EXERCISES 10

Please submit solutions by 3pm on Thursday 3rd May to the pigeonholes in 4W (ground floor).

(W) = Warm-up; (H) = Homework; (A) = Additional.

1. (W) Let $\alpha, \beta \colon V \to V$ be linear operators on an n-dimensional space V such that $\alpha\beta = \beta\alpha$. Show that both $\operatorname{Ker} \beta$ and $\operatorname{Im} \beta$ are α-invariant.

2. (W) Write sketch proofs of Proposition 5.24 and Corollary 5.27.

3. (W) Let $A \in M_n(\mathbb{R})$ be a matrix whose minimal polynomial is of the form $(t-\lambda)^k$ for some integer $k \geq 1$, and let $N = A - \lambda I$. Show that

$$A^\ell = \sum_{i=0}^{k-1} \binom{\ell}{i} \lambda^{\ell-i} N^i,$$

for $\ell > 0$, and compute A^ℓ for $\ell > 0$ for the matrix

$$A = \begin{pmatrix} 3 & -2 & 0 \\ 2 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. (H) Let $\alpha \colon V \to V$ be a linear operator satisfying $\alpha^2 = \alpha$. Show that α is diagonalisable and that $V = \operatorname{Ker}(\alpha) \oplus \operatorname{Im}(\alpha)$. [Hint: the assumption on α gives you an element in $\operatorname{Ker}(\Phi_\alpha)$ such that there are only three options for m_α; investigate each one.]

5. (H) Let $\alpha \colon \mathbb{C}^3 \to \mathbb{C}^3$ be the linear map given by left multiplication by the matrix

$$A := \begin{pmatrix} 5 & -8 & 8 \\ -1 & 8 & -5 \\ -5 & 10 & -9 \end{pmatrix}.$$

Determine for α the characteristic polynomial, the minimal polynomial, the Jordan Normal Form J, the algebraic and the geometric multiplicities, the generalised eigenspaces, a basis for $\mathbb{C}^3$ such that the matrix for α with respect to this basis is J and, finally, a change of basis matrix P such that $J = P^{-1}AP$.

6. (A) Let α and β be diagonalisable linear operators on an n-dimensional space V. Assume in addition that $\alpha\beta = \beta\alpha$.

(1) Let λ be an eigenvector of β. Show that $E_\beta(\lambda)$ is α-invariant. [Hint: consider a slight variant of Exercise 10.1 above using $\beta - \lambda\mathrm{id}$.]

(2) Show that one can find a basis $v_1, \ldots, v_n$ for V such that each v_i is an eigenvector of both α and β; in this case we say that α and β are *simultaneously diagonalisable*. [Hint: consider the decomposition $V = E_\beta(\mu_1) \oplus E_\beta(\mu_2) \oplus \cdots \oplus E_\beta(\mu_s)$ that results from diagonalisability of β.]

The course website is: `http://people.bath.ac.uk/dmjc20/Alg2B/`

Solutions 10

1. If $w \in \operatorname{Ker}\beta$ then

$$\beta(\alpha(w)) = \alpha(\beta(w)) = \alpha(0) = 0.$$

hence $\alpha(w) \in \operatorname{Ker}\beta$. This shows that $\operatorname{Ker}\beta$ is α-invariant. To see that $\operatorname{Im}\beta$ is α-invariant, notice that if $v = \beta(u)$ then $\alpha(v) = \alpha(\beta(u)) = \beta(\alpha(u)) \in \operatorname{Im}\beta$.

2. No solution given.

3. (1) Using the binomial formula, we have

$$A^n = (\lambda I + N)^n = \sum_{i=0}^{n} = \binom{n}{i}\lambda^{n-i} I \cdot N^i = \sum_{i=0}^{k-1}\binom{n}{i}\lambda^{n-i}N^i,$$

where in the last equation we have used the fact that $N^k = 0$.

(2) We have

$$\Delta_A(t) = \begin{vmatrix} 3-t & -2 & 0 \\ 2 & -1-t & 1 \\ 0 & 0 & 1-t \end{vmatrix} = [(t-3)(t+1)+4]\cdot(1-t) = (t^2-2t+1)(1-t) = -(t-1)^3.$$

As

$$(A-I)^2 = \begin{pmatrix} 2 & -2 & 0 \\ 2 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & -2 & 0 \\ 2 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

is not zero, we have $m_A(t) = (t-1)^3$.

Let $N = A - I$. We have $N^3 = 0$ and therefore

$$A^n = (I+N)^n = I + nN + \binom{n}{2}N^2 = \begin{pmatrix} 1+2n & -2n & -2\binom{n}{2} \\ 2n & 1-2n & n-2\binom{n}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

4. Notice that $t^2 - t \in \operatorname{Ker}(\Phi_\alpha)$, so the minimal polynomial of α is $t, t-1$ or $t^2 - t = (t-1)t$. In each case, the minimal polynomial is a product of distinct linear polynomials, so α is diagonalisable by Corollary 5.27. There are three cases:

- If $m_\alpha = t$ then $\alpha = 0$ and $V = \operatorname{Ker}(\alpha)$ wheras $\operatorname{Im}(\alpha) = \{0\}$. Thus $V = V \oplus \{0\} = \operatorname{Ker}(\alpha) \oplus \operatorname{Im}(\alpha)$.
- If $m_\alpha = t-1$ then $\alpha = \mathrm{id}$ and $\operatorname{Ker}(\alpha) = \{0\}$ whereas $\operatorname{Im}(\alpha) = V$, so $V = \{0\} \oplus V = \operatorname{Ker}(\alpha) \oplus \operatorname{Im}(\alpha)$.
- Otherwise $m_\alpha = t(t-1)$. Let $p_1 = t$ and $p_2 = t-1$. Proposition 5.24 and its proof gives that

$$V = \operatorname{Ker}(p_1(\alpha)) \oplus \operatorname{Ker}(p_2(\alpha)) = \operatorname{Ker}(p_1(\alpha)) \oplus \operatorname{Im}(p_1(\alpha)) = \operatorname{Ker}(\alpha) \oplus \operatorname{Im}(\alpha).$$

