Coarsening dynamics in condensing stochastic particle systems

Watthanan "Mim" Jatuviriyapornchai

Joint work with Dr.Stefan Grosskinsky & Dr.Dario Spano
Mathematics Institute, University of Warwick

Condensation phenomena in stochastic systems,
University of Bath
4 July 2016

Outline

Setting

- Lattice $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$
- State space $\Omega = \mathbb{N}^{\Lambda_L}$
- Configuration $\boldsymbol{\eta} = (\eta_x : x \in \Lambda_L) \in \Omega$
- Jump probability $q(x,y) = \frac{1}{L-1}, \ \forall x \neq y$
- Dynamics are given by the generator

$$(\mathcal{L}f)(\boldsymbol{\eta}) = \sum_{x,y\in\Lambda_L} q(x,y)u(\eta_x)v(\eta_y)(f(\boldsymbol{\eta}^{x\to y}) - f(\boldsymbol{\eta})), \tag{1}$$

where $\eta_z^{x\to y} = \eta_z - \delta(z,x) + \delta(z,y)$.

Stationary measures

Under certain conditions[1], the processes admit stationary product measure with marginal

$$\nu_\phi[\eta_x = n] = \frac{1}{z(\phi)} w(n)\phi^n \tag{2}$$

is stationary, provided that

$$z(\phi) := \sum_{n=0}^{\infty} w(n)\phi^n < \infty,$$

for all $x \in \Lambda_L$. For fixed number of particles,

$$\pi_{L,N} = \nu_\phi[\cdot \mid \sum_{x\in\Lambda_L} \eta_x = N] \tag{3}$$

is the unique stationary measure on $\{\eta : \sum_{x\in\Lambda_L} \eta_x = N\}$.

[1] Chleboun, P. and Grosskinsky, S., 2014. Condensation in stochastic particle systems with stationary product measures. Journal of Statistical Physics, 154(1-2), pp.432-465.

Empirical processes

Define two empirical processes :

site empirical process

$$F_k(\eta(t)) := \frac{1}{L} \sum_{x \in \Lambda_L} \delta_{\eta_x(t),k}. \quad (4)$$

size-biased empirical process

$$P_k(\eta(t)) := \frac{1}{N} \sum_{x \in \Lambda_L} k\delta_{\eta_x(t),k}. \quad (5)$$

Relation

$$kF_k(\eta) = \rho P_k(\eta) \text{ for all } \eta \in \Omega_{L,N} \text{ and } k \geq 1.$$

Zero-range processes

$$u(k) = g(k),\ v(k) \equiv 1$$

Jump rate $g : \mathbb{N} \to [0, \infty)$

$$g(k) = \begin{cases} 0 & \text{if} \quad k = 0, \\ 1 + \frac{b}{k^\gamma} & \text{otherwise}, \end{cases} \tag{6}$$

for any constant $b > 0$ and $\gamma \in (0, 1]$.

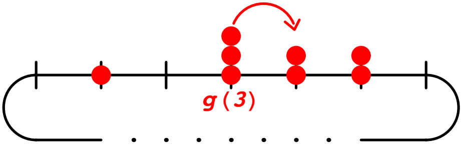
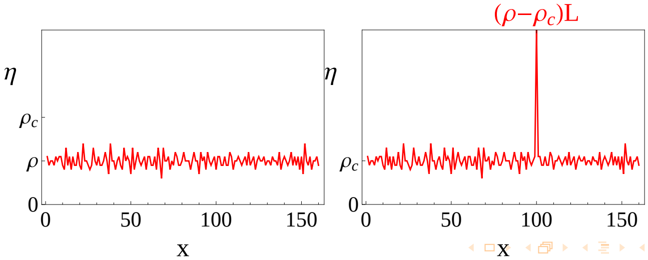


Figure : ZRP

Condensation

For our specific jump rate, the system exhibits a phase transition in the thermodynamic limit $N, L \to \infty$. If the particle density $\rho = \frac{N}{L}$ is above some critical value ρ_c, the system separates into

1. a homogeneous background
2. a condensate, which is the excess mass accumulated on a single randomly located lattice site.



Theorem ([2])

If $\rho > \rho_c$ *then for any* $\epsilon > 0$,

$$\lim_{N,L\to\infty,\frac{N}{L}\to\rho} \pi_{L,N}\left(\mid \frac{1}{L}\max_{x\in\Lambda_L}\eta_x - \rho - \rho_c \mid > \epsilon\right) = 1.$$

Critical density

$$\rho_c := \mathbb{E}_{\nu_1}[\eta_x].$$

$\gamma = 1$

$b > 2, \quad \rho_c = \frac{1}{b-2} < \infty$

$\gamma \in (0,1)$

$b > 0, \quad \rho_c < \infty$

[2] Grosskinsky, S., Schutz, G.M. and Spohn, H., 2003. Condensation in the zero range process: stationary and dynamical properties. Journal of statistical physics, 113(3-4), pp.389-410.

