Subexponential LD & Condensing ZRP

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joint work with Inés Armendáriz and Stefan Grosskinsky

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A toy example

Suppose $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of i.i.d. r.v.'s with

$$X_i \sim \mathcal{N}(m, \sigma^2).$$

Then, $$\bar{X}_n := \frac{X_1 + \cdots + X_n}{n} \sim \mathcal{N}(m, \frac{\sigma^2}{n})$$

and for any $x > m$

$$\mathbb{P}[\,\bar{X}_n \geq x] = \int_{\frac{\sqrt{n}(x-m)}{\sigma}}^{\infty} e^{-\frac{u^2}{2}} \frac{du}{\sqrt{2\pi}} \sim \frac{\sigma}{\sqrt{2\pi n}(x-m)} e^{-\frac{n(x-m)^2}{2\sigma^2}}.$$

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That is,

$$\frac{1}{n}\log\mathbb{P}[\,\bar{X}_n \geq x\,] \longrightarrow -\frac{(x-m)^2}{2\sigma^2}.$$

A toy example II

Suppose $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of i.i.d. r.v.'s with

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Then,

$$X_i - \bar{X}_n \sim \mathcal{N}(0, \frac{(n-1)\sigma^2}{n})$$

and

$$X_i - \bar{X}_n,\ \bar{X}_n \text{ are independent.}$$

Hence,

$$\mathcal{L}\big[X_i \mid \bar{X}_n = x\big] = \mathcal{N}(x, \frac{(n-1)\sigma^2}{n}) \longrightarrow \mathcal{N}(x, \sigma^2).$$

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With a bit more work we find

$$\mathrm{Cov}(X_i - \bar{X}_n, X_j - \bar{X}_n) = \sigma^2 \delta_{ij} - \frac{\sigma^2}{n}$$

and, if we set $\mathbf{e}^\top = (1, 1, \ldots, 1)$, then for any $k \in \mathbb{N}$

$$\mathcal{L}\big[(X_1, \cdots, X_k) \mid \bar{X}_n = x\big] = \mathcal{N}\big(x\mathbf{e}, \sigma^2 \mathbb{I}_k - \frac{\sigma^2}{n}\mathbf{e}\,\mathbf{e}^\top\big) \longrightarrow \mu^k$$

where the law μ is $\mathcal{N}(x, \sigma^2)$.

Cramér's theorem

Suppose $\{X_n\}_{n\in\mathbb{N}}$ are i.i.d. r.v.'s with common law μ and

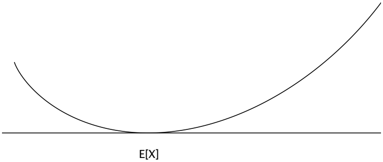
$$M(\lambda) = \mathbb{E}\left[e^{\lambda X_i}\right] < \infty, \text{ for } |\lambda| \leq \lambda_0.$$

If $x > \mathbb{E}\left[X_i\right]$, then

$$\frac{1}{n}\log\mathbb{P}\left[\bar{X}_n \geq x\right] \longrightarrow -I(x),$$

where

$$I(x) = \sup_{\lambda}\left(\lambda x - \log M(\lambda)\right)$$



[Cramér 1938].

Gibbs conditioning

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μ_* solves the following variational problem

$$\inf_{\nu \in \mathcal{I}} H(\nu|\mu), \qquad \text{where} \qquad \mathcal{I} = \{\nu \in M_1(\mathbb{R}) : \int u\,\nu(du) \geq x\}$$

and

$$H(\nu|\mu) = \begin{cases} \int f \log f \; d\mu & \text{if } \nu \ll \mu \text{ with } f = \frac{d\nu}{d\mu}, \\ +\infty & \text{otherwise} \end{cases}$$

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We may take $k = k(n) \to \infty$ slowly, but the result is no longer true if $k = O(n)$. [Dembo-Zeitouni 1996].

