Monotonicity and condensation in stochastic particle systems.

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joint work with T. Rafferty and S. Grosskinsky

6/07/2016

Outline

- Motivation and background.
- Recap zero-range process (ZRP):
  - Definition.
  - Stationary measures.
- Other examples with product stationary measures.
- Condensation…
  - On finite lattices and in the thermodynamic limit.
- Result
  - Condensation and product stationary measure => non-monotone dynamics.
- Idea of proof.
- Monotone condensing systems.

Motivation

- Monotonicity is a useful tool:
  - » Coupling techniques used to derive hydrodynamic limits.
    [e.g. T. Gobron, E. Saada, Ann. I. H. Poincare (2010)]
  - » Dynamics of condensation in inhomogeneous systems.
    [e.g. C. Landim, Ann. Probab. (1996)]
- Known examples of (homogeneous) condensing systems are non-monotone;
  - » for example the ZRP.
- Non-monotonicity indicates a canonical overshoot of relevant observables.
  - » Possible links with metastability at the critical point.

Setup

Lattice: $\Lambda = \{1, \ldots, L\}$

State space: $\Omega_L = \mathbb{N}_0^{\Lambda_L} = \{0, 1, 2, \ldots\}^L$

(a priori no upper bound on local occupancy)

Configuration: $\boldsymbol{\eta} = (\eta_x)_{x\in\Lambda}$ with $\eta_x \in \mathbb{N}_0$

- Dynamics: continuous time Markov process which conserves the total number of particles.
  - » Ergodic on $\Omega_{L,N} = \{\eta \in \Omega_L \, : \, \sum_{x\in\Lambda} \eta_x = N\}$.
  - » Unique stationary measure on $\pi_{L,N}$ called canonical measures.

Assume throughout spatially homogeneous:

$$\pi_{L,N}[\eta_x \in \cdot] = \pi_{L,N}[\eta_y \in \cdot] \quad \forall \, x, y \in \Lambda\,.$$

Heuristic

Lattice: $\Lambda = \{1, \ldots, L\}$

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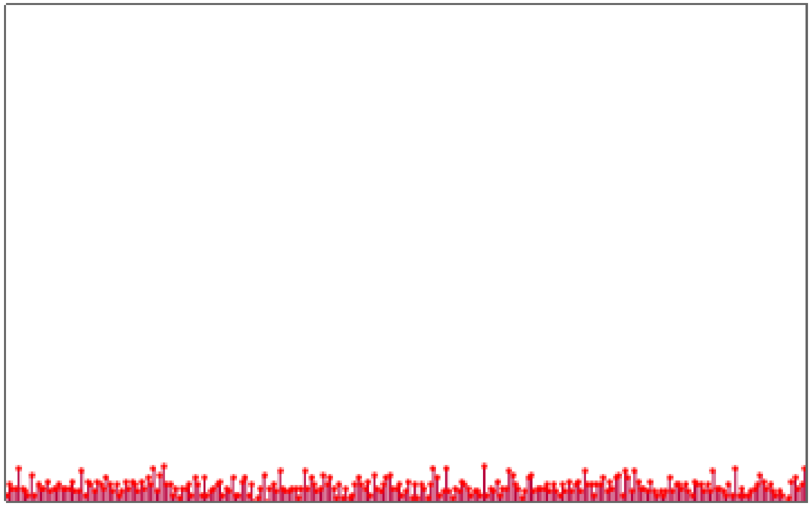

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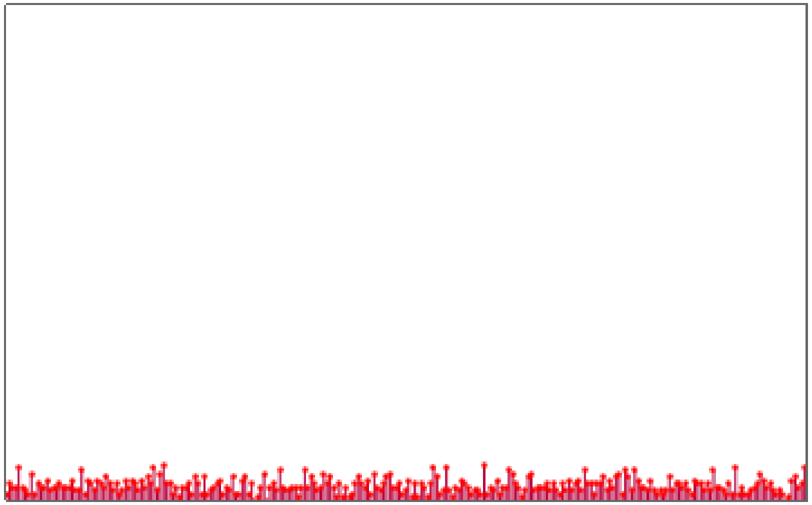

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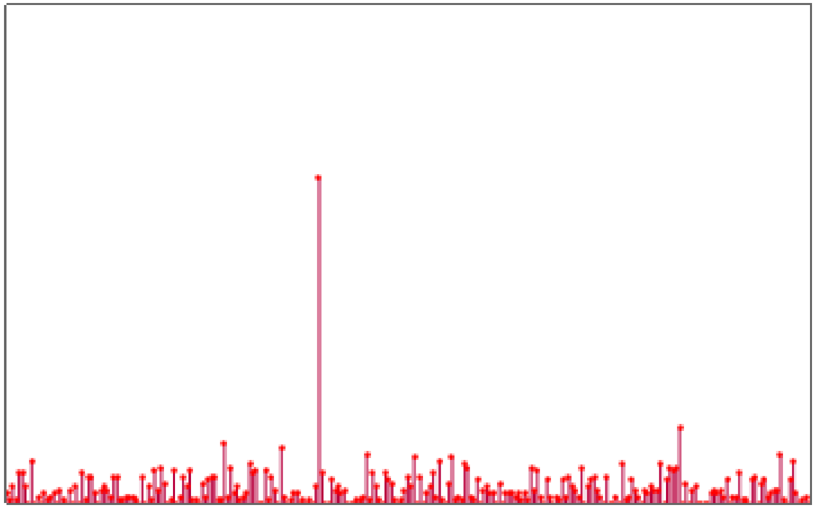