Coarsening

Coarsening Regime

The cluster sites exchange particles through the bulk. This leads to a decreasing number of cluster sites of increasing size.

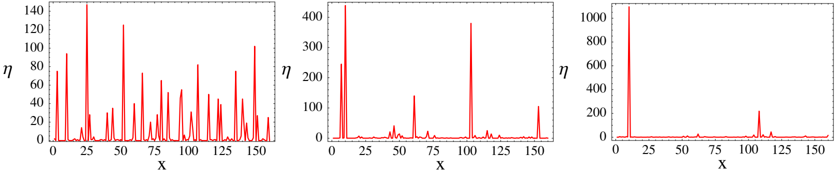


Figure : Dynamics of ZRP.

1. Introduction
2. Zero-range processes
   - Dynamics of empirical processes
   - Simulation Results
3. Inclusion processes
4. Conclusion

$f_k(t)$

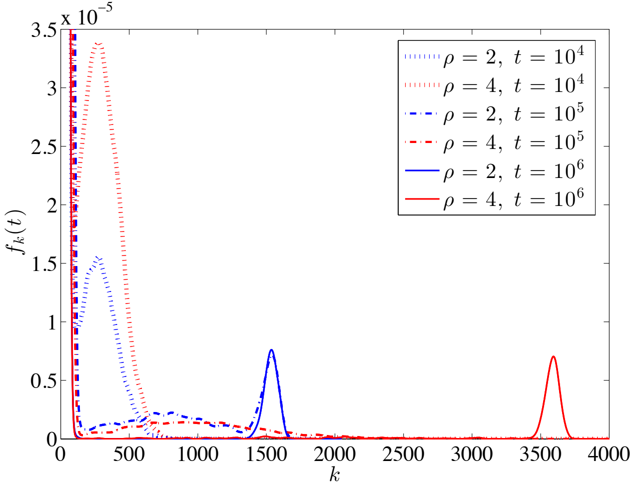


Figure : $f_k(t)$. Parameter values are $\gamma = 1$ with $b = 4$ and $L = 1024$.

$p_k(t)$

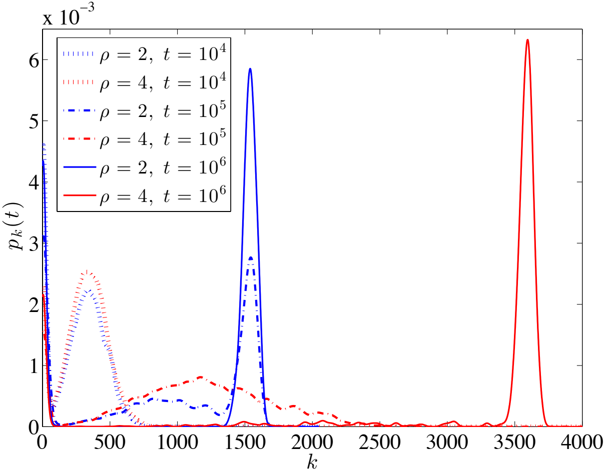


Figure : $p_k(t)$. Parameter values are $\gamma = 1$ with $b = 4$ and $L = 1024$.

Analysis of $F_k(\eta)$

$$
\begin{aligned}
(\mathcal{L}F_k)(\boldsymbol{\eta}) &= \sum_{x,y\neq x} \frac{1}{L-1} g(\eta_x)\left[F_k(\boldsymbol{\eta}^{x\to y}) - F_k(\boldsymbol{\eta})\right] \\
&= -g(k)F_k(\boldsymbol{\eta}) - \frac{1}{L-1}\sum_{\substack{x\in\Lambda\\ y\neq x}} g(\eta_x)\frac{\delta_{k,\eta_y}}{L} \\
&\quad + \frac{1}{L-1}\sum_{\substack{x\in\Lambda\\ y\neq x}} g(\eta_x)\frac{\delta_{k-1,\eta_y}}{L} + g(k+1)F_{k+1}(\boldsymbol{\eta}) \\
&= -\left(g(k) + \langle g\rangle_\eta\right)F_k(\boldsymbol{\eta}) \\
&\quad + \langle g\rangle_\eta F_{k-1}(\boldsymbol{\eta}) + g(k+1)F_{k+1}(\boldsymbol{\eta}) \\
&\quad + \frac{1}{L-1}\left(g(k) - \langle g\rangle_\eta\right)\left(F_k(\boldsymbol{\eta}) - F_{k-1}(\boldsymbol{\eta})\right) .
\end{aligned}
$$

