Metastability in condensing zero-range processes

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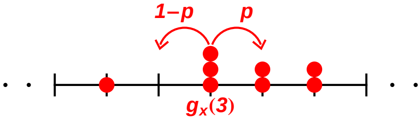
Condensation phenomena in stochastic systems, BATH

The zero-range process



Lattice: Λ of size L

State space: $X_L = \{0, 1, ..\}^\Lambda$

Jump probabilities: $p(x, y) \in [0, 1]$

Jump rates: $g_x : \{0, 1, ..\} \to [0, \infty)$, $g_x(k) = 0 \iff k = 0$

Generator: $f \in C_0(X_L)$

$$\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda} g_x(\eta_x) p(x, y) \big(f(\eta^{x,y}) - f(\eta)\big)$$

[Spitzer (1970), Andjel (1982)]

The zero-range process

$g_x(k) = k \quad \Rightarrow \quad$ independent identical particles

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Condensation phenomena

- spatial heterogeneity

  $\Rightarrow$ condensation on the 'slowest' site

  [Evans (1996), Krug, Ferrari (1996), Benjamini, Ferrari, Landim (1996), Ferrari, Sisko (2007)]

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Condensation phenomena

- spatial heterogeneity
  $\Rightarrow$ condensation on the 'slowest' site

  [Evans (1996), Krug, Ferrari (1996), Benjamini, Ferrari, Landim (1996), Ferrari, Sisko (2007)]

- effective attraction of particles due to $g(k) \searrow$
  $\Rightarrow$ condensation on a random site

  $g(n) \simeq 1 + \frac{b}{n}\,, \quad b > 2$ [Evans (2000)]

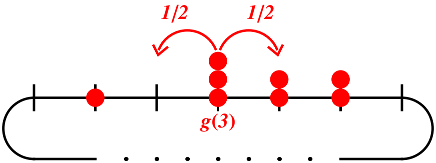
  $g(n) \simeq n e^{-2n}$

Evans model

Lattice: Λ of size L

State space: $X_L = \{0, 1, ..\}^\Lambda$

$$\eta = (\eta_x)_{x\in\Lambda}$$



Jump rates: $p(x,y)\,g(\eta_x)$

choose $g(k) = \left(\frac{k}{k-1}\right)^b \simeq 1 + \frac{b}{k}$ with $b > 0$

$g(0) = 0,\ g(1) = 1$

choose $p(x,y) = \frac{1}{2}\delta_{y,x+1} + \frac{1}{2}\delta_{y,x-1}$

Generator: $\mathcal{L}f(\eta) = \sum_{x\in\Lambda_L} g(\eta_x)\left(\frac{1}{2}f(\eta^{x,x+1}) + \frac{1}{2}f(\eta^{x,x-1}) - f(\eta)\right)$

[Spitzer '70; Andjel '82; Evans '00]

Canonical measures and condensation

fixed number of particles N: $\mu_{L,N}[\,\cdot\,] = \frac{\nu_\phi[\,\cdot\,]}{\nu_\phi[\sum_x \eta_x = N]}$, $\nu_\phi\big[\eta_x = k\big] \propto \frac{\phi^k}{k^b}$

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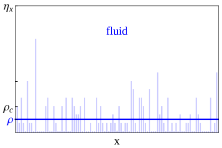
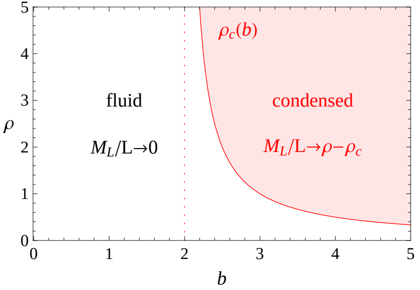
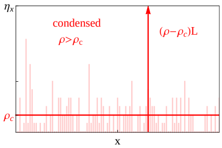
Equivalence of ensembles

