Motivating Crystals - The yoga of roots & Weights

① $sl_2(\mathbb{F})$

Lie algebra, 3-dim^l vector space over $\mathbb{F}$ with basis $e, f, h \in sl_2(\mathbb{F})$ & brackets

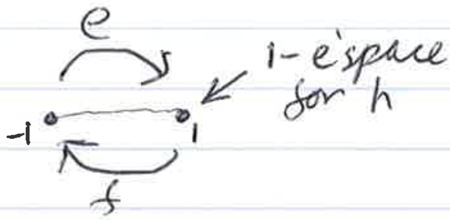
$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h$$

A rep$^{\underline{n}}$ of a Lie alg $\mathfrak{g}$ is a linear map $\rho: \mathfrak{g} \to gl(\mathfrak{g})$ s.t.

$$\rho([x,y]) = [\rho(x), \rho(y)] := \rho(x)\circ\rho(y) - \rho(y)\circ\rho(x)$$

<u>Example 1</u> $\rho: sl_2(\mathbb{F}) \to gl(\mathbb{F}^2)$

$$e \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$



$$f \longmapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$h \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

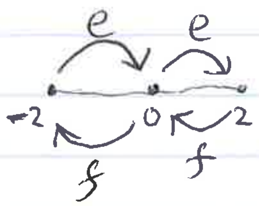
$sl_2(\mathbb{F}) \cong$ 2x2 traceless matrices

<u>Example 2</u> Any Lie alg has an adjoint rep$^{\underline{n}}$ $ad: \mathfrak{g} \to gl(\mathfrak{g})$; $x \mapsto ad(x)$ s.t. $ad(x)(y) = [x,y]$

e.g. $sl_2(\mathbb{F}) \longrightarrow gl(sl_2(\mathbb{F})) \cong gl(\mathbb{F}^{\{e,h,f\}})$

$$e \longmapsto \begin{array}{c} \begin{array}{ccc} e & h & f \end{array} \\ \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \qquad f \longmapsto \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$h \mapsto \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ diagonal with ~~entries~~ e'vals $-2, 0, 2$



General theory

For an irred rep^n ρ on V $\rho(h)$ is diagonalisable

e'vals are called 'weights'

e'vectors/spaces are called "weight vectors/spaces"

$V_\lambda = \{v \in V \mid \rho(h)v = \lambda v\}$

Lemma $\rho(e)V_\lambda \subseteq V_{\lambda+2}$, $\rho(f)V_\lambda \subseteq V$

V fin dim^L $V = \bigoplus V_\lambda$ $\exists \lambda$ highest wt w/ $V_\lambda \neq 0$, $V_{\lambda+2} = 0$

$v_0 \in V_\lambda$ $v_i = \frac{1}{i!} \rho(f)^i v_0$

$\lambda \in \mathbb{Z}$ $\dim V_\lambda = 1$ & we have a basis of wt'vecs related by e's and f's

② Semisimple Lie Algebras

$\mathfrak{g}$ semisimple, Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ max abelian & semisimple

$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$

$\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \ \forall h \in \mathfrak{h} \}$
root space for α

$\Delta = \{ \alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0 \}$ roots

$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ rep^n $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$

$V = \bigoplus_\lambda V_\lambda$, $V_\lambda = \{ v \in V \mid \rho(h)(v) = \lambda(h)v \}$

$V_\lambda \neq 0$ λ-weight
V_λ-weight space

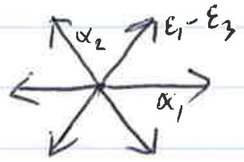
<u>Example 3</u>

$\mathfrak{sl}_3(\mathbb{F})$ $\mathfrak{h} = \{$diagonal traceless matrices$\}$

$\mathfrak{h} = \text{span}\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$

$\varepsilon_i\left(\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \right) = a_i$

$\exists$ 6 roots w/ 1-dim^L root spaces
$\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$