Evolution equation

Using

$$\frac{d}{dt}\mathbb{E}[F_k(\eta(t))] = \mathbb{E}[(\mathcal{L}F_k)(\boldsymbol{\eta}(t))]$$

with notation $f_k(t) = \mathbb{E}[F_k(\boldsymbol{\eta})]$ and $\langle g \rangle = \sum_{k=1}^{\infty} g(k) f_k(t)$.

$$\begin{aligned}\frac{df_k(t)}{dt} &= g(k+1)f_{k+1}(t) + \langle g \rangle f_{k-1}(t) \\ &\quad - (g(k) + \langle g \rangle) f_k(t),\end{aligned} \tag{7}$$

for all $k \geq 0$ with $f_{-1}(t) = 0$.

Birth death process $(Y_t : t \geq 0)$

This is a birth death chain with state space $\mathbb{N}_0$ with

$$\begin{aligned} \text{birth rate} &= \langle g \rangle \\ \text{death rate} &= g(k) \end{aligned}$$

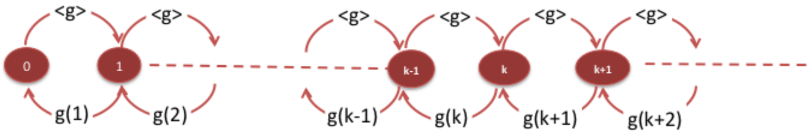


Figure : Birth-Death Processes Y_t Diagram.

Separated state

Ansatz:

$$f_k(t) = \underbrace{f_k(t)\,\mathbb{I}_{[0,1/\sqrt{\epsilon_t}]}(k)}_{:=f_k^{\text{bulk}}(t)} + \underbrace{f_k(t)\,\mathbb{I}_{(1/\sqrt{\epsilon_t},\infty)}(k)}_{:=f_k^{\text{cond}}(t)} \tag{8}$$

Scaling forms[3]

$$f_k^{\text{cond}}(t) = \epsilon_t^2 h(u), \quad \text{with } u = k\epsilon_t \text{ and } \epsilon_t = t^{-\frac{1}{\gamma+1}}. \tag{9}$$

$$\langle g \rangle \approx 1 + A\epsilon_t^{\gamma}, \tag{10}$$

where ϵ_t is the time scale and A is a constant.

[3] Godreche, C., 2003. Dynamics of condensation in zero-range processes. Journal of Physics A: Mathematical and General, 36(23), p.6313.

Analysis of $P_k(\eta)$

For $k = 1$,

$$\frac{d}{dt}p_1(t) = -g(1)p_1(t) - \langle g\rangle p_1(t) + \frac{1}{\rho}\langle g\rangle f_0(t) + \frac{1}{2}g(2)p_2(t)$$
$$= \frac{1}{2}g(2)p_2(t) - 2\langle g\rangle p_1(t) + \sum_{k\geq 2}\frac{1}{k}(g(k) - \langle g\rangle)p_k(t).$$

For $k > 1$,

$$\frac{d}{dt}p_k(t) = \frac{k}{k+1}g(k+1)p_{k+1}(t) + \frac{k}{k-1}\langle g\rangle p_{k-1}(t)$$
$$- \left(\frac{k-1}{k}g(k) + \frac{k+1}{k}\langle g\rangle\right)p_k(t)$$
$$+ \frac{1}{k}(\langle g\rangle - g(k))p_k(t).$$

Birth death with killing/cloning $(X_t : t \geq 0)$

birth rate	$\frac{k+1}{k}\langle g\rangle, \quad \text{for } k > 0 \,,$
death rate	$\frac{k-1}{k} g(k), \quad \text{for } k > 1 \,,$
rate from k to 1	$\frac{1}{k}(g(k) - \langle g\rangle)_+, \quad \text{for } k > 1 \,,$
cloning rate	$\frac{1}{k}(\langle g\rangle - g(k))_+, \quad \text{for } k > 1 \,,$
killing rate	$\sum_{k>1} \frac{1}{k}(\langle g\rangle - g(k))_+, \quad \text{for } k = 1 \,,$

where we denote by $(\cdot)_+ = \max\{0, (\cdot)\}$ the positive part of the expression and $\langle g\rangle = \rho \sum_{k\geq 1} \frac{g(k)}{k} p_k(t)$.