Condensation on fixed L

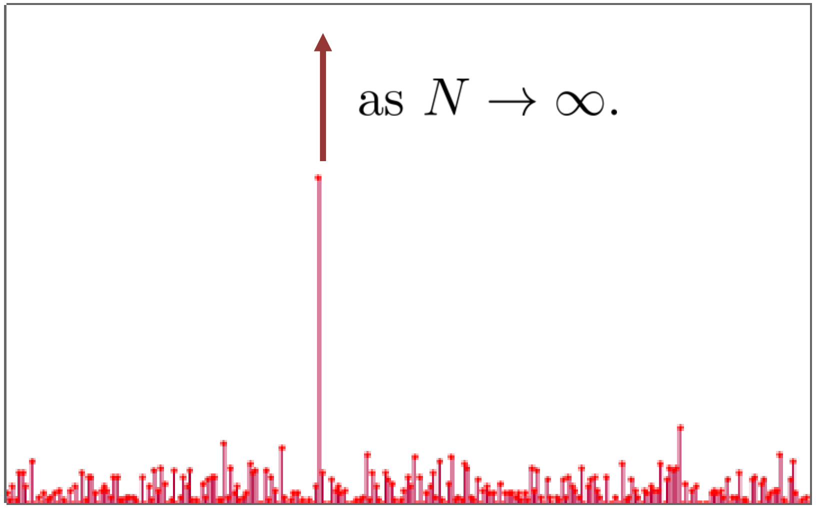
Let $M_L(\eta) := \max_{x\in\Lambda} \eta_x$.

Definition

Condensation occurs on $\Omega_{L,N}$ for $L \geq 2$ (fixed) iff

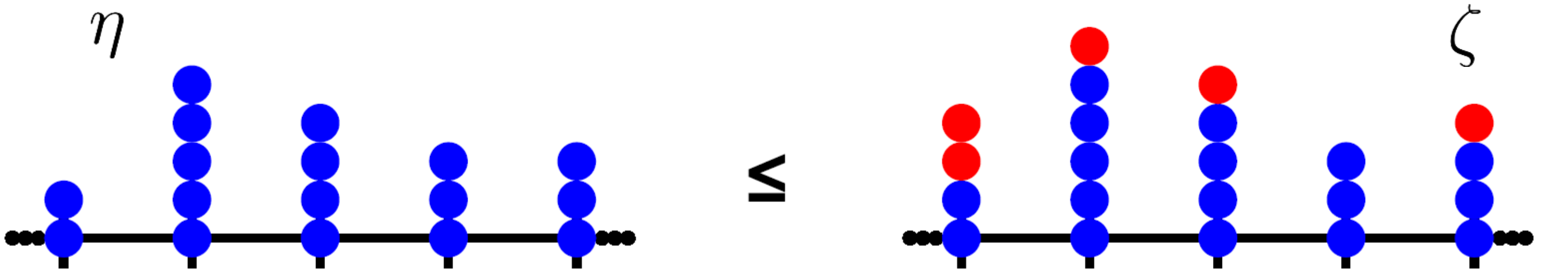
$$\lim_{K\to\infty} \lim_{N\to\infty} \pi_{L,N}[M_L \geq N-K] = 1 \ .$$

"All but a finite number of particles typically accumulate on a single site."



Monotonicity

- Partial order on Ω_L : $\eta \leq \zeta \iff \eta_x \leq \zeta_x \quad \text{for all} \quad x \in \Lambda$



- $f : \Omega_L \to \mathbb{R}$ is increasing if $\eta \leq \zeta$ implies $f(\eta) \leq f(\zeta)$.

- Partial order on measures:

$$\mu \leq \pi \text{ if } \mu(f) \leq \pi(f) \text{ for all } f \text{ increasing.}$$

Monotonicity

- A process is monotone (attractive) if

$$\mu_0 \leq \pi_0 \implies \mu_t \leq \pi_t \quad \text{for all} \quad t \geq 0\,.$$

- In particular, for any initial conditions with $\eta \leq \zeta$ and f increasing

$$\mathbb{E}_\eta\, f(\eta_t) \leq \mathbb{E}_\zeta\, f(\zeta_t)$$

  » i.e. the process preserves monotonicity.

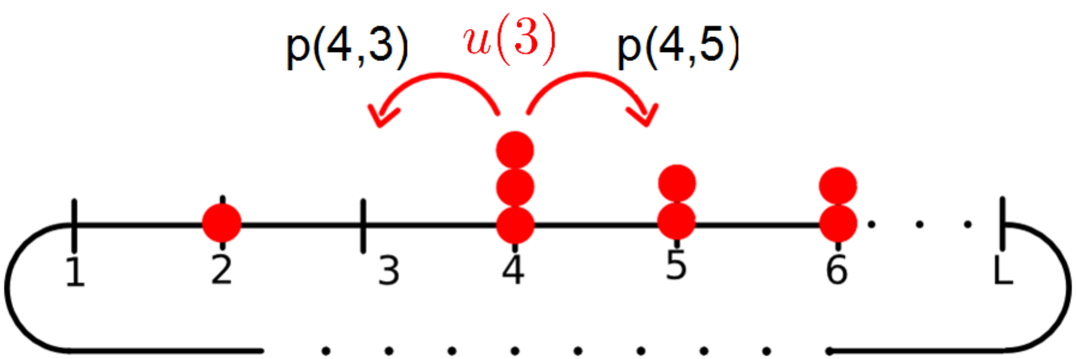
- For example zero-range process is monotone iff the jump rates are increasing in the site occupation (no condensation!).

Zero-range process

- Generator:

$$\mathcal{L}f(\eta) = \sum_{x \in \Lambda} p(x,y)u(\eta_x)\left[f(\eta^{x,y}) - f(\eta)\right]$$

  - » $p(x,y)$ irreducible RW on Λ.
  - » Assume $p(x,y) = q(x-y)$.