5. (1) The characteristic polynomial of A is $\chi_A(t) = (1-t)^2(2-t)$. The minimal polynomial divides $\Delta_A(t)$ and has the same roots, so $m_A(t) = (t-1)^s(t-2)$ where $1 \le s \le 2$. Notice first that

$$(A - \mathbb{I}_3)(A - 2\mathbb{I}_3) = \begin{pmatrix} -20 & 0 & -16 \\ 15 & 0 & 12 \\ 25 & 0 & 20 \end{pmatrix} \neq 0.$$

Thus the minimal polynomial is $m_A(t) = (t-1)^2(t-2)$.

(2) As the multiplicity of $t-1$ in $m_A(t)$ is two we must have a Jordan block of size two, i.e., a $J(1,2)$; and as the multiplicity of $t-2$ in $m_A(t)$ is only one we must have that the largest dimension of a Jordan block with respect to the eigenalue 2 is one. Since A is a 3×3 matrix, there is only one possible Jordan normal form, namely $J(1,2) \oplus J(2,1)$ (or you can swap the order of these).

(3) The characteristic polynomial shows that the algebraic multiplicity of 1 equals two and 2 equals one. There is one Jordan block for both eigenvalues, hence $\mathrm{gm}\,(1) = \mathrm{gm}\,(2) = 1$. (Notice that we've calculated this without having to compute any eigenspaces!)

(4) For $\lambda = 1$, we first compute a basis for $\mathrm{Ker}(A - \mathbb{I}_3) = E_\alpha(1)$. To solve $(A - \mathbb{I}_3)v_1 = 0$, we perform ERO's on

$$A - \mathbb{I}_3 = \begin{pmatrix} 4 & -8 & 8 \\ -1 & 7 & -5 \\ -5 & 10 & -10 \end{pmatrix} \text{ to obtain } \begin{pmatrix} 1 & -2 & 2 \\ 0 & 5 & -3 \\ 0 & 0 & 0 \end{pmatrix},$$

and we solve $5y - 3z = 0$ and $x - 2y + 2z = 0$ to see that $v_1 = (-4, 3, 5)^T$ is a basis for this eigenspace. To compute generalised eigenspace $G_\alpha(1)$, one approach is to now solve

$$\begin{pmatrix} 4 & -8 & 8 \\ -1 & 7 & -5 \\ -5 & 10 & -10 \end{pmatrix} v_2 = \begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix}.$$

The vector $v_2 = (-1, 1, 1)^T$ will do, so $G_\alpha(1)$ has basis $\{(-4, 3, 5)^T, (-1, 1, 1)^T\}$. Notice that we list the eigenvector v_1 before v_2, just as in Proposition 5.28.

For $\lambda = 2$, we determine $G_\alpha(2) = E_\alpha(2)$ by performing ERO's on the matrix

$$A - 2\mathbb{I}_3 = \begin{pmatrix} 3 & -8 & 8 \\ -1 & 6 & -5 \\ -5 & 10 & -11 \end{pmatrix} \text{ to obtain } \begin{pmatrix} 1 & -6 & 5 \\ 0 & 10 & -7 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, we solve $10y - 7z = 0$ and $x - 6y + 5z = 0$. The space of solutions $G_\alpha(2)$ is spanned by the eigenvector $(-8, 7, 10)^T$.

(5) The matrix for α with respect to the basis

$$\big((-4, 3, 5)^T, (-1, 1, 1)^T, (-8, 7, 10)^T\big)$$

is the matrix

$$\mathrm{JNF}(A) = J(1,2) \oplus J(2,1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(6) The matrix whose columns are these basis vectors

$$P = \begin{pmatrix} -4 & -1 & -8 \\ 3 & 1 & 7 \\ 5 & 1 & 10 \end{pmatrix}.$$

gives the Jordan matrix of A via

$$P^{-1}AP = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \mathrm{JNF}(A).$$

6. (1) As β commutes with α we have that $\beta - \lambda\mathrm{id}$ also commutes with α. Exercise 10.1 now shows that $\mathrm{Ker}(\beta - \lambda\mathrm{id})$ is α-invariant. We're now done, because $E_\alpha(\lambda) = \mathrm{Ker}(\beta - \lambda\mathrm{id})$.

(2) Suppose that $\mu_1, \ldots, \mu_s$ are the eigenvalues of β and that $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of α. As β is diagonalisable we have that

$$V = E_\beta(\mu_1) \oplus E_\beta(\mu_2) \oplus \cdots \oplus E_\beta(\mu_s).$$

From (1) we know that $E_\beta(\mu_i)$ is α-invariant. If we write α_i for the restriction of α to $E_\beta(\mu_i)$, then $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_s$. As $m_\alpha(\alpha_i) = 0$, we have that the minimal polynomial of α_i divides m_α and is thus a product of distinct linear factors. Thus α_i is diagonalisable and we can find a basis $\mathcal{V}_i$ for $E_\beta(\mu_i)$ consisting of eigenvectors for α_i (and thus α). Notice that $\beta(v) = \mu_i v$ for all $v \in \mathcal{V}_i$ and thus $\mathcal{V}_i$ consists of vectors that are eigenvectors for both β and α. It follows that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s$ is a basis for V that consists of vectors that are eigenvectors for both α and β.