Subexponential Distributions

The picture is completely different when μ has no exponential moments, i.e. $M(\lambda) = \mathbb{E}\left[e^{\lambda X_i}\right] = \infty$ for all $\lambda > 0$. A distribution μ supported on the positive half-line is called subexponential if

$$\lim_{x \to \infty} \frac{\mathbb{P}\left[X + Y > x\right]}{\mathbb{P}\left[X > x\right]} = 2,$$

where X, Y are independent μ-distributed r.v.'s.

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Heavy-tailed distributions typically used in applications are all in this class.

- Regularly varying tails: $\mathbb{P}\big[X_1 > x\big] \sim x^{-\gamma}L(x)$ with $\gamma > 0$,
- Lognormal type tails: $\mathbb{P}\big[X_1 > x\big] \sim x^{-\beta}e^{-\gamma(\log x)^{\lambda}}$, $\lambda > 1$.
- Weibull type tails: $\mathbb{P}\big[X_1 > x\big] \sim x^{-\beta}e^{-\gamma x^{\lambda}}$, $\lambda < 1$.

Large Deviation Probabilities

When $X_1, X_2, \cdots, X_n$ are i.i.d. and subexponential the large deviations probabilities of their sum are typically given by

$$\mathbb{P}\big[X_1 + \cdots + X_n > x\big] \sim n\mathbb{P}\big[X_1 > x\big]$$

[Heyde 1968, Nagaev 1969,..., Denisov et al 2009]

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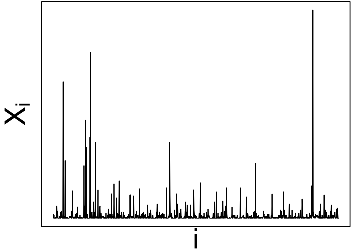
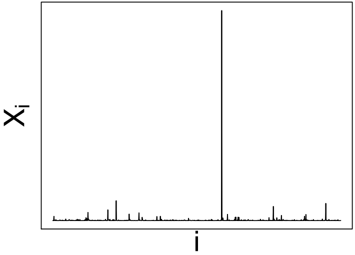
Note that since it is always true that

$$\mathbb{P}\big[\max\{X_1, \ldots, X_n\} > x\big] \sim n\mathbb{P}\big[X_1 > x\big],$$

subexponentiality implies that a large deviation of the sum is typically realised by a single big jump.





Gibbs conditioning for Subexponential r.v.'s

Theorem (I. Armendáriz, ML)

Let $X_1, X_2 \ldots$ be i.i.d. r.v.'s with subexponential distribution μ. Define $\mu_x = \mathcal{L}\big[X_i \mid X_i > x\big]$, and

$$\mu_{n,x} = \mathcal{L}\big[(X_1, \ldots, X_n) \mid X_1 + \cdots + X_n > x\big].$$

Then,

$$\lim_{x \to \infty} \sup_{n \leq A(x)} \left\| \mu_{n,x} - \frac{1}{n} \sum_{j=1}^{n} \sigma^j (\mu^{n-1} \times \mu_x) \right\|_{\text{t.v.}} = 0.$$

The maximum entirely absorbs the correlations introduced by conditioning- the bulk becomes asymptotically independent.

Idea of the proof

Proof: (for nonnegative r.v.'s) Note that if μ is a probability measure, $\mu\big[A\big] > 0$, and $\mu_A\big[\,\cdot\,\big] = \mu\big[\,\cdot\mid A\big]$, then μ_A is the solution to the minimization problem

$$\min_{\nu[A]=1} H(\nu|\mu).$$

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Candidate distribution: $\mu^*_{n,x} = \frac{1}{n}\sum_{j=1}^n \sigma^j(\mu^{n-1} \times \mu_x)$. By Csiszár's parallelogram identity and Pinsker's inequality

$$\begin{aligned}\|\mu_{n,x} - \mu^*_{n,x}\|^2_{\text{t.v.}} &\le H\left(\mu^*_{n,x} \,|\, \mu^n\right) - H\left(\mu_{n,x} \,|\, \mu^n\right) \\ &\le \log\left(\frac{\mathbb{P}\big[S_n > x\big]}{n\mathbb{P}\big[X_1 > x\big]}\right) + n\mathbb{P}\big[X_1 > x\big].\end{aligned}$$