[Spitzer (1970), Andjel (1982)]

Stationary measures

- Grand canonical measures:
  - » Product measures on Ω_L with marginals

$$\nu_\phi\left[\eta_x = n\right] = \frac{1}{z(\phi)} w(n)\phi^n$$

    - $\phi \leq \phi_c$ the radius of convergence of $z(\phi) = \sum_n w(n)\phi^n$
    - single site weights (for ZRP):

$$w(n) = \prod_{k=1}^{n} \frac{1}{u(k)}$$

  - » Density $R(\phi) = \nu_\phi(\eta_x)$ increasing in ϕ .
  - » Critical density $\rho_c = R(\phi_c) \in [0, \infty]$.

    $\rho_c < \infty$ implies condensation in the thermodynamic limit.

Stationary measures

- Canonical measures (fixed number of particles N).

  - $$\pi_{L,N}[\,\cdot\,] = \nu_\phi[\,\cdot\;|\sum_x \eta_x = N]$$

  - $$\pi_{L,N}[\eta] = \frac{1}{Z_{L,N}} \prod_x w(\eta_x) \quad \text{for} \quad \eta \in \Omega_{L,N}$$

Other examples

- Misanthrope process

$$\mathcal{L}^{\mathrm{mis}} f(\eta) = \sum_{x,y\in\Lambda} r(\eta_x,\eta_y)p(x,y)\left(f(\eta^{x,y} - f(\eta)\right)$$

[Cocozza-Thivent '85]

Product measures iff $\forall n \geq 1, m \geq 0$

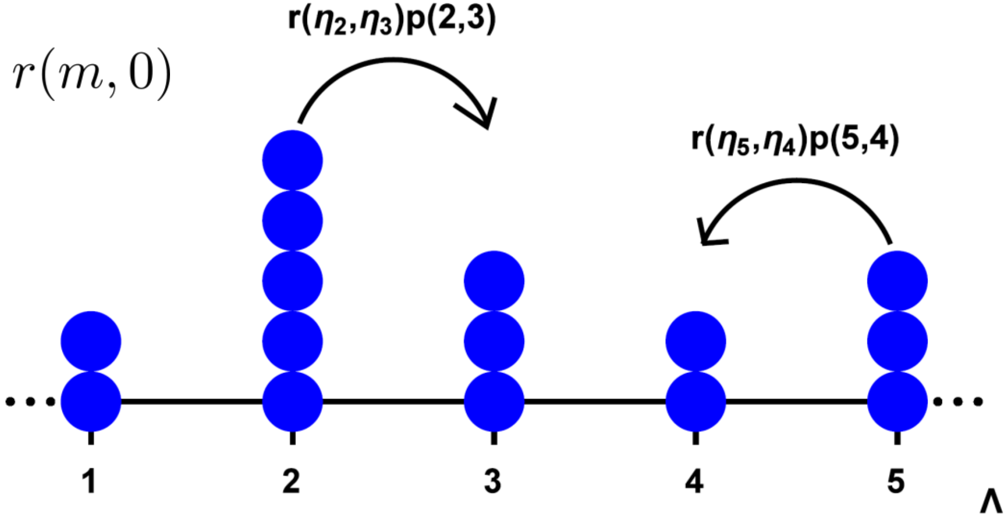
$$\frac{r(n,m)}{r(m+1,n-1)} = \frac{r(n,0)r(1,m)}{r(m+1,0)r(1,n-1)},$$

and symmetric, or

$$r(n,m) - r(m,n) = r(n,0) - r(m,0)$$

then

$$w(n) = \prod_{k=1}^{n} \frac{r(1,k-1)}{r(k,0)}$$



Other examples

- Generalised ZRP

$$\mathcal{L}^{\mathrm{gZRP}} f(\eta) = \sum_{x,y\in\Lambda} \alpha_k(\eta_x) p(x,y) \left(f(\eta^{x \xrightarrow{k} y}) - f(\eta) \right)$$

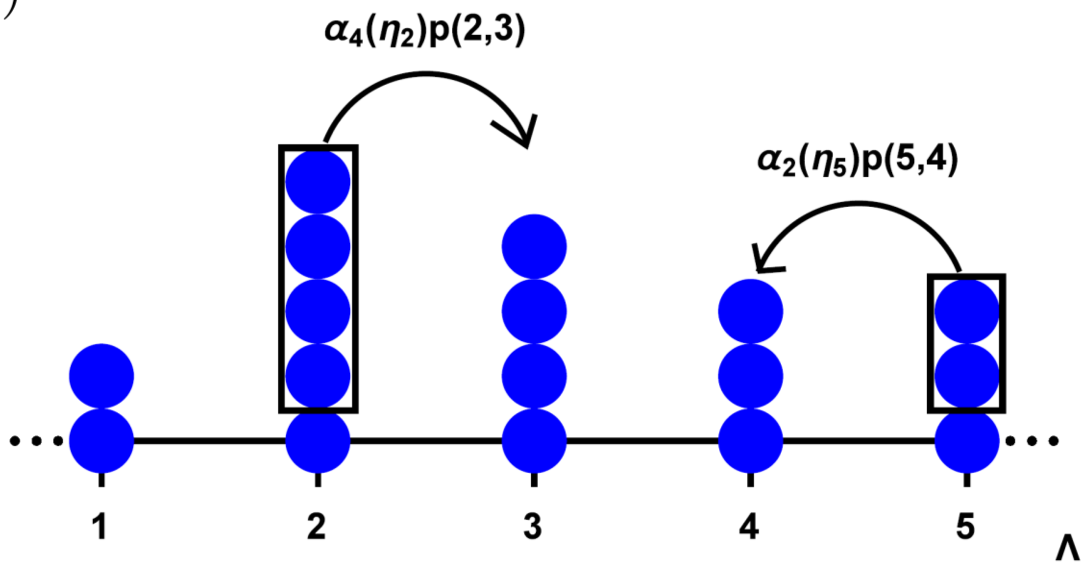
[Evans *et al.* 2004]

