV (ii) $\Pi^+ := \{\pi \in \Pi \mid \operatorname{Im}\pi \subset W\}$

Interior of Weyl chamber

Not needed till III (iii) $\Pi^+_0 := \{\pi \in \Pi \mid \pi(t) \in W_0, \forall t > 0\}$

(Weyl Chamber: $W := \{\beta \in h^* \mid \beta(H_\alpha) \geq 0, \forall \alpha$ positive root$\}$
$W$ acts simply transitively on the possible Weyl chambers)

Let $\pi \in \Pi$, and let $\alpha$ be a simple root. We define

$$h_{\pi,\alpha} : [0,1] \to \mathbb{R}$$
$$t \longmapsto (\pi(t), \alpha^\vee) \quad \left(= \frac{2}{(\alpha,\alpha)}(\pi(t),\alpha)\right)$$

we then let $m_{\pi,\alpha} := \min h_{\pi,\alpha}$, and define

$$l_{\pi,\alpha} : [0,1] \to [0,1]$$
$$t \longmapsto \min_{s \in [t,1]} \{1, h_{\pi,\alpha}(s) - m_{\pi,\alpha}\}$$

non-decreasing

We can now define the following root operators:

$$f_\alpha : \Pi \sqcup \{\circ\} \longrightarrow \Pi \sqcup \{\circ\}$$

$$\pi \longmapsto \begin{cases} t \mapsto \pi(t) - \ell_{\pi,\alpha}(t)\cdot\alpha & \text{if } \ell_{\pi,\alpha}(1) = 1 \\ \Theta & \text{otherwise} \end{cases}$$

$$\Theta \longmapsto \Theta$$

This is a bit cheeky

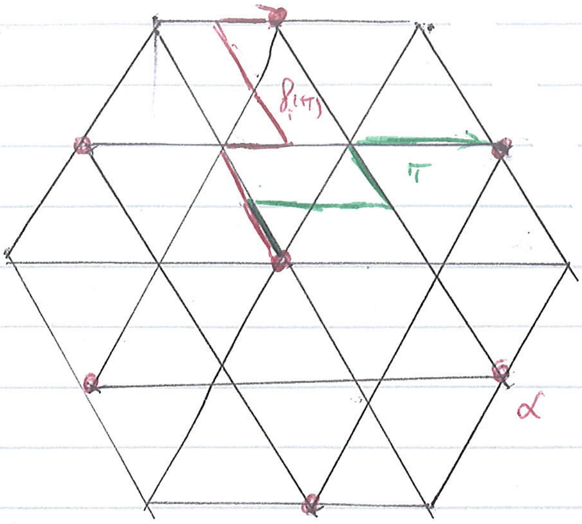
$$e_\alpha : \Pi \sqcup \{\circ\} \longrightarrow \Pi \sqcup \{\circ\}$$

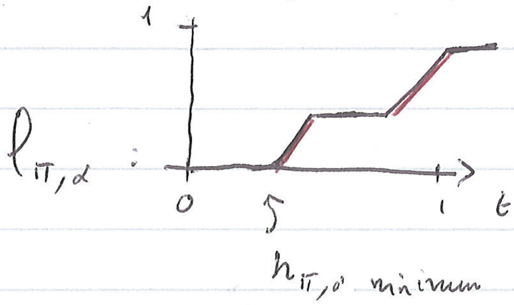
$$\pi \longmapsto \begin{cases} \pi' & \text{if } \exists!\, \pi' \text{ s.t. } f_\alpha(\pi') = \pi \\ \Theta & \text{else} \end{cases}$$

$$\Theta \longmapsto \Theta$$

* Do Example *

## Example




$\ell_{\pi,\alpha}$ will always be 0 until the last time $m_{\pi,\alpha}$ is obtained!

The root operator reflects pieces of the path, to move the endpoint by $-\alpha$ as opposed to a full reflexion which moves it by

$$(\pi(1), \alpha^\vee)\cdot\alpha$$

$$\|$$

$$n_\alpha^\pi \in \mathbb{Z} \text{ (as } \pi(1) \in \Lambda_W$$

# III An Initial Character Formula

Let $B \subset \Pi$ be stable under the root operators

$B$ defines a $\mathfrak{g}$-crystal via

$$\begin{aligned} &\text{vertices} : B \\ &\text{edges} : \{ \pi \xrightarrow{\ i\ } f_i(\pi) \}_{\pi, i} \\ &\text{wt} : B \longrightarrow \Lambda \\ &\qquad \pi \longmapsto \pi(1) \end{aligned}$$

Weyl Group

**Prop** Char $B$ is stable under $W$.

Sketch Proof For $\alpha$ a simple root we define

A tweaked reflection on paths

$$s_\alpha(\pi) := \begin{cases} f_\alpha^{n_\alpha^\pi}(\pi) & \text{if } n_\alpha^\pi > 0 \\ e_\alpha^{-n_\alpha^\pi}(\pi) & \text{otherwise} \end{cases}$$

Then check

$$s_\alpha^2 = id \qquad \text{and} \qquad s_\alpha(\pi)(1) = s_\alpha(\pi(1))$$

$\pi(1) \pm (\pi(1), \alpha^\vee)\alpha$

moves the end pt as a full reflection would

This gives a bij between $\{\pi \in B \mid \pi(1) = \beta\}$ and $\{\pi \in B \mid \pi(1) = s_\alpha(\beta)\}$ $\square$