Gibbs conditioning for Subexponential r.v.'s (local case)

Theorem

Let $X_1, X_2 \ldots$ be i.i.d. lattice r.v.'s with subexponential distribution μ. For admissible values of x, define

$$\mu_{n,x} = \mathcal{L}\big[(X_1, \ldots, X_n) \mid S_n = x\big].$$

If $\nu^j_{n,x}$ is a distribution on $\mathbb{R}^n$ with marginal on the co-ordinates other that j equal to μ^{n-1} and conditional distribution of the j-th co-ordinate given the others $\delta_{x-\sum_{i\neq j} x_i}$ then

$$\lim_{x\to\infty} \sup_{n \leq A(x)} \|\mu_{n,x} - \frac{1}{n}\sum_{j=1}^{n} \nu^j_{n,x}\|_{\text{t.v.}} = 0.$$

The condensing zero-range process

State space: $\mathbb{X} = \{0, 1, \ldots\}^{\Lambda} \qquad \boldsymbol{\eta} = (\eta_x)_{x \in \Lambda}$.

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Dynamics: If there are k particles at a site x, one of them leaves after an exponential time with rate $g(k)$, where

$$g : \{0, 1, 2, \dots\} \to [0, \infty)$$

and goes to $y \in \mathbb{T}$ with probability $p(x, y)$. [Spitzer, 1970)]

Jump rates: $g \downarrow$: effective attraction.
A standard model for condensation [Evans, 2000]

$$\begin{cases} g(k) = 1 + \frac{b}{k^\lambda} & k \in \mathbb{N} \\ g(0) = 0 \end{cases} \qquad \text{for } \lambda \in (0, 1],\ (b > 2 \text{ if } \lambda = 1.)$$

Jump probabilities: $p(x, y) \in [0, 1]$
$\displaystyle\sum_y p(x, y) = \sum_x p(x, y) = 1$, walk irreducible.

Invariant product measures

Conservation of the number of particles $\sum_{x \in \Lambda} \eta_x(t) = const.$

leads to a family of invariant product measures

Grand-canonical measures (fugacity ϕ) Product measures over Λ with marginals for η_x

$$\nu_\phi[k] = \frac{1}{z(\phi)} \frac{\phi^k}{g!(k)} \quad \text{where} \quad g!(k) = \prod_{n=1}^{k} g(n)$$

defined when the partition function $z(\phi) = \sum_k \frac{\phi^k}{g!(k)}$ converges.

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$$\frac{1}{g!(k)} \sim \begin{cases} k^{-b} & , \ \lambda = 1, \ b > 2 \\ \exp(-\frac{b}{1-\lambda} k^{1-\lambda}) & , \ \lambda \in (0,1) \end{cases}$$

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Density $R(\phi) = \langle \eta_x \rangle_{\nu_\phi} = \phi \, \partial_\phi \log z(\phi) \quad \uparrow \quad$ in ϕ

Critical density $\rho_c = \lim_{\phi \nearrow \phi_c} R(\phi) < +\infty.$

Canonical ensembles

Now consider the ZRP with N particles on $|\Lambda_L| = L$ sites.