Product measures iff

$$\alpha_k(n) = u(k) \frac{h(n-k)}{h(n)} \, ,$$

then

$$w(n) = h(n)$$



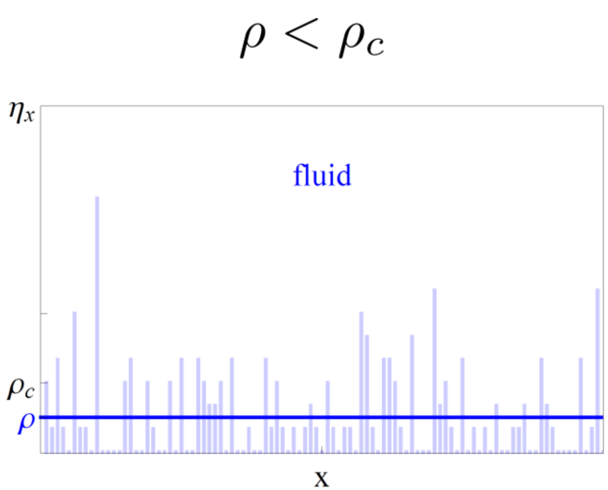
Condensation

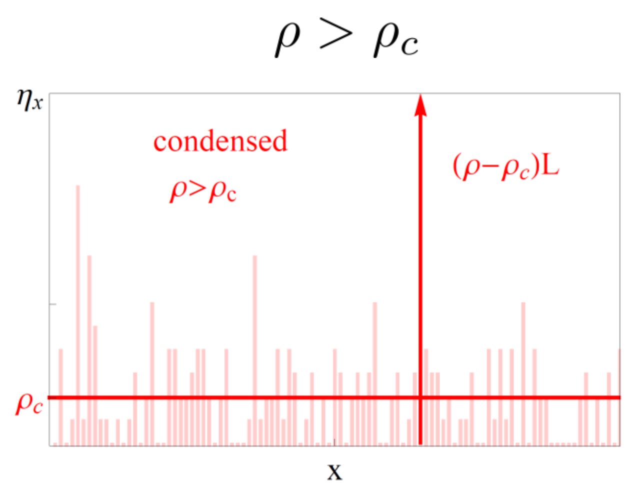
- Recall condensation in the thermodynamic limit.

Equivalence of ensembles

In the thermodynamic limit $N, L \to \infty$ with $N/L \to \rho$

$$\pi_{L,N} \to \nu_\phi \quad \text{where} \quad \begin{cases} R(\phi), & \text{if} \rho \le \rho_c \\ \phi = \phi_c, & \text{if} \rho > \rho_c \end{cases} .$$





[Jeon, March, Pittel '00; G., Schütz, Spohn '03; Ferrari, Landim, Sisko '07; Armendáriz, Loulakis '09]

Condensation

- On finite L

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"All but a finite number of particles typically accumulate on a single site."

[e.g. Ferrari *et al.* 2007]

- Product stationary measure and condensation on finite L iff sub-exponential grand canonical critical measures

Condensation

- On finite L

Assume $\lim_{n\to\infty} w(n-1)/w(n) \in (0,\infty]$.

Proposition [PC., T. Rafferty, S. Grosskinsky (2016)]

If there are stationary product measures then there is condensation for fixed L iff $\phi_c < \infty$, ν_{ϕ_c} exists and

$$\lim_{N\to\infty} \frac{\nu_{\phi_c}[\eta_1+\eta_2=N]}{\nu_{\phi_c}[\eta_1=N]} \in (0,\infty) \quad \text{exists.}$$

- Product stationary measure and condensation on finite L iff sub-exponential grand canonical critical measures

Condensation

- On finite L

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» Outside the max the canonical distribution converges in TV to the critical product measure,

$$\pi_{L,N}[\eta_1=n_1,\ldots,\eta_{L-1}=n_{L-1} \mid M_L=\eta_L] \to \prod_{i=1}^{L-1} \nu_{\phi_c}[\eta_i=n_i] \ \text{ as } N\to\infty\,.$$

Condensation

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» Examples

- power law tails $w(n) \sim n^{-b}$ for $b > 1$.
- stretched exponential tails $w(n) \sim e^{-Cn^{\gamma}}$ for $0 < \gamma < 1$, $C > 1$.
- almost exponential tails $w(n) \sim e^{-Cn/\log(n)^{\beta}}$ for $\beta > 0$.

Condensation

- Connection with the thermodynamic limit.
  - » Product measure and condensation on fixed L => sub-exponential.
  - » Sub-exponential and $\rho_c < \infty$

    => condensation in the thermodynamic limit.

[Armendariz, Loulakis (2011)]

- All well studied condensing systems condense both in the thermodynamic limit and on finite L.

- Example condensing for fixed L but not in the thermodynamic limit:
  - » power law tails $w(n) \sim n^{-b}$ for $b \in (1, 2)$.
  - » First moment not finite.

Result

Theorem [PC., T. Rafferty, S. Grosskinsky (2016)]

A (spatially homogeneous) process which condenses for fixed L and has stationary product measures with $\rho_c < \infty$ is necessarily **non monotone**.

- Surprisingly general.
  - » Statement about the dynamics (monotonicity) from hypothesis on the stationary measures.

The same is true if $w(n) \sim n^{-b}$ with $b \in (3/2, 2]$.
This case has $\rho_c = \infty$ so does not condense in TD limit.

Preliminary result

Lemma

If the process is monotone then canonical distirbutions $\pi_{L,N}$ are ordered in N,

$$\pi_{L,N} \leq \pi_{L,N+1} \quad \text{for all} \quad N \geq 0\,.$$

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Proof

Fix two intial distributions μ and μ' concentrating on $\Omega_{L,N}$ and $\Omega_{L,N+1}$ respectively, by

$$\mu[\eta] = \begin{cases} 1 & \text{if } \eta_1 = N, \eta_x = 0 \text{ for } x \neq 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu'[\eta] = \begin{cases} 1 & \text{if } \eta_1 = N+1, \eta_x = 0 \text{ for } x \neq 1 \\ 0 & \text{otherwise,} \end{cases}$$

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Fix two intial distributions μ and μ' concentrating on $\Omega_{L,N}$ and $\Omega_{L,N+1}$ respectively. Clearly $\mu \leq \mu'$.

ergodicity

$$\pi_{L,N} = \lim_{t\to\infty} \mu_t \leq \lim_{t\to\infty} \mu'_t = \pi_{L,N+1}$$

monotonicity

The heuristic

- The idea of the proof comes from the observation of a 'canonical overshoot' in the ZRP.
  - Turns out to be more general.
  - Examine the background density

$$R_L^{\mathrm{bg}}(N) := \frac{\pi_{L,N}(N - M_L)}{L-1} \,.$$

$$R_L^{\mathrm{bg}}(N) = \pi_{L,N}(\text{“density outside max”})$$

An increasing function.