Relations

$$\rho p_k^{\text{cond}}(t) = k f_k^{\text{cond}}(t).$$

$$\sum_k p_k^{\text{cond}}(t) = \frac{1}{\rho} \sum_k k f_k^{\text{cond}}(t) = \frac{\rho - \rho_c}{\rho}.$$

Scaling form

$$p_k^{\text{cond}}(t) = \frac{1}{\rho} k f_k^{\text{cond}}(t) = \frac{1}{\rho} u h(u) \epsilon_t.$$

1 Introduction

2 Zero-range processes
- Dynamics of empirical processes
- Simulation Results

3 Inclusion processes

4 Conclusion

Simulation of BD chains: $\langle g \rangle \approx \langle g \rangle_m$

$f_k(t)$: Y_t^i

$$\langle g \rangle_m = \frac{1}{m} \sum_{i=1}^{m} g(Y_t^i)$$

$p_k(t)$: X_t^i

$$\langle g \rangle_m = \rho \sum_{i=1}^{m} \frac{g(X_t^i)}{X_t^i}$$

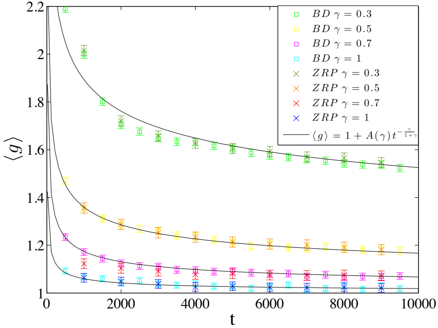


Figure : $\langle g \rangle$. Parameter values are $b = 4$, $\rho = 2$, and $L = m = 1024$.

Subcritical case

Size-biased marginals of stationary measure

$$\bar{\nu}_\phi(k) := \frac{k}{R(\phi)} \nu_\phi[\eta_x = k]$$

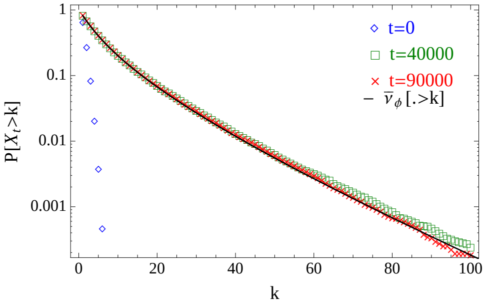


Figure : Convergence to the tail distribution of the size-biased marginal. Parameter values are $\gamma = 1$ with $b = 2.5$, $\rho = 1 < \rho_c = 2$ and $m = 10^5$.

Supercritical case : Phase separation

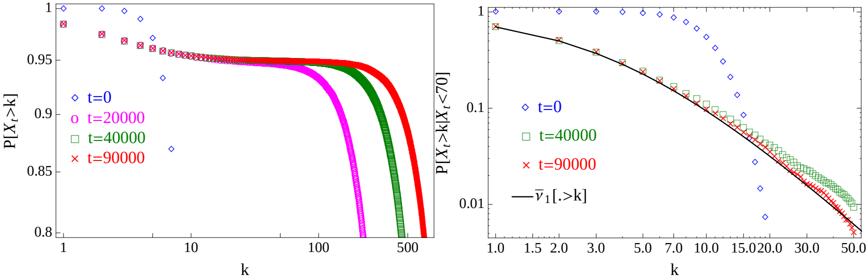


Figure : X_t ensemble size is $m = 10^5$ with parameter values are $\gamma = 1,\ b = 4$ and $\rho = 10 > \rho_c = 0.5$.

Dynamics of X_t

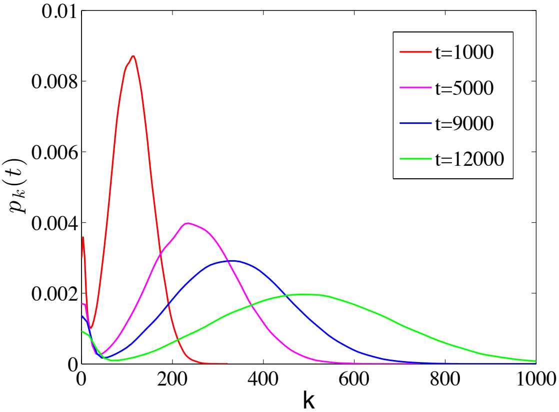


Figure : X_t ensemble size is $m = 10^5$ with parameter values are $\gamma = 1,\ b = 4$ and $\rho = 10 > \rho_c = 0.5$.