**Thm** (We introduce an abuse of notation where $\beta \in \mathfrak{h}^*_{\mathbb{R}}$ denote the straight path to $\beta$)

$$\text{Char } B = \sum_{\substack{\pi \in B \\ \rho * \pi \in \Pi_0^+}} \text{Char } V_{\pi(1)}$$

Path concatenation

$\pi(t) \in W_0 \ \forall\, t > 0$

where $\rho$ is the Weyl vector (half sum of the positive roots)

## Sketch Proof

We show that

$$\left(\sum_{w\in W} \operatorname{sgn}(w) e^{w(\rho)}\right) \overbrace{\left(\sum_{\pi\in B} e^{\pi(1)}\right)}^{\text{char } B} = \sum_{\substack{\pi\in B \\ \rho*\pi\in\Pi_0^+}} \sum_{w\in W} \operatorname{sgn}(w) e^{w(\rho+\pi(1))}$$

By Weyl's character formula this proves the Theorem

$\Big\Updownarrow$ by W invariance

Only comparing coeffs of the form $e^\beta$ where $\beta \in \Lambda^+$

$$\sum_{(w,\pi)\in\Omega} \operatorname{sgn}(w) e^{w(\rho)+\pi(1)} = \sum_{\substack{\pi\in B \\ \rho*\pi\in\Pi_0^+}} e^{\rho+\pi(1)} \qquad (*)$$

dominant weights: $\Lambda_w^+ \cap W$

$$\Omega := \{(w,\pi)\in W\oplus B \mid w(\rho)+\pi(1)\in\Lambda_w^+\}$$

We consider

$\Omega_0 \subset \Omega$ given by

$$\Omega_0 := \{(w,\pi)\in W\oplus B \mid w = id, \rho*\pi\in\Pi_0^+\}$$

If the sum on the LHS of $(*)$ was over $\Omega_0$ the equality would be trivial.

Need to show:

$$\sum_{\Omega\setminus\Omega_0} \operatorname{sgn}(w) e^{w(\rho)+\pi(1)} = 0$$

Step 1 $\forall (w,\pi) \in \Omega \setminus \Omega_0$, $\exists F$ a face of $W$ s.t. $w(\rho) * \pi$ intersects $F$.

If $w(\rho) * \pi \notin \Pi_0^+$ then, as $w(\rho) * \pi(1) \in \Lambda^+$, the result is obvious. If $w \neq id$ then we have $w(\rho) \notin W$ and the result follows in the same way.

Step 2

We define

$$\mathcal{E} : \Omega \setminus \Omega_0 \longrightarrow \Omega \setminus \Omega_0$$

$$(w,\pi) \longmapsto \begin{cases} (s_\alpha w, f^{-n_\alpha^{w(\rho)}}(\pi)) & \text{if } n_\alpha^{w(\rho)} < 0 \\ (s_\alpha w, e^{n_\alpha^{w(\rho)}}(\pi)) & \text{else} \end{cases}$$

where $\alpha$ is the simple root corresponding to the last face of $W$ intersected by $w(\rho) * \pi$.

Exercise: check $\mathcal{E}^2 = id$. Let $(w', \pi')$ denote $\mathcal{E}(w,\pi)$. Then

$$sgn(w') = -sgn(w)$$

and

$$\begin{aligned} w'(\rho) * \pi'(1) &= w(\rho) - n_\alpha^{w(\rho)} \cdot \alpha + n_\alpha^{w(\rho)} \cdot \alpha + \pi(1) \\ &= \pi(\rho) * \pi(1) \end{aligned}$$

Therefore

$$\sum_{(w,\pi) \in \Omega \setminus \Omega_0} sgn(w) e^{w(\rho) + \pi(1)} = \sum_{(w',\pi') \in \Omega \setminus \Omega_0} sgn(w') e^{w(\rho) + \pi'(1)} = -\sum_{(w,\pi) \in \Omega \setminus \Omega_0} sgn(w) e^{w(\rho) + \pi(1)}$$

$\square$

## III.V The way forward

dominant weights
$W \cap \Lambda_w$

In view of our motivation in I and the theorem in III we would like to find, for every $\lambda \in \Lambda_w^+$, a set $B \subset \Pi$, stable under the root operators s.t

$\{ \pi \in B \mid p * \pi \in \Pi_0^+ \} = \{ \pi_0 \}$ where

$\pi_0(1) = \lambda$.

The next step is to define a special class of path called a 'locally integral concatenation' and show that if $\pi_0$ is a l.i.c and $\pi_0(1) = \lambda$ (and another condition)

$$\{ \pi \in B_{\pi_0} \mid p * \pi \in \Pi_0^+ \} = \{ \pi_0 \}$$

where $B_{\pi_0}$ is the smallest set that contains $\pi_0$ and is invariant under the root operators. This would be IV.
This is the Littelmann Path model.

V would then be to show that certain continuity properties of the root allow us to start with a more general path.

In fact, any path in $\Pi^+$ with end point $\lambda$.

(One can exhibit explicit isomorphism of the underlying pre crystals.)