- If the process is monotone, $\pi_{L,N}$ are ordered in N, so

$$R_L^{\mathrm{bg}}(N) \leq R_L^{\mathrm{bg}}(N+1) \,.$$

The heuristic

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  - Turns out to be more general.
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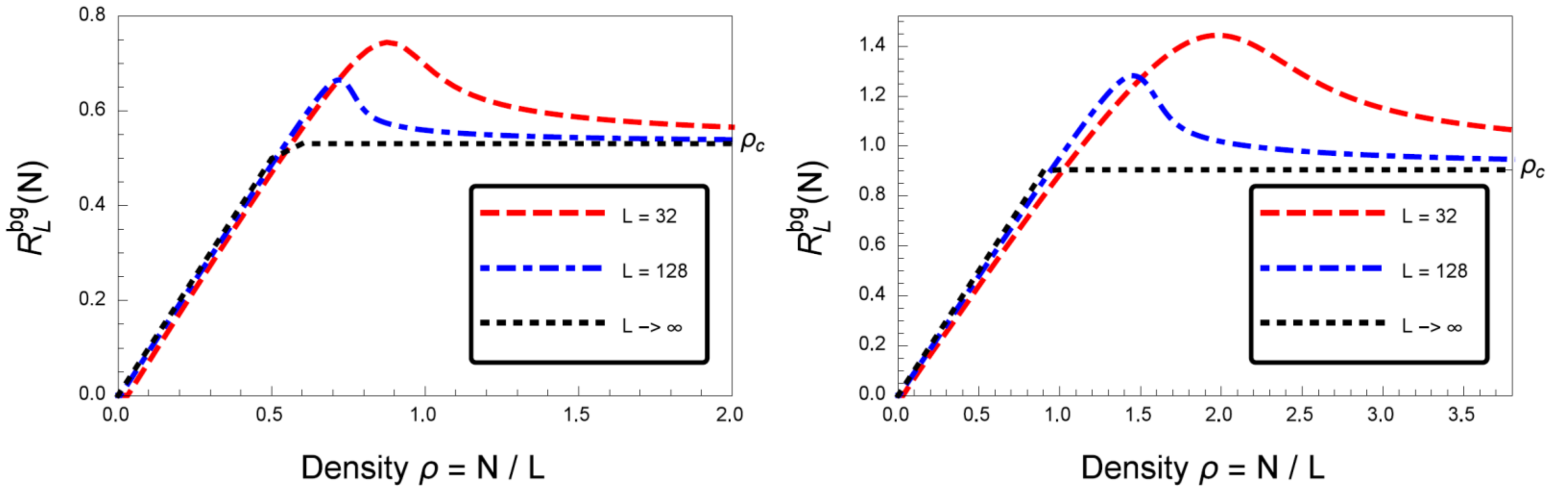
$$R_L^{\mathrm{bg}}(N) := \frac{N - \pi_{L,N}(M_L)}{L-1} .$$



- (Left) Power law weights $w(n) = n^{-b}$ with $b = 5$.
- (Right) Log-normal weights $w(n) = \exp\{-(\log(n))^2\}$.

Proof idea

- It turns out that we are unable to check monotonicity of the background density…
  - » There is a simpler monotone observable

$$f(\eta) = \mathbb{1}\left(\eta_1 = \ldots = \eta_{L-1} = 0\right) \ .$$

    - Which is decreasing.

$$\pi_{L,N}(f) = \pi_{L,N}[\text{“all particles on site L”}] = \frac{w(0)^{L-1} w(N)}{Z_{L,N}} \, .$$

  - » If the process is monotone then

$$\pi_{L,N}(f) \geq \pi_{L,N+1}(f) \, .$$

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$$\frac{w(N)}{Z_{L,N}} \geq \frac{w(N+1)}{Z_{L,N+1}}$$

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  - Recall, condensation on fixed L and product stationary measures implies ν_{ϕ_c} is sub-exponential.
  - In particular

$$\frac{Z_{L,N}}{w(N)} \to Lz(\phi_c)^{L-1} \ \ as\ N \to \infty\ for\ all\ L \geqslant 2\ .$$

[Chover, Ney, Wainger (1973)]

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Proof idea

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  - » Monotone implies:

$$\frac{Z_{L,N}}{w(N)} \leq \frac{Z_{L,N+1}}{w(N+1)} \quad \text{for all} \quad N \geq 0\,.$$

  - » Condenses implies:

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[Chover, Ney, Wainger (1973)]

  - » We are able to construct a subsequence on which the convergence is actually from above, which gives a contradiction.