The process is irreducible over

$$X_{L,N} = \{\boldsymbol{\eta} \in X_L \;:\; S_L(\boldsymbol{\eta}) := \sum_{x \in \Lambda_L} \eta_x = N\}.$$

Canonical measures Invariant measures $\mu_{L,N}$ supported on $X_{L,N}$

$$\mu_{L,N}[\boldsymbol{\eta}] = \frac{\prod_{x \in \Lambda_L} \frac{1}{g!(\eta_x)}}{\sum_{\boldsymbol{\eta}: S_L(\boldsymbol{\eta}) = N} \prod_{x \in \Lambda_L} \frac{1}{g!(\eta_x)}}$$

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with $\phi \leq \phi_c = 1$ and

$$\nu_\phi[k] \sim \frac{\phi^k}{g!(k)} = \phi^k \times \begin{cases} k^{-b} & \text{if } \lambda = 1,\ b > 2 \\ e^{-\frac{b}{1-\lambda} k^{1-\lambda}} & \text{if } \lambda \in (0,1) \end{cases} \quad .$$

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$$\mu_{L,N}\left[\,\cdot\,\right] = \nu_{\phi}^{L}\left[\ \cdot\ \middle|\ S_L(\boldsymbol{\eta}) = N\right]$$

Question: How large is $M_L = \max_{x \in \Lambda_L} \eta_x$ under $\mu_{L,N}$?

Subcritical Densities

If $N/L \to \rho < \rho_c$ there exists a fugacity $\phi(\rho) < \phi_c \; (= 1)$ such that $\langle \eta_x \rangle_{\nu_{\phi(\rho)}} = \rho$.

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Think of $$\mu_{L,N}[\,\cdot\,] = \nu^L_{\phi(\rho)}\Big[\,\cdot\,\Big|\, S_L(\boldsymbol{\eta}) = N\Big].$$

The event we are conditioning upon is not so unlikely, and locally $\mu_{L,N}$ behaves as a product of $\nu_{\phi(\rho)}$ in the limit (equivalence of ensembles.)

With a bit more work one sees that the typical size of M_L under $\mu_{L,N}$ is the same as under $\nu^L_{\phi(\rho)}$.

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$$\mu_{L,N}\left[\,\left|\frac{M_L}{\log L} - c(\rho)\right| > \epsilon\right] \longrightarrow 0, \qquad 0 < c(\rho) < +\infty.$$

Supercritical Densities

If $N/L \to \rho > \rho_c$ there exists no fugacity that corresponds to this density.

Now think of $\quad \mu_{L,N}\big[\,\cdot\,\big] = \nu^{L}_{\phi_c}\Big[\;\cdot\;\Big|\;S_L(\boldsymbol{\eta}) = N\Big].$

The event we are conditioning upon is a rare event and we need to understand how this large deviation of the sum is typically realised.

Recall that ν_{ϕ_c} is subexponential:

$$\nu_{\phi_c}[k] \sim \begin{cases} k^{-b} & \lambda = 1, b > 2 \\ e^{-\frac{b}{1-\lambda}\,k^{1-\lambda}} & \lambda \in (0,1) \end{cases}$$

Understanding the invariant measures $\mu_{L,N}$ reduces to Gibbs conditioning for subexponential r.v's

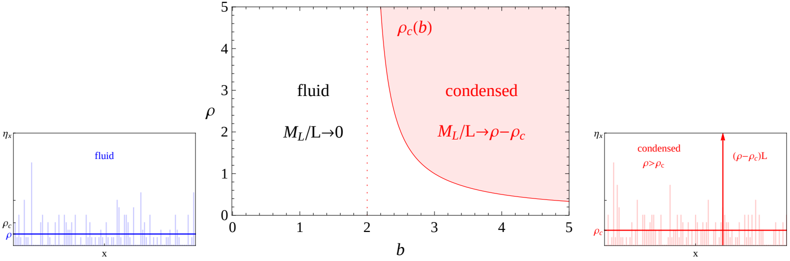
$$\mu_{L,N}\big[\,\cdot\,\big] = \nu_{\phi_c}\Big[\;\cdot\;\Big|\;\sum_{x\in\mathbb{T}} \eta_x = N\Big]$$