Scaling behavior

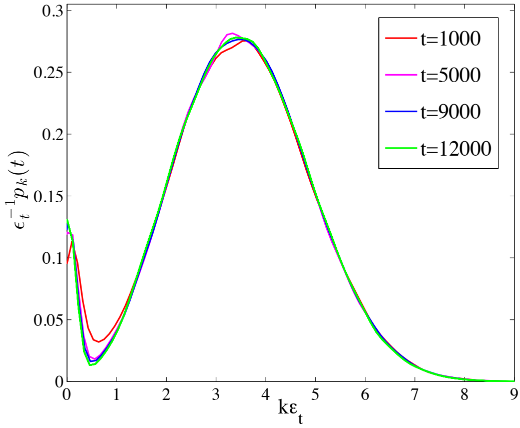


Figure : X_t ensemble size is $m = 10^5$ with parameter values are $\gamma = 1,\ b = 4$ and $\rho = 10 > \rho_c = 0.5$.

Theoretical comparison $\gamma = 0.5$

$$t^{-\frac{1-\gamma}{1+\gamma}} h''(u) + \left(\frac{u}{(\gamma+1)} + \frac{b}{u^{\gamma}} - A \right) h'(u) + \left(\frac{2}{(\gamma+1)} - \frac{b\gamma}{u^{\gamma+1}} \right) h(u) = 0$$

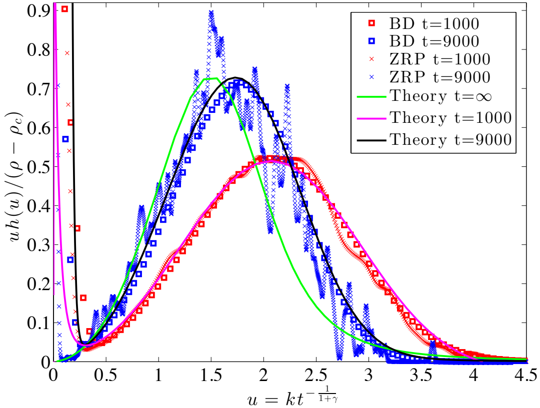


Figure : Parameter values are $b = 4$, $\rho = 2$ with $\gamma = 0.5$ and ensemble size $L = m = 1024$.

Theoretical comparison $\gamma = 1$

$$h''(u) + \left(\frac{1}{2}u - A + \frac{b}{u}\right) h'(u) + \left(1 - \frac{b}{u^2}\right) h(u) = 0.$$

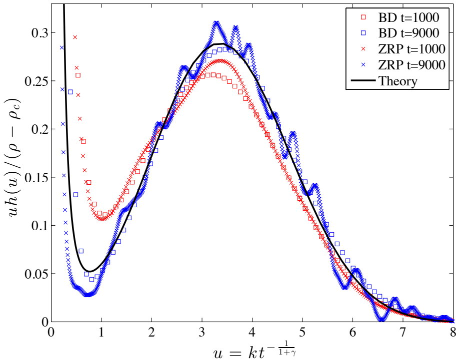


Figure : Parameter values are $b = 4$, $\rho = 2$ with $\gamma = 1$ and ensemble size $L = m = 1024$.

$\sigma^2(t)$

$\sigma^2(t)$

$$\sigma^2(t) = \rho\mathbb{E}[p_k(t)] = \rho\sum_k kp_k(t) = \sum_k k^2 f_k(t).$$

Time evolution of $\sigma^2(t)$

$$\frac{d}{dt}\sigma^2(t) = \frac{d}{dt}\sum_{k\geq 1} k^2 f_k(t)$$

$$= 2\rho(\langle g\rangle - 1) + 2\left(\langle g\rangle - b\sum_{k\geq 1} k^{1-\gamma} f_k(t)\right).$$

$f_k(t)$ and $p_k(t)$

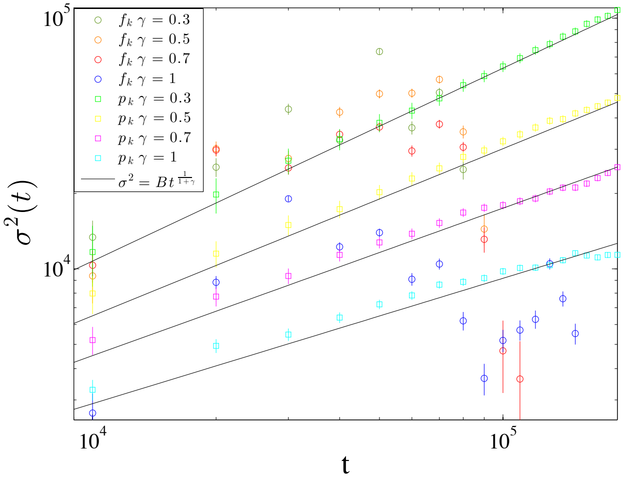


Figure : Parameter values are $b = 4$, $m = 1000$ and $\rho = 10$,

$p_k(t)$ and ZRP

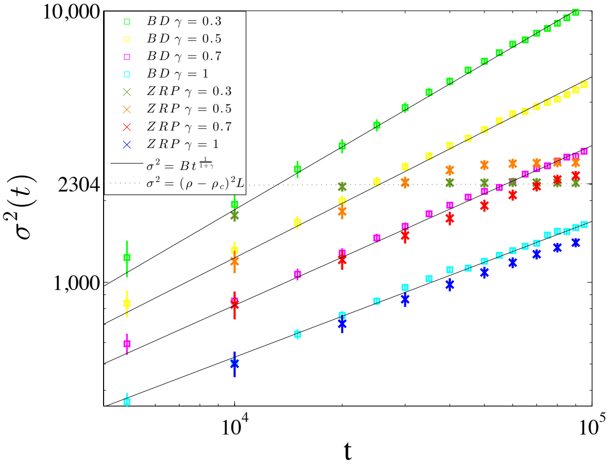


Figure : Parameter values are $b = 4$, $\rho = 2$ and system size $L = m = 1024$.

Inclusion processes (IP)

$$u(n) = n,\ v(n) = d + n,\ d > 0$$

$$(\mathcal{L}f)(\eta) = \sum_{x,y\in\Lambda} \frac{1}{L-1}\eta_x(d+\eta_y)(f(\eta^{x\to y}) - f(\eta)). \quad (11)$$

Under the condition of $d \to 0$[4], the critical density of IP is $\rho_c = 0$. The condensate contains all particles and can be localised on any site of the lattice.

[4] Grosskinsky, S., Redig, F. and Vafayi, K., 2011. Condensation in the inclusion process and related models. Journal of Statistical Physics, 142(5), pp.952-974.

$p_k(t)$ of IP

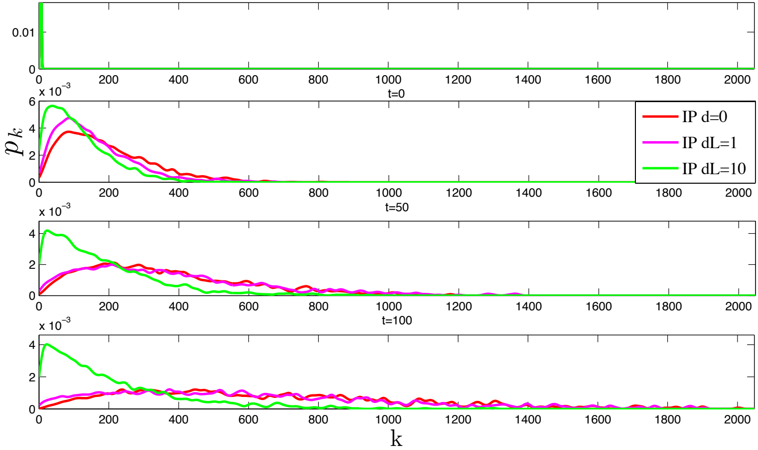


Figure : IP with $L = 1024$ and $\rho = 2$.

$f_k(t)$ of IP

With $\langle \eta \rangle = \sum_{k=1}^{\infty} k f_k(t) = \rho$,

$$\frac{d}{dt} f_k(t) = (k+1)(d+\rho) f_{k+1}(t) + \rho(d+(k-1)) f_{k-1}(t) - (dk + 2\rho k + \rho d) f_k(t),$$

valid for all $k \geq 0$ with the convention $f_{-1}(t) \equiv 0, \forall t \geq 0$.

This is a birth death chain with state space $\mathbb{N}_0$ with

$$\begin{aligned} \text{birth rate} &= \rho(d+k) \\ \text{death rate} &= (d+\rho)k. \end{aligned}$$

Case d=0

When $d = 0$, this leads to a linear birth death chain with birth rate = death rate = ρk

$$\frac{d}{dt} f_k(t) = \rho(k+1) f_{k+1}(t) + \rho(k-1) f_{k-1}(t) - 2\rho k f_k(t). \tag{12}$$

We assume that $f_k(t)$ takes the scaling form

$$f_k(t) = \epsilon_t^2 h(u), \quad \text{with } u = k\epsilon_t. \tag{13}$$

With $\epsilon_t = \frac{1}{\rho t}$, we have

$$u h''(u) + (2+u) h'(u) + 2h(u) = 0. \tag{14}$$

P_k d=0

When $d = 0$ in p_k,

$$\frac{d}{dt}p_k(t) = \rho k p_{k+1}(t) + \rho k p_{k-1}(t) - 2\rho k p_k(t), \tag{15}$$

for all $k \geq 1$ with the convention $p_0(t) = p_{-1}(t) = 0$.

$$\sum_k k p_k(t) = 2\rho t + C,$$

where $C = \rho + 1$ as it is simply the sized-biased initial condition of $Poi(\rho)$. Hence,

$$\sigma^2(t) = \mathbb{E}[f_k] = \rho\mathbb{E}[p_k] = 2\rho^2 t + \rho(\rho + 1)$$

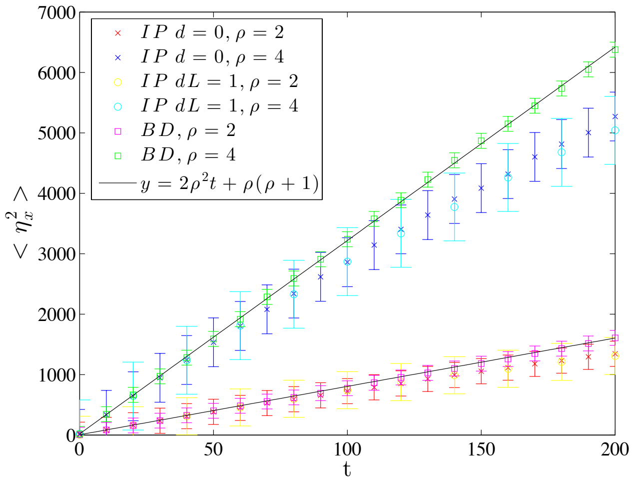


Figure : $\sigma^2(t)$ of system size 1024 from simulation of CGIP $d = 0$, $dL = 1$ and the birth-death p_k chain.

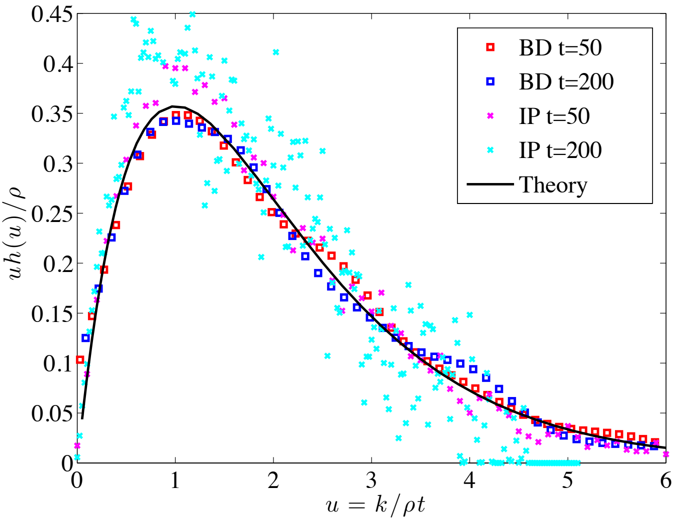


Figure : Normalised $uh(u) = \epsilon_t^{-1}\rho p_k(\eta)$ birth-death and IP simulation for $L = 1024, d = 0, \rho = 4$. Plotting against the solution of (14).

Self-duality

The SIP[5] is self-dual with the duality function :

$$D(\boldsymbol{\xi}, \boldsymbol{\eta}) = \prod_x d(\xi_x, \eta_x),$$

where $d(k,n) = \frac{n!}{(n-k)!}\frac{\Gamma(d)}{\Gamma(d+k)}$.

The self-duality of the SIP is then given by

$$\mathbb{E}_{\boldsymbol{\eta}}[D(\boldsymbol{\xi}, \boldsymbol{\eta}(t))] = \mathbb{E}_{\boldsymbol{\xi}}[D(\boldsymbol{\xi}(t), \boldsymbol{\eta})].$$

[5] Giardina, C., Kurchan, J., Redig, F. and Vafayi, K., 2009. Duality and hidden symmetries in interacting particle systems. Journal of Statistical Physics, 135(1), pp.25-55.

Time dependent variances $\sigma^2(t)$

Proposition

For $x \neq y \in \Lambda$, and for every initial product measure ν_ρ with density ρ and second moment σ_0^2 we have

$$\sigma^2(t) = \sigma_0^2 \mathbb{P}_{x,x}[X_t = Y_t] + \left(\frac{d\rho(1+\rho)+\rho^2}{d}\right) \mathbb{P}_{x,x}[X_t \neq Y_t], \quad (16)$$

where X_t and Y_t denote the particle positions for two SIP-particles.

Exact computations for two dual particles

Consider the process with only two particles called Z_t which has only 2 states which is either both particles are on the same site i.e. $Z_t = 0$ or they are on two different sites i.e. $Z_t = 1$. This process has Q-matrix :

$$Q = \begin{pmatrix} -2d(L-1) & 2d(L-1) \\ 2(d+1) & -2(d+1) \end{pmatrix}$$

Diagonalise Q which has eigenvalues 0 and $-2(1+dL)$ to obtain $Q = U\Lambda U^{-1}$ where

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & -2(1+dL) \end{pmatrix}.$$

Therefore,

$$P_t = \frac{1}{(1+dL)} \begin{pmatrix} (d+1) + d(L-1)e^{-2(1+dL)t} & d(L-1)[1 - e^{-2(1+dL)t}] \\ (d+1)[1-e^{-2(1+dL)t}] & d(L-1) + 2(d+1)e^{-2(1+dL)t} \end{pmatrix}.$$

$$\sigma^2(t) = \sigma_0^2 \mathbb{P}_0[Z_t = 0] + \left(\frac{d\rho(1+\rho) + \rho^2}{d} \right) \mathbb{P}_0[Z_t = 1].$$

Using $\mathbb{P}_0(Z_t = 0) = \frac{1}{1+dL}[(d+1) + d(L-1)e^{-2(1+dL)t}]$,

$$\sigma^2(t) = \rho(\rho+1) + \frac{\rho^2(L-1)}{1+dL}(1 - e^{-2(1+dL)t}). \qquad (17)$$

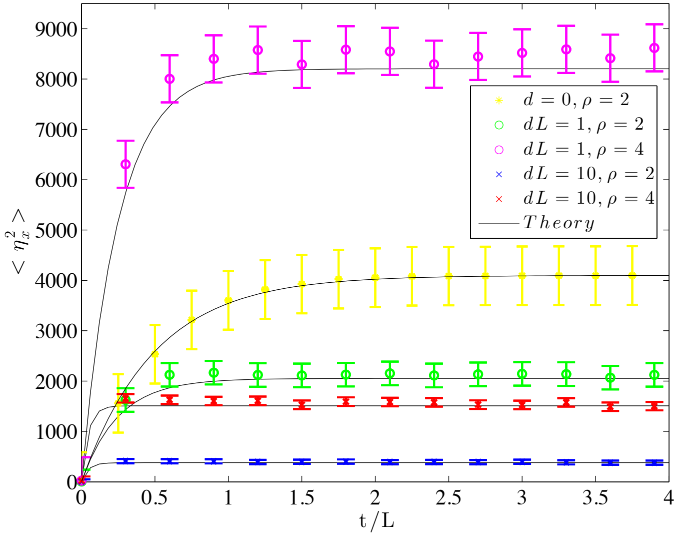


Figure : The second moment $\sigma^2(t)$ of IP $L = 1024$.

Conclusion

- The coarsening time scale of CGZRP is $\epsilon_t = t^{-\frac{1}{1+\gamma}}$ for $\gamma \in (0,1]$.
- The coarsening time scale of CGIP $d = 0$ is $\epsilon_t = \frac{1}{\rho t}$
- The use of the size-biased birth death chain provides a strong tool to analyze the dynamics without finite size effects and significantly improves statistics.
- This approach is generic and can be adapted to other condensing particle systems such as Inclusion processes (work in progress).

For Further Reading

C. Godreche.
Dynamics of condensation in zero-range processes.
Journal of Physics A: Mathematical and General, vol. 36, no.23, p.6313, 2003.

P. Chleboun and S. Grosskinsky.
Condensation in stochastic particle systems with stationary product measure.
Journal of Statistical Physics, vol. 154, no 1-2, pp. 432–465,2014.

M.Evans and B.Waclaw
Condensation in stochastic mass transport models: beyond the zero-range process.
Journal of Physics A: Mathematical and Theoretical, vol. 17, no. 9, p. 095001,2014

J.Cao, P. Chleboun, and S. Grosskinsky.
Dynamics of condensation in the totally asymmetric inclusion process. Journal of Statistical Physics vol. 155, no 3, pp. 523–543, 2014.

W. J. and S. Grosskinsky,
Coarsening dynamics in condensing zero-range processes and size-biased birth death chains.
Journal of Physics A: Mathematical and Theoretical, vol. 40, no. 18, 2016