$$H_L(N) := \frac{1}{Lz(\phi_c)^{L-1}} \frac{Z_{L,N}}{w(N)}$$

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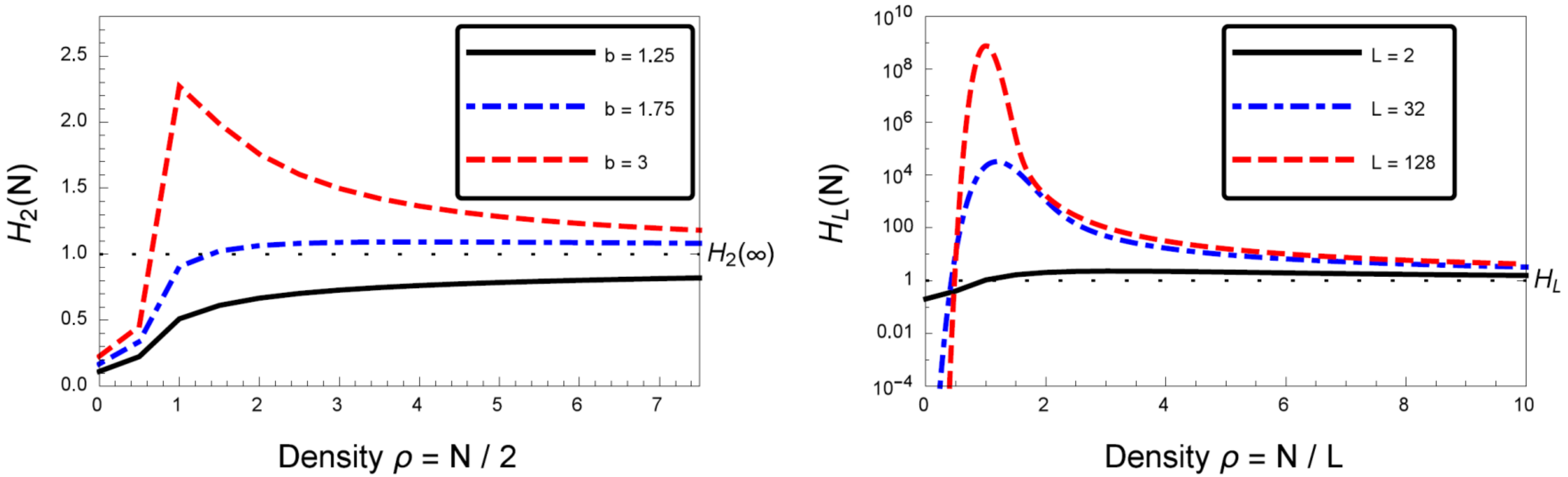


- (Left) Power law weights $w(n) = n^{-b}$ on two sites $L = 2$.
- (Right) Log-normal weights $w(n) = \exp\{-(\log(n))^2\}$.

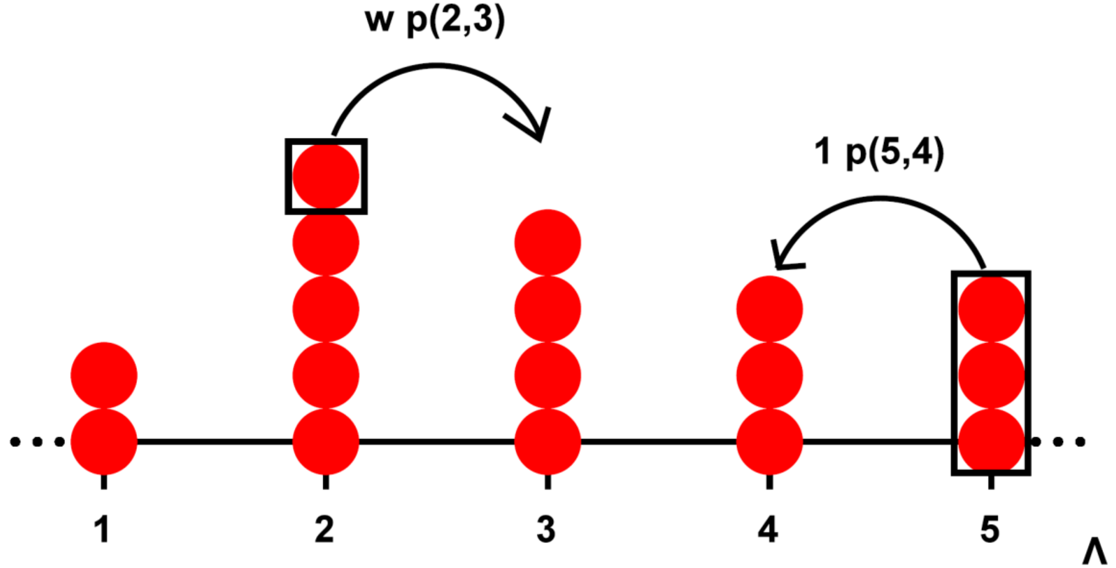
Monotone condensing examples

- Non-homogenous processes can be monotone and condense due to site disorder, e.g. ZRP with one slow site.

Monotone condensing examples

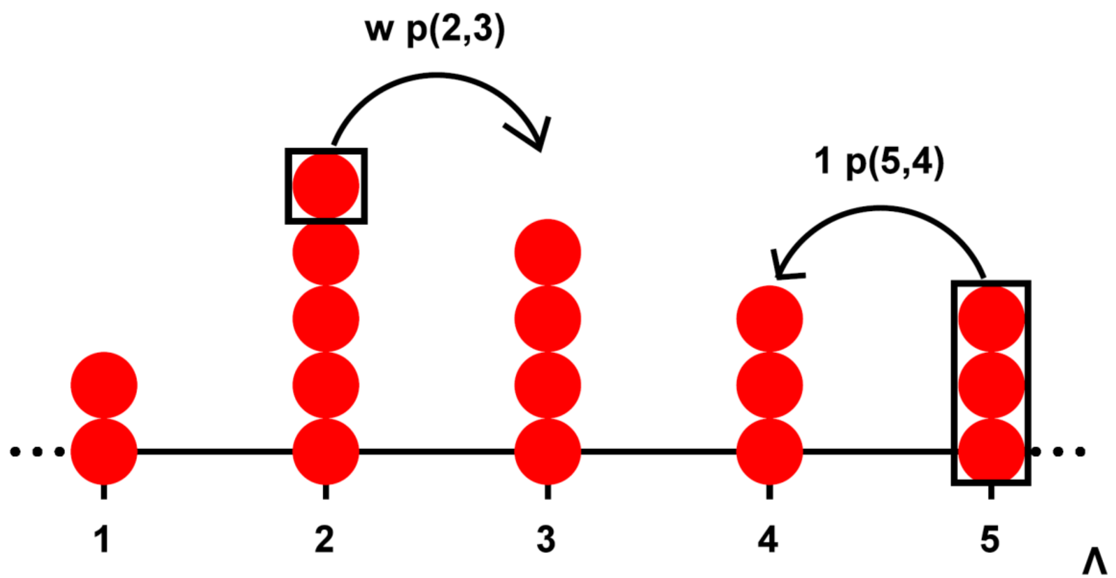
- Chipping model [Rajesh, Majumdar (2011)]

$$\mathcal{L}^{chip} f(\eta) = \sum_{x,y\in\Lambda_L} w\mathbb{1}(\eta_x > 0)p(x,y)\big(f(\eta^{x,y}) - f(\eta)\big)$$
$$+ \sum_{x,y\in\Lambda_L} \mathbb{1}(\eta_x > 0)p(x,y)\big(f(\eta + \eta_x(\delta_y - \delta_x)) - f(\eta)\big) \ .$$