Canonical measures and condensation

Equivalence of ensembles [Grosskinsky, Schütz, Spohn '03]

In the thermodynamic limit $L, N \to \infty$, $N/L \to \rho$

$$\mu_{L,N} \xrightarrow{w} \nu_\phi \quad \text{where} \quad \begin{cases} R(\phi) = \rho \ , & \rho \le \rho_c \\ \phi = \phi_c \ , & \rho \ge \rho_c \end{cases} .$$



Corollaries to the strong invariance principle

Let $M_L = \max_{x\in\mathbb{T}} \eta_x$.

1. Since M_L+ mass in the bulk $= N$, $\mu_{L,N}$-a.s we get a conditional stable LT for the maximum from the stable LT for i.i.d. variables. If ν_{ρ_c} has finite variance $(b > 3)$

$$\mu_{L,N}\left[\frac{M_L - (N - \rho_c L)}{\sigma\sqrt{L}} \le x\right] \longrightarrow \int_{-\infty}^{x} e^{-y^2/2}\frac{dy}{\sqrt{2\pi}}.$$

(confirming the conjecture obtained from numerical simulations by Godréche & Luck.)

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2. Conditional k-order statistics converge to unconditional $(k-1)$-order statistics. E.g. if $M_L^{(2)}$ is the second largest of the $\{\eta_x\}_{x\in\mathbb{T}}$ then

$$\mu_{L,N}\left[M_L^{(2)} \le x\big(\Gamma(b)L\big)^{\frac{1}{b-1}}\right] \to e^{-x^{1-b}}.$$

Around criticality [Armendáriz, Grosskinsky, L, 2013]

Subcritical density: The maximum $M_L(\boldsymbol{\eta}) = \max_{x\in\Lambda_L} \eta_x$ is $O(\log L)$ with Gumbel fluctuations.

$$\mu_{L,N}\left[\frac{M_L(\boldsymbol{\eta}) - \alpha(\rho)\log L}{\beta_L} \leq x\right] \longrightarrow e^{-e^{-x}}.$$

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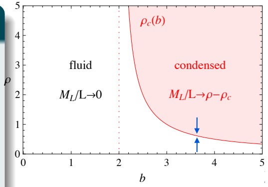
Question

How does $M_L(\boldsymbol{\eta})$ behave as we go through the critical density? In particular, when does the condensate emerge?

Ferrari/ Evans, Majumdar 2008



Small deviations

Let's focus on the case $\lambda = 1$ and $b > 3$ so that $\sigma^2 < +\infty$.

$$\nu_{\phi_c}\big[\eta_x = k\big] \sim k^{-b}.$$

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Proposition

If $-L^{\frac{b-2}{b-1}} \ll N - \rho_c L \ll \sqrt{L \log L}$, then

$$\mu_{L,N}\left[M_L \leq xL^{\frac{1}{b-1}}\right] \sim \nu_{\phi_c}^L\left[M_L \leq xL^{\frac{1}{b-1}}\right] \to e^{-ux^{1-b}} \quad \forall x > 0.$$

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Critical behaviour: Typical size of the maximum is $L^{\frac{1}{b-1}}$ with Frechét fluctuations.

Large Deviations

Above the critical scale, the bulk variables become asymptotically independent and the maximum satisfies a law of large numbers and a central limit theorem. That is,

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If $N - \rho_c L \gg \sqrt{L \log L}$, then

$$\frac{M_L}{N - \rho_c L} \xrightarrow{\mu_{L,N}} 1, \quad \text{and}$$

$$\frac{M_L - (N - \rho_c L)}{\sqrt{\sigma^2 L}} \xrightarrow{d} \mathcal{N}(0, 1).$$

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Supercritical behaviour: Typical size of the maximum is $N - \rho_c L$ with Gaussian fluctuations.