In the thermodynamic limit $L, N \to \infty\ , \quad N/L \to \rho$

$$\mu_{L,N} \to \nu_\phi \quad \text{where} \quad \begin{cases} \phi \leftrightarrow \rho\ ,\ \rho \le \rho_c \\ \phi = \phi_c\ ,\ \rho \ge \rho_c \end{cases} .$$







[Jeon, March, Pittel '00; Grosskinsky, Schütz, Spohn '03; Ferrari, Landim, Sisko '07; A., Loulakis '09]

Metastability: dynamics of the condensate

Potential theoretic approach: [Bovier, Gayrard, Eckhoff, Klein '01, '02,...]
[Bovier, den Hollander, Metastability - a potential theoretic approach (2016)]

Martingale approach: [Beltrán, Landim '10, '11, '15]

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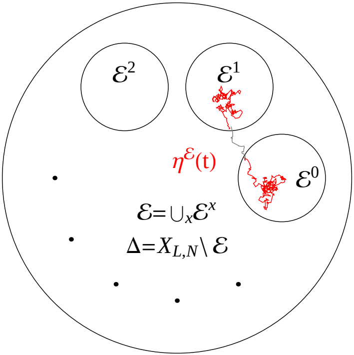
Trace process • metastable wells

$$\mathcal{E}^x := \left\{ \eta_x \geq N - \rho_c L - \alpha_L,\, \eta_y \leq \beta_L,\, y \neq x \right\} ;$$



Trace process $\eta^{\mathcal{E}}$

- $\eta^{\mathcal{E}}$ is a Markov process on $\mathcal{E} = \cup_{x \in \Lambda} \mathcal{E}^x$ with generator $\mathcal{L}^{\mathcal{E}}$ and rates

$$r^{\mathcal{E}}(\eta, \xi) = r(\eta, \xi) + \sum_{\zeta \in \Delta} r(\eta, \zeta)\, \mathbb{P}_\zeta[T_{\mathcal{E}} = T_\xi]$$

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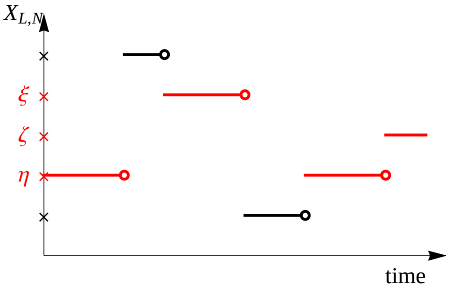

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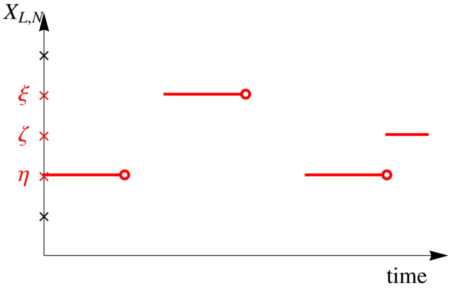

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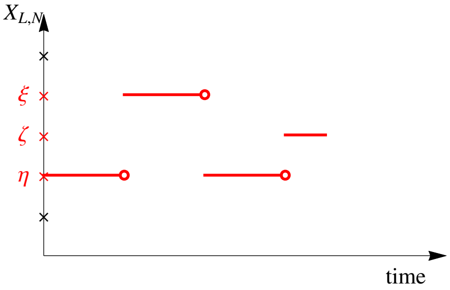

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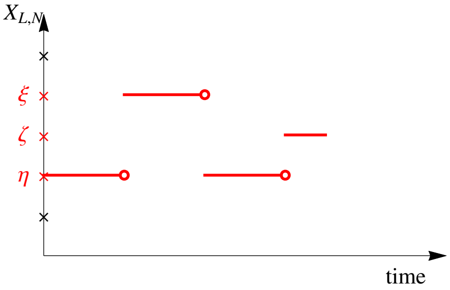
- invariant measure

$$\mu[\cdot] = \mu_{L,N}[\ \cdot \mid \mathcal{E}]$$



Main result

Theorem. A., Grosskinsky, Loulakis [arXiv:1507.03797]