Monotone condensing examples

- Chipping model [Rajesh, Majumdar (2011)]
  - » No product stationary measures.
  - » Proving anything on more than 2 sites is hard, but there are heuristics. If w < 1 it condenses.



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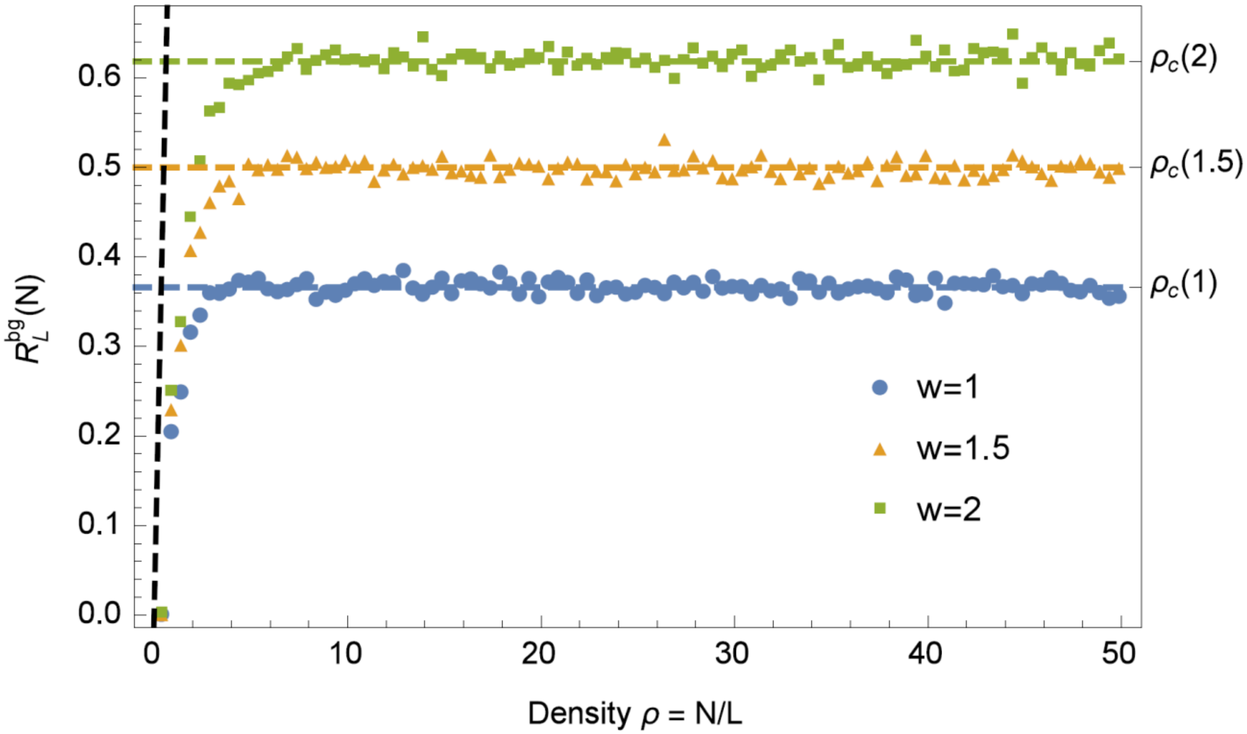
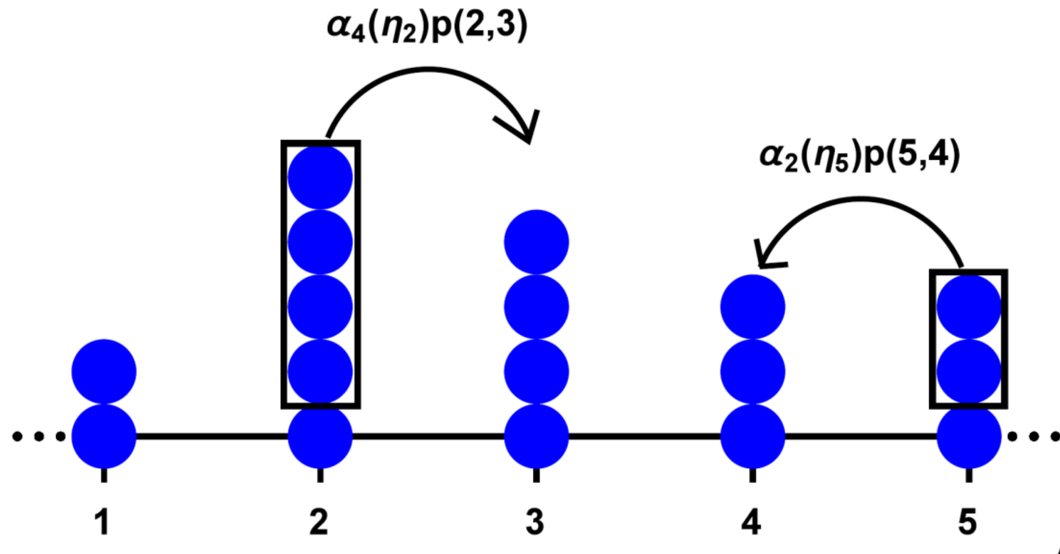


Monotone condensing examples

- The generalised ZRP with $w(n) \sim n^{-b} \quad \text{with} \quad b \in (1, 3/2]\,,$

i.e. $\rho_c = \infty$ but condenses on fixed L.

