Exercises 9

Please submit solutions by 3pm on Thursday 26th April to the pigeonholes in 4W (ground floor).

(W) = Warm-up; (H) = Homework; (A) = Additional.

1. (W) Let $\lambda \in \mathbb{C}$ and let $f(t) \in \mathbb{C}[t]$ be of degree n.

(1) Find $a_1, \ldots, a_n \in \mathbb{C}$ such that $f(t) = a_0 + a_1(t-\lambda) + a_2(t-\lambda)^2 + \cdots a_n(t-\lambda)^n$.

(2) For $\lambda = 2$ and $f(t) = t^4 - 3t^3 + 2t - 1$, compute the scalars $a_0, a_1, \ldots, a_4 \in \mathbb{C}$ from part (1).

2. (W) Write sketch proofs of Theorem 5.7 and Proposition 5.19.

3. (W) Let V be a finite dimensional vector space, and let $v \in V$ be nonzero. Let s be the smallest integer such that $\alpha^s v = 0$. Show that $v, \alpha v, \ldots, \alpha^{s-1}v$ are linearly independent.

4. (H) For $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$, consider the $n \times n$ complex matrix

$$J(\lambda, n) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}$$

in which every entry that is not shown is equal to zero. Compute the algebraic and geometric multiplicities of the unique eigenvalue λ. Also, compute the minimal polynomial.

5. (H) Consider the real-valued matrix

$$A = \begin{pmatrix} 1 & -5 & -7 \\ 1 & 4 & 2 \\ 0 & 1 & 4 \end{pmatrix}.$$

Find an invertible real-valued matrix P such that

$$P^{-1}AP = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

6. (A) For $k > 0$ and for $1 \leq i \leq k$, let $\alpha_i \colon V_i \to V_i$ be a $\Bbbk$-linear map for some field $\Bbbk$. Prove by induction that

$$\det(\alpha_1 \oplus \cdots \oplus \alpha_k) = \det(\alpha_1) \cdot \det(\alpha_2) \cdots \det(\alpha_k).$$

[Hint: it suffices to show that for $A_1 \in M_m(\Bbbk)$ and $A_2 \in M_n(\Bbbk)$, we have $\det(A_1 \oplus A_2) = \det(A_1) \cdot \det(A_2)$. You might try proving this by induction on m.]

The course website is: `http://people.bath.ac.uk/dmjc20/Alg2B`

Solutions 9

1. (1) We proceed by induction on the degree n of f. If $n = 0$ then the result is clear. Suppose $n > 0$. Since $\mathbb{C}[t]$ is a Euclidean domain, we have $f = (t-\lambda)q + r$ for some $q, a_0 \in \mathbb{C}[t]$, where q has degree $n-1$ and $\deg(a_0) < 1$. By induction,

$$q(t) = a_1 + a_2(t-\lambda) + \cdots + a_n(t-\lambda)^{n-1}$$

for some $a_i \in \mathbb{C}$ $(1 \leq i \leq n)$. Thus for $a_0 := r$, we get $f(t) = a_0 + a_1(t-\lambda) + \cdots + a_n(t-\lambda)^n$

(2) We seek $a_i \in \mathbb{C}$ such that $f(t) = a_0 + a_1(t-2) + \cdots + a_4(t-2)^4$. Cleary $a_0 = f(2) = 16 - 24 + 4 - 1 = -5$. Also taking derivatives gives

$$4t^3 - 9t^2 + 2 = \frac{d}{dt} f(t) = a_1 + 2a_2(t-2) + \cdots + 4a_4(t-2)^3.$$

Evaluating this at 2 yields $a_1 = 32 - 36 + 2 = -2$. Arguing in the same manner yields

$$a_2 = \frac{1}{2}(12t^2 - 18t)|_{t=2} = 6; \quad a_3 = \frac{1}{6}(24t - 18)|_{t=2} = 5; \quad a_4 = \frac{1}{24}(24) = 1.$$

Therefore $f(t) = -5 - 2(t-2) + 6(t-2)^2 + 5(t-2)^3 + (t-2)^4$.

2. No solution given.

3. Suppose

$$a_0 v + a_1 \alpha v + \cdots + a_{n-1}\alpha^{s-1} v = 0.$$

Applying α^{s-1} to both sides gives $a_0 \alpha^{s-1} v = 0$ and as $\alpha^{s-1} v \neq 0$ it follows that $a_0 = 0$. Next apply α^{s-2} to both sides and this gives $a_1 \alpha^{s-1} v = 0$ that imples that $a_1 = 0$. Continuing like this with $\alpha^{s-3}, \alpha^{s-4}, \ldots \alpha^0 = \mathrm{id}$, gives that $a_2 = \ldots = a_{n-1} = 0$.