The ZRP with $b > 20$, as $L, N \to \infty$, $N/L \to \rho > \rho_c$,
exhibits metastability w.r.t. the rescaled condensate location

$$Y_t^L := \psi_L(\eta^{\mathcal{E}}(\theta_L t)) := \frac{1}{L}\sum_{x\in\Lambda} x\mathbb{1}_{\mathcal{E}^x}\big(\eta^{\mathcal{E}}(\theta_L t)\big) \in \mathbb{T} \quad \text{on the scale } \theta_L = L^{1+b} \ .$$

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For all initial conditions $\eta^L(0) \in \mathcal{E}^0$ we have weakly on pathspace

$$\big(Y_t^L : t \geq 0\big) \Rightarrow (Y_t : t \geq 0) \quad \text{with} \quad Y_0 = 0 \ ,$$

where $(Y_t : t \geq 0)$ is a Lévy-type jump process on $\mathbb{T}$ with generator

$$\mathcal{L}^{\mathbb{T}} f(u) = K_{b,\rho} \int_{\mathbb{T}\setminus\{0\}} \frac{1}{d(v,u)}\big(f(v) - f(u)\big)\, dv \ ,$$

where $d(v,u) = |v-u|\big(1 - |v-u|\big)$ is the distance in $\mathbb{T}$.

Proof

- $\big(Y_t^L : t \geq 0\big)$ is **tight** on $D\big([0,T],\mathbb{T}\big)$
- identify limit points $\big(Y_t : t \geq 0\big)$ as solutions of the **martingale problem**

$$f(Y_t) - f(Y_0) - \int_0^t \mathcal{L}^{\mathbb{T}} f(Y_s)\, ds \quad \text{is a martingale .} \tag{1}$$

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Introduce the **auxiliary process** $\mathcal{L}^{\Lambda}$ on Λ with averaged rates

$$r^{\Lambda}(x,y) = \frac{1}{\mu[\mathcal{E}^x]} \sum_{\eta\in\mathcal{E}^x,\, \xi\in\mathcal{E}^y} \mu[\eta]\, r^{\mathcal{E}}(\eta,\xi)$$

and write

$$\int_0^t \Big(\mathcal{L}^{\mathbb{T}} f(Y_s^L) - \theta_L \mathcal{L}^{\mathcal{E}}(f\circ\psi_L)(\eta^{\mathcal{E}}(\theta_L s))\Big) ds$$

$$= \int_0^t \Big(\mathcal{L}^{\mathbb{T}} f(Y_s^L) - \theta_L \mathcal{L}^{\Lambda} f(Y_s^L)\Big) ds + \theta_L \int_0^t \Big(\mathcal{L}^{\Lambda} f(Y_s^L) - \mathcal{L}^{\mathcal{E}}(f\circ\psi_L)(\eta^{\mathcal{E}}(\theta_L s))\Big) ds$$

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1. Prove equilibration within wells on a scale $t_{\text{mix}} \ll \theta_L = L^{1+b}$
2. Prove convergence of averaged dynamics on the scale θ_L
3. central Lemma: **uniform bounds** on exit rates

1 – Equilibration within a well

Restricted process to a well $\mathcal{E}^x$ by ignoring jumps outside, $\mu^x = \mu[\,\cdot\,|\mathcal{E}^x]$

- bound on relaxation time t_{rel}, mixing time $t_{\mathsf{mix}}(\epsilon)$

$$t_{\mathsf{rel}} \leq CL^4 \quad \text{and} \quad t_{\mathsf{mix}}(\epsilon) \leq t_{\mathsf{rel}} \log\left(\frac{1}{\epsilon\mu_{\mathsf{min}}}\right) \leq CL^5 \log\left(1/\epsilon\right)$$

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- ergodic L^2 bound for functions with $\mu^x(h) = 0$, $x \in \Lambda$

$$\mathbb{E}_\mu \Big| \int_0^t h(\eta_u^{\mathcal{E}})\, du \Big|^2 \leq 24t\, t_{\mathsf{rel}} \sum_{x\in\Lambda} \mu\big[\mathcal{E}^x\big]\, \mu^x\big(h^2\big), \tag{2}$$