Large Deviations (downside)

Below the lower critical scale, the asymptotic distribution of the maximum under $\mu_{L,N}$ becomes Gumbel.

Proposition

If $\rho_c L - N \gg L^{\frac{b-2}{b-1}}$, then

$$\mu_{L,N}\Big[M_L \leq A_L + xB_L\Big] \to e^{-e^{-x}},$$

where

$$A_L \sim (b-1)L^{\frac{1}{b-1}}\frac{\log J_L}{J_L}, \quad B_L \sim \frac{L^{\frac{1}{b-1}}}{J_L}, \quad J_L = \frac{\rho_c L - N}{L^{\frac{b-2}{b-1}}} \to \infty.$$

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Subritical behaviour: The size of the maximum gradually reduces from $L^{\frac{1}{b-1}}$ to $\log L$ with Gumbel fluctuations.

Lower Critical Scale

At the lower critical scale the asymptotic distribution of the maximum under $\mu_{L,N}$ changes from Fréchet to Gumbel.

Proposition

If $\dfrac{\rho_c L - N}{L^{\frac{b-2}{b-1}}} \to \omega > 0$, then

$$\mu_{L,N}\left[M_L \leq x L^{\frac{1}{b-1}}\right] \longrightarrow \exp\left\{-C \int_x^{+\infty} e^{-\omega t}\, \frac{dt}{t^b}\right\}, \forall x > 0.$$

Critical Scale

$N - \rho_c L = O(\sqrt{L \log L})$ is the scale at which the condensate emerges. Define

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$M_L \sim L^{\frac{1}{b-1}}$, Frechét fluctuations.

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Phase transition

If $N - \rho_c L = (1-\epsilon)\Delta_L$

- The conditional distribution of the maximum is asymptotically equivalent to that of the maximum of independent r.v.'s drawn from ν_{ϕ_c}.
- $M_L = o\big(\Delta_L\big)$, and the number of particles in the bulk is $\rho_c L + O\big(\Delta_L\big)$.

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The condensate appears with the removal of particles from the bulk. The correlations shift from being entirely absorbed by the bulk, to being entirely absorbed by the maximum.

At the critical point

Suppose now

$$N - \rho_c L = \Delta_L \Big(1 + \frac{b}{2(b-3)} \frac{\log \log L}{\log L} + \frac{\gamma_L}{\log L} \Big).$$

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Proposition

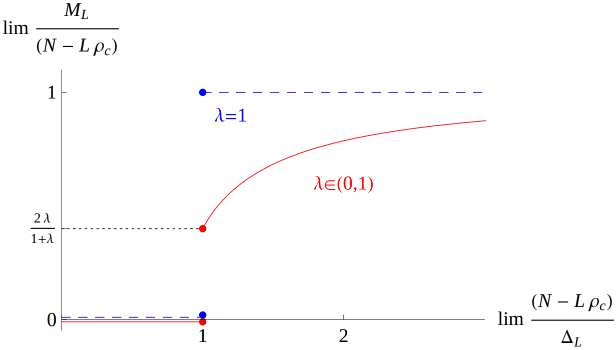
If $\gamma_L \to \gamma \in \mathbb{R}$,

$$\frac{M_L}{N - \rho_c L} \xrightarrow{\mu_{L,N}} Be(p_\gamma),$$

where $p_\gamma \in (0,1)$ is such that $p_\gamma \to 0(1)$ as $\gamma \to -\infty(+\infty)$.

The law of large numbers when $\lambda < 1$.

Comparison of the law of large numbers for the power law and the stretched exponential case.



Conclusion

- We saw how large deviations for subexponential r.v.'s are typically realised: one variable assumes a big value and the bulk becomes asymptotically independent.
- We used this knowledge to obtain refined results for the typical size of the condensate and its fluctuations at supercritical densities.
- Emergence of the giant component: LLN & fluctuations for the maximum if $N = \rho_c L + o(L)$
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Thank you for your attention