4. Note that $\Delta_J(t) = (\lambda - t)^n$. The algebraic multiplicity of λ is n by definition. To determine the geometric multiplicity notice that $v = (a_1, \ldots, a_n)$ is in the λ-eigenspace if and only if

$$\begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & \\ & \cdot & \cdot & & \\ & \cdot & \cdot & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} \\ a_n \end{pmatrix} = \begin{pmatrix} a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \\ 0 \end{pmatrix},$$

which happens if and only if $a_2 = a_3 = \cdots = a_n = 0$, that is, if and only if $v = a_1 e_1$. Therefore the λ-eigenspace is $\langle e_1 \rangle$ which has dimension 1, so the geometric multiplicity of λ is 1.

For $A = J(\lambda, n)$, we compute that $\Delta_A(t) = (\lambda - t)^n$, so $m_A(t) = (t-\lambda)^s$ for some $1 \leq s \leq n$. Let E_{ij} be the matrix with 1 in position (i,j) and 0 elsewhere. Notice that

$$A - \lambda I = E_{12} + E_{23} + \cdots + E_{(n-1)n}$$

and that

$$(A - \lambda I)^{n-1} = E_{12}E_{23} \cdots E_{(n-1)n} = E_{1n} \neq 0,$$

whereas $(A - \lambda I)^n = E_{1n}(E_{12} + E_{23} + \cdots + E_{(n-1)n}) = 0$. Hence $m_A(t) = (t-\lambda)^n$.

5. The characteristic polynomial of A is $\Delta_A(t) = (3-t)^3$, so the eigenvalue $\lambda = 3$ has multiplicity three. We compute that

$$A - 3I \neq 0; \quad (A-3I)^2 \neq 0; \quad (A-3I)^3 = 0,$$

so the minimal polynomial is $m_A(t) = (t-3)^3$. Thus, there is at least one Jordan block $J(3,3)$, and since A is a 3×3 matrix, we see that the Jordan normal form of A must be $J(3,3)$. It remain to find the basis in which the corresponding linear map α is represented by this matrix.

Perform ERO's on the matrix

$$A - 3I = \begin{pmatrix} -2 & -5 & -7 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \text{ to obtain } \begin{pmatrix} -2 & -5 & -7 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, we solve $y + z = 0$ and $-2x - 5y - 7z = 0$. The set of such solutions is spanned by the eigenvector

$$v_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

so the eigenspace of $\lambda = 3$ has dimension one (confirming that $\lambda = 3$ determines only one block). This is the vector v_1 from Proposition 5.19. To compute v_2, Proposition 5.19 shows that

$$(\alpha - \lambda id)v_2 = v_1, \quad i.e., \quad (A-3I)v_2 = v_1.$$

That is, to compute v_2 we must solve

$$\begin{pmatrix} -2 & -5 & -7 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \quad \text{to get} \quad v_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

Repeat for the system

$$\begin{pmatrix} -2 & -5 & -7 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{to get} \quad v_3 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

Thus, putting the vectors *in the correct order* from left to right gives the invertible matrix

$$P = \begin{pmatrix} -1 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix},$$

and the Jordan normal form of A is

$$P^{-1}AP = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} = J.$$

To check this you needn't compute the inverse of P; just check that $AP = PJ$. [The question didn't ask you to check that your answer P was correct, but doing this is a good habit to get in to.]

6. Let $A_1 \in M_m(\mathbb{k})$ and $A_2 \in M_n(\mathbb{k})$. If $m = 1$, then $A_1 = (a_{11})$ and expanding about the top row of the matrix $A_1 \oplus A_2$ immediately gives $\det(A_1 \oplus A_2) = a_{11} \cdot \det(A_2) = \det(A_1) \cdot \det(A_2)$. Assume the result holds for the direct sum of any two square matrices where the first matrix lies in $M_\ell(\mathbb{k})$ with $\ell < m$. Now for $A_1 \in M_m(\Bbbk)$, write A_{ij} for the cofactor of the entry a_{ij} of $A_1 = (a_{ij})$. Then expand along the top row of $A_1 \oplus A_2$ to see that

$$\begin{aligned} \det(A_1 \oplus A_2) &= a_{11} \det(A_{11} \cdot A_2) - \cdots + (-1)^{m+1} a_{1m} \det(A_{1m} \cdot A_2) \\ &= a_{11} \det(A_{11}) \det(A_2) - \cdots + (-1)^{m+1} a_{1m} \det(A_{1m}) \det(A_2) \qquad \text{by induction} \\ &= \big(a_{11} \det(A_{11}) - \cdots + (-1)^{m+1} a_{1m} \det(A_{1m})\big) \det(A_2) \\ &= \det(A_1) \det(A_2) \end{aligned}$$

as required. This proves the statement in the hint. Induction on k gives

$$\det(A_1 \oplus \cdots \oplus A_k) = \det(A_1) \cdot \cdots \det(A_k)$$

which is the matrix version of the statement $\det(\alpha_1 \oplus \cdots \oplus \alpha_k) = \det(\alpha_1) \cdot \det(\alpha_2) \cdots \det(\alpha_k)$.