Exercises 8

Please submit solutions by 3pm on Thursday 19th April to the pigeonholes in 4W (ground floor).

(W) = Warm-up; (H) = Homework; (A) = Additional.

1. (W) Use the fact that $\mathbb{C}$ is a normed $\mathbb{R}$-algebra to show that the set $\{a^2+b^2 \mid a,b\in\mathbb{Z}\}$ is closed under multiplication, and hence write 85 as a sum of two integer squares.

2. (W) Write sketch proofs of: Theorem 2.14 (including 2.11), Theorem 3.11; Theorem 3.17 (including 3.15); Proposition 3.20 (including 3.7); Theorem 3.21 (existence only); Theorem 4.13 and Theorem 4.15.

3. (W) Find the characteristic polynomials of the following complex matrices, and determine the algebraic and geometric multiplicity of each of the eigenvalues.

$$A=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}, \quad B=\begin{pmatrix}0 & 2\\ -2 & 4\end{pmatrix}.$$

4. (H) For each quaternionic number $z=a+bi+cj+dk\in\mathbb{H}$, define the *algebraic conjugate* of z to be the quaternionic number $\overline{z}=a-bi-cj-dk\in\mathbb{H}$. Show that for all $z,w\in\mathbb{H}$, we have:

$$z\overline{z}=\|z\|^2, \quad \overline{w\cdot z}=\overline{z}\cdot\overline{w}, \quad \text{and} \quad \|z\cdot w\|=\|z\|\cdot\|w\|.$$

Deduce that $\mathbb{H}$ is a (noncommutative!) division ring that is also a normed $\mathbb{R}$-algebra.

5. (H) Use the fact that $\mathbb{H}$ is a normed $\mathbb{R}$-algebra to write 273 as a sum of four integer squares.

6. (A) Show that the set of all 2×2 matrices of the form $\begin{pmatrix}\alpha & -\overline{\beta}\\ \beta & \overline{\alpha}\end{pmatrix}$ for some $\alpha,\beta\in\mathbb{C}$ defines a subring of $M_2(\mathbb{C})$, and use the matrices

$$\mathbf{1}:=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}; \quad \mathbf{i}:=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} \quad \mathbf{j}:=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \quad \mathbf{k}:=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$$

to help you to write down a ring isomorphism from $\mathbb{H}$ to this subring of $M_2(\mathbb{C})$. In addition, show that the set of all unit quaternions (those $z=a+bi+cj+dk\in\mathbb{H}$ satisfying $\|z\|=1$) coincide under this isomorphism with the subgroup of $M_2(\mathbb{C})$ given by

$$SU(2):=\left\{A=\begin{pmatrix}\alpha & -\overline{\beta}\\ \beta & \overline{\alpha}\end{pmatrix}\in M_2(\mathbb{C}) \mid \det(A)=1\right\}.$$

The course website is: `http://people.bath.ac.uk/dmjc20/Alg2B/`

SOLUTIONS 8

1. We have

$$\begin{aligned}(a^2+b^2)(c^2+d^2) &= \|a+ib\|^2\,\|c+id\|^2\\ &= \|(a+ib)(c+id)\|^2\\ &= \|(ac-bd)+i(ad+bc)\|^2\\ &= (ac-bd)^2+(ad+bc)^2.\end{aligned}$$

Now compute that

$$85 = 5\cdot 17 = \|1+2i\|^2\,\|1+4i\|^2 = \|-7+6i\|^2 = 7^2+6^2.$$

2. No solution given.

3. We have that $\Delta_A(t) = (2-t)^2-1 = (t-2)^2-1 = (t-3)(t-1)$. As both the eigenspaces must be one dimensional, it is clear that $\mathrm{am}(3) = \mathrm{gm}(3) = \mathrm{am}(1) = \mathrm{gm}(1) = 1$.

We then have $\Delta_B(t) = t(t-4)+4 = t^2-4t+4 = (t-2)^2$, so $\mathrm{am}(2) = 2$. To determine the geometric multiplicity, we must determine the eigenspace. We have

$$\begin{pmatrix}-2 & 2\\ -2 & 2\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix},$$

which holds if and only if $x=y$. Thus the eigenspace is one-dimensional spanned by $\begin{pmatrix}1\\1\end{pmatrix}$, so $\mathrm{gm}(2)=1$.

4. Note first that

$$\begin{aligned}z\overline{z} &= (a+bi+cj+dk)(a-bi-cj-dk)\\ &= a^2-(bi+cj+dk)^2\\ &= a^2-(bi)^2-(cj)^2-(dk)^2-bc(ij+ji)-bd(ik+ki)-cd(jk+kj)\\ &= a^2+b^2+c^2+d^2\\ &= \|z\|^2\end{aligned}$$

For the next equality, both sides are bilinear, so we need only show that the equality holds when $z,w\in\{i,j,k\}$. Now $\overline{i^2} = \overline{-1} = -1 = (-i)(-i) = \overline{i}^2$, and similarly $\overline{j^2} = \overline{j}^2$ and $\overline{k^2} = \overline{k}^2$. Then

$$\overline{ij} = \overline{k} = -k = (-j)(-i) = \overline{j}\cdot\overline{i}$$

and

$$\overline{ji} = \overline{-k} = k = (-i)(-j) = \overline{i}\cdot\overline{j}.$$

By symmetry we also have $\overline{jk} = \overline{j}\cdot\overline{j}$, $\overline{kj} = \overline{j}\cdot\overline{k}$, $\overline{ki} = \overline{i}\cdot\overline{k}$ and $\overline{ik} = \overline{k}\cdot\overline{i}$ as required.