[J. Beltrán and C. Landim '15, *Martingale approach to metastability*]

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- Apply (2) + 3. + bounds on $\sum_{y\neq x} r^\Lambda(x,y)$ from 2. to $h = r^{\mathcal{E}} - r^\Lambda$ to get

$$\sup_{\eta\in\mathcal{E}} \mathbb{E}_\eta \Big|\theta_L \int_0^t \Big(\mathcal{L}^\Lambda f(Y_s^L) - \mathcal{L}^{\mathcal{E}}(f\circ\psi_L)(\eta^{\mathcal{E}}(\theta_L s))\Big) ds\Big| \to 0$$

2 – Mean rates as capacities

$$\mu[\mathcal{E}^{A_1}]r^\Lambda(A_1,A_2) = \mu[\mathcal{E}^{A_1}]\,\frac{1}{|A_1|}\sum_{\substack{x\in A_1\\ y\in A_2}} r^\Lambda(x,y) \qquad A_1,\,A_2\subset\Lambda$$

$$= \frac{1}{2}\Big(\mathrm{cap}\big(\mathcal{E}^{A_1},\mathcal{E}\setminus\mathcal{E}^{A_1}\big) + \mathrm{cap}\big(\mathcal{E}^{A_2},\mathcal{E}\setminus\mathcal{E}^{A_2}\big) - \mathrm{cap}\big(\mathcal{E}^{A_1\cup A_2},\mathcal{E}\setminus\mathcal{E}^{A_1\cup A_2}\big)\Big)$$

Prove bounds

$$\theta_L\,\mathrm{cap}\big(\mathcal{E}^{A_1},\mathcal{E}\setminus\mathcal{E}^{A_1}\big) \le K(b,\rho)\,(1+\bar{\epsilon}_L)\sum_{\substack{x\in A\\ y\notin A}}\mathrm{cap}_\Lambda(x,y)$$

$$\theta_L\,\mathrm{cap}\big(\mathcal{E}^{A_1},\mathcal{E}\setminus\mathcal{E}^{A_1}\big) \ge K(b,\rho)\,(1-\underline{\epsilon}_L)\sum_{\substack{x\in A\\ y\notin A}}\mathrm{cap}_\Lambda(x,y)$$

where $\mathrm{cap}_\Lambda(x,y) = \frac{1}{|x-y|\,(L-|x-y|)}$ capacities of symmetric rw on Λ.

2 – Regularization

- Total exit rate from a well $\propto \log L$
- Upper and lower bounds for rates $r^{\Lambda}(x, y)$ do not match

see also [A. Bovier, R. Neukirch '14]

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- Coarse graining in Λ & Lipschitz test functions to regularize

$$\theta_L \mathcal{L}^\Lambda f(x) = \sum_{m=1}^{\bar{L}} r^\Lambda(V_0, V_m) \left(f\left(\frac{x+\ell m}{L}\right) - f\left(\frac{x}{L}\right)\right) + o(1)$$

with $|V_i| = \ell \propto \alpha_L \log^3 L \to \infty$, $\quad \bar{L} = L/\ell$.

($\to$ leads to choice of $\alpha_L = L^{1/2+5/(2b)}$)

- matching bounds from capacity representation for $r^\Lambda(V_0, V_m)$

$$\sup_{\eta\in\mathcal{E}} \mathbb{E}_\eta \left| \int_0^t \left(\mathcal{L}^{\mathbb{T}} f(Y_s^L) - \theta_L \mathcal{L}^\Lambda f(Y_s^L)\right) ds \right| \to 0$$

3 – Coupling to a branching system of BD processes

$m = \lceil 2^b \rceil$ largest possible arrival rate for ZRP

$x \in \Lambda$, couple $\big(\eta_x(t) : t \geq 0\big)$ with a growing system of BD chains $\zeta_x^{\mathbf{k}}$,
indexed by the m-regular tree $\mathcal{R}_m$

- Each chain ζ_x has birth rate 1 and death rate $g(\zeta_x)$.
  Arrival events for $\eta_x(t)$ are used only for one of the coupled chains
- At any time t, only m of the chains are coupled to $\eta_x(t)$, and the rest are evolving independently.