$$\mathcal{L}^{\mathrm{gZRP}} f(\eta) = \sum_{x,y \in \Lambda} \alpha_k(\eta_x) p(x,y) \left(f(\eta^{x \xrightarrow{k} y}) - f(\eta) \right)$$



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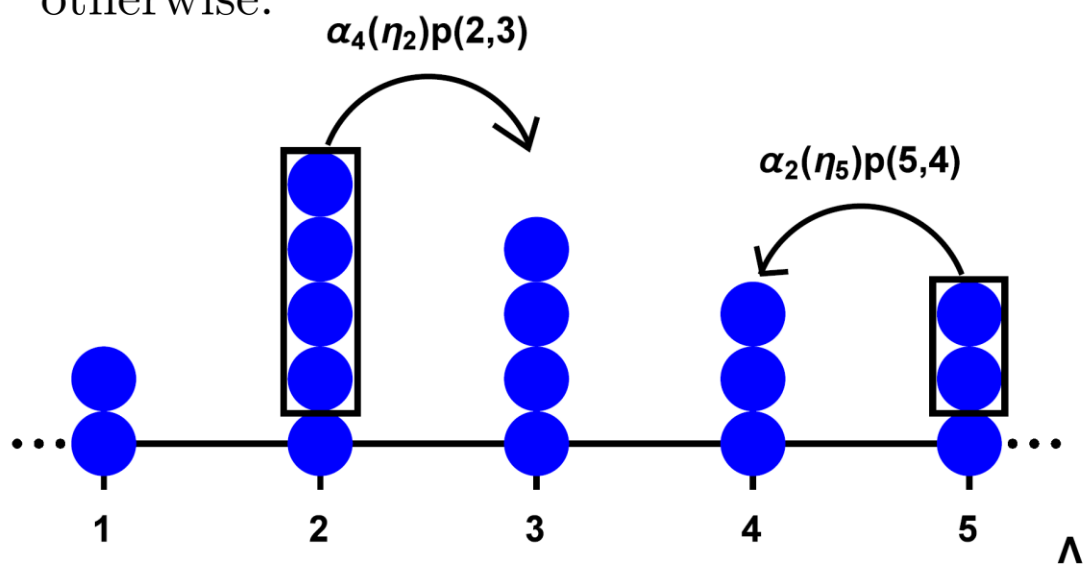
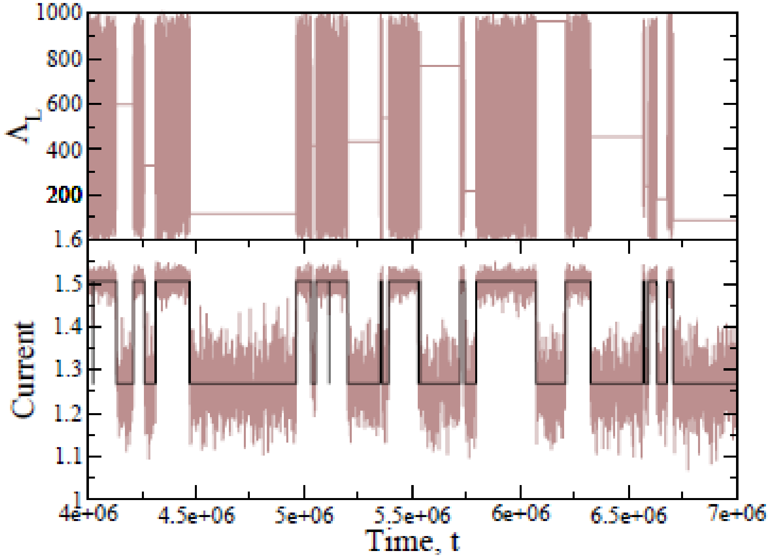
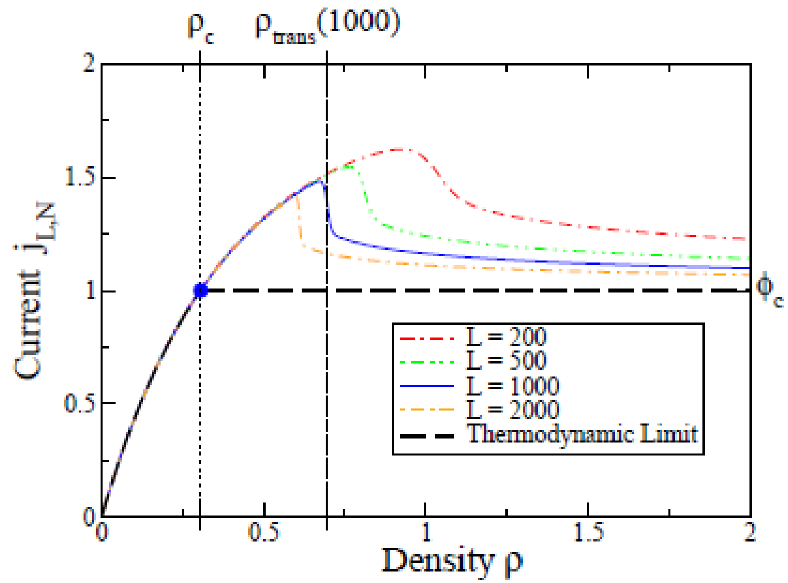
$$\alpha_k(n) = \begin{cases} 0 & \text{if } k = 0 \text{ or } n = 0, \\ k^{-b}(1 - \frac{k}{n})^{-b} & \text{if } k \in \{1, \ldots, n-1\} \\ 1 & \text{otherwise.} \end{cases}$$



Overshoot and metastability

- For power law and stretched exponential tails the overshoot has been observed to be related to metastability in the ZRP.





Summary

- For homogenous systems
  - Condensation and product stationary measure => non-monotone dynamics.
  - General result on dynamics.
- Implications for metastability.
- Connections between condensation on finite L and in thermodynamic limit.
- Examples of homogeneous condensing process with product stationary measures.
  - They don't condense in the thermodynamic limit.

Thank you.