In general roots
$\Delta = \Delta_+ \sqcup \Delta_-$

& $\exists$ simple roots $\alpha_1, \dots, \alpha_k$ s.t. $\alpha = \sum_{i=1}^{k} n_i \alpha_i$

$\begin{cases} n_i \geq 0 \ \forall i & \alpha \in \Delta_+ \\ n_i \leq 0 \ \forall i & \alpha \in \Delta_- \end{cases}$

$\mathfrak{g}_{\alpha_i} = \langle e_i \rangle$ $\qquad$ $[h, e_i] = \alpha_i(h)\, e_i \quad \forall h \in \mathfrak{h}$

$\mathfrak{g}_{-\alpha_i} = \langle f_i \rangle$ $\qquad$ $[h, f_i] = -\alpha_i(h)\, f_i$

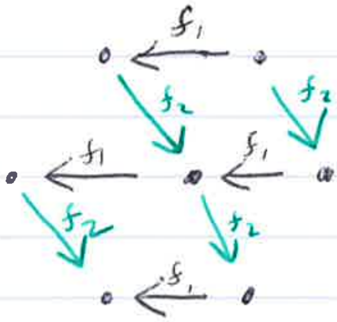
$$[e_i, f_j] = \begin{cases} \alpha_i^\vee \in \mathfrak{h} & i = j \\ 0 & \text{o.w.} \end{cases}$$

Now $\mathfrak{g}$ is determined by cartan matrix

$(a_{ij}) = \alpha_i(\alpha_j^\vee)$

<u>Punchline</u> For each $i \in \{1, \ldots, k\}$ have an $\mathfrak{sl}_2$ subalg : span $\{e_i, f_i, \alpha_i^\vee\}$

So repⁿs of $\mathfrak{g}$ are repⁿs of several $\mathfrak{sl}_2$'s stitched together nicely



Crystals ask the question : Can I find a nice / canonical basis for repⁿs related by raising/lowering ops

In this case choose any highest wt vec $0 \neq v_{\alpha_1+\alpha_2} \in \mathfrak{g}_{\alpha_1+\alpha_2}$

& define $v_{\alpha_2} = f_1 v_{\alpha_1+\alpha_2}$, $v_{\alpha_1} = f_2 v_{\alpha_1+\alpha_2}$

$f_2 f_1 v_{\alpha_1+\alpha_2} \neq f_1 f_2 v_{\alpha_1+\alpha_2}$

?

③ Pre-crystals & crystals

Def^n A pre-crystal is an edge coloured quiver (directed graph or digraph) s.t. at each vertex b & for each colour i there is at most 1 incoming and 1 outgoing arrow w/ colour i

Key example the weight lattice Λ of a root system $\Delta = \Delta_+ \sqcup \Delta_-$

vertices : $\lambda \in \Lambda \subseteq \mathfrak{h}^*$ $\quad (\lambda(\alpha_i^*) \in \mathbb{Z})$

colours : $I = \{\text{simple roots}\} \cong \{\text{nodes of Dynkin}\}$

$\cong \{f_1, \dots, f_n\}$
$\quad \alpha_1, \dots, \alpha_n$

edges : $f_i : \lambda \mapsto \lambda - \alpha_i$

for any pre-crystal B we have operators:

$$f_i : B \sqcup \{0\} \to B \sqcup \{0\}$$
$$b \mapsto \begin{cases} b' & \text{if } b \xrightarrow{f_i} b' \\ 0 & \text{o.w.} \end{cases}$$

Def$^{\underline{n}}$ A morphism of pre-crystals $\phi : B \to B'$ is a morphism of quivers preserving the edge colouring

A crystal for $\mathfrak{g}$ ~~is~~ (or a root system Δ) is a pre-crystal B w/ a morphism $B \to \Lambda$ to the pre-crystal of $\mathfrak{g}$ (or Δ) (satisfying some properties