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- At any time t, only m of the chains are coupled to $\eta_x(t)$, and the rest are evolving independently.
- Number of chains grows linearly with time
- $\max_{\mathbf{k}} \zeta_x^{\mathbf{k}}(t) \geq \eta_x(t)$ for all times $t \geq 0$.

Uniform exit rate bound: $$\sup_{\eta \in \mathcal{E}^x} \sum_{\xi \notin \mathcal{E}^x} r^{\mathcal{E}}(\eta, \xi) \leq C \, \frac{1}{L^5 (\log L)^2}$$

3 – Coupling to a branching system of BD processes

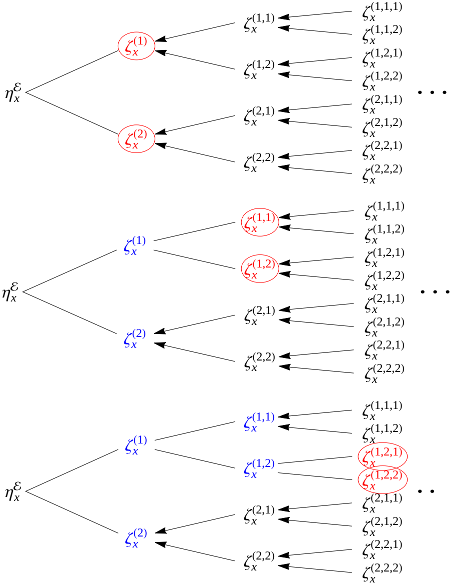
Example for $m = 2$

arrows $\to$: identical copies

coupled chains : red encircled

independent chains : in blue

- coupled at generation $n = 1$ (top)
- particle arrives at x (middle)
  chains in 1st gen. turn independent
  2 descendants on top coupled
- second particle arrives, etc.



Condensing zero-range with vanishing density

Lattice: complete graph Λ of size L

State space: $X_L = \{0, 1, ..\}^\Lambda$

$$\eta = (\eta_x)_{x \in \Lambda}$$

Jump rates: $p(x, y)\, g(\eta_x)$

choose $\quad g(k) = ke^{-2(k-1)\,\frac{L-1}{L}} \quad$ and $\quad p(x, y) = \frac{1}{L-1}$

Grand canonical measures: $1/g!(k) = e^{k^2-k}/k!,$ do not exist

Canonical measure: $\frac{1}{Z} \prod_x \frac{e^{\eta_x^2}}{\eta(x)!} \mathbf{1}_{\sum \eta_x = N}$

Number of particles: $N = (1 + \gamma) \log L, \quad N/L \to 0.$

Metastability

- **Metastable wells**

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Theorem. A., de Masi, Presutti

The ZRP with as $L, N \to \infty$, $N = (1+\gamma)\log L$, $\gamma \in \left(0, \frac{1}{\sqrt{2}}\right)$,
exhibits metastability w.r.t. the rescaled condensate location

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For all $\eta^L(0) \in \mathcal{E}^1$, $\left(Y_t^L : t \geq 0\right) \Rightarrow (Y_t : t \geq 0,\, Y_0 = 0)$, rate 1, uniform on $\mathbb{T}$.

Metastability

- Condensates $\eta^x : \eta^x_x = N, \eta^x_y = 0, y \neq x$

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Theorem continued

A., de Masi, Presutti

The time scale is

$$\theta_L = \frac{1}{r^\Lambda(\eta^1, \mathcal{F})}, \quad r^\Lambda(\eta^1, \mathcal{F}) = \frac{1}{\mu_{L,N}(\eta^1)} \operatorname{cap}(0, N),$$

where $\operatorname{cap}(0, N)$ are the capacities of a BD chain on