Finally, using the equality $\overline{w\cdot z} = \overline{z}\cdot\overline{w}$, we have

$$\|z\cdot w\|^2 = zw\overline{zw} = zw\overline{w}\cdot\overline{z} = z\|w\|^2\overline{z} = z\overline{z}\|w\|^2 = \|z\|^2\cdot\|w\|^2.$$

This last equality shows that $\mathbb{H}$, equipped with the standard inner product on $\mathbb{R}^4$, is a normed $\mathbb{R}$-algebra. For $z\neq 0$, the first equality above shows that $\frac{\overline{z}}{\|z\|^2}$ is a multiplicative inverse of z, so $\mathbb{H}$ is a division algebra.

5. We have

$$\begin{aligned}273 = 3\cdot 7\cdot 13 = 21\cdot 13 = (4^2+2^2+1^2)(3^2+2^2) &= \|4+2i+j\|^2\,\|3+2i\|^2\\ &= \|8+14i+3j-2k\|^2 = 8^2+14^2+3^2+2^2.\end{aligned}$$

6. Given two such matrices, say

$$\begin{pmatrix}\alpha & -\overline{\beta}\\ \beta & \overline{\alpha}\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}\gamma & -\overline{\delta}\\ \delta & \overline{\gamma}\end{pmatrix},$$

we have that

$$\begin{pmatrix}\alpha & -\overline{\beta}\\ \beta & \overline{\alpha}\end{pmatrix} - \begin{pmatrix}\gamma & -\overline{\delta}\\ \delta & \overline{\gamma}\end{pmatrix} = \begin{pmatrix}\alpha-\gamma & -\overline{(\beta-\delta)}\\ \beta-\delta & \overline{\alpha-\gamma}\end{pmatrix}$$

and that

$$\begin{pmatrix}\alpha & -\overline{\beta}\\ \beta & \overline{\alpha}\end{pmatrix} \cdot \begin{pmatrix}\gamma & -\overline{\delta}\\ \delta & \overline{\gamma}\end{pmatrix} = \begin{pmatrix}\alpha\gamma-\delta\overline{\beta} & -\overline{(\beta\gamma+\overline{\alpha}\delta)}\\ \beta\gamma+\overline{\alpha}\delta & \overline{\alpha\gamma-\delta\overline{\beta}}\end{pmatrix}.$$

Therefore the given set is a subring of $M_2(\mathbb{C})$. To write down an isomorphism, define $\phi\colon \mathbb{H}\to M_2(\mathbb{C})$ by

$$\phi(a+bi+cj+dk) = a\mathbf{1}+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}.$$

The image of this map is precisely the subring introduced above, because

$$a\mathbf{1}+b\mathbf{i}+c\mathbf{j}+d\mathbf{k} = \begin{pmatrix}a+bi & -\overline{(-c+di)}\\ -c+di & \overline{a+bi}\end{pmatrix},$$

and because a given matrix $\begin{pmatrix}\alpha & -\overline{\beta}\\ \beta & \overline{\alpha}\end{pmatrix}$ is of the form $\phi(a+bi+cj+dk)$ for

$$a = \frac{\alpha+\overline{\alpha}}{2};\quad b = \frac{\alpha-\overline{\alpha}}{2i};\quad c = -\frac{\beta+\overline{\beta}}{2};\quad d = \frac{\beta-\overline{\beta}}{2i}$$

To check that ϕ is a ring homomorphism, note that

$$\phi(i\cdot i) = \phi(-1) = -\mathbf{1} = \mathbf{i}\cdot\mathbf{i} = \phi(i)\cdot\phi(i),$$

(and similarly for $\phi(j\cdot j) = \phi(j)\cdot\phi(j)$ and $\phi(k\cdot k) = \phi(k)\cdot\phi(k)$), that

$$\phi(i\cdot j) = \phi(k) = \mathbf{k} = \mathbf{i}\cdot\mathbf{j} = \phi(i)\cdot\phi(j),$$

(and similarly for $\phi(j\cdot k) = \phi(j)\cdot\phi(k)$ and $\phi(k\cdot i) = \phi(k)\cdot\phi(i)$), and that

$$\phi(j\cdot i) = \phi(-k) = -\mathbf{k} = \mathbf{j}\cdot\mathbf{i} = \phi(j)\cdot\phi(i),$$

(and similarly for $\phi(k\cdot j) = \phi(k)\cdot\phi(j)$ and $\phi(i\cdot k) = \phi(i)\cdot\phi(k)$). Finally, note that ϕ has kernel equal to zero, so the fundamental isomorphism theorem shows that ϕ induces an isomorphism

$$\phi\colon \mathbb{H} \longrightarrow \left\{\begin{pmatrix}\alpha & -\overline{\beta}\\ \beta & \overline{\alpha}\end{pmatrix} \mid \alpha,\beta\in\mathbb{C}\right\}.$$

For the latter part about unit quaternions, we need only notice that under the above isomorphism, an element $z = a+bi+cj+dk\in\mathbb{H}$ satisfying $\|z\| = 1$ is sent to

$$\phi(z) = \begin{pmatrix}a+bi & -\overline{(-c+di)}\\ -c+di & \overline{a+bi}\end{pmatrix},$$

which has determinant

$$(a+bi)(\overline{a+bi}) - (-c+di)\cdot\big(-\overline{(-c+di)}\big) = a^2+b^2-(-c^2-d^2) = a^2+b^2+c^2-d^2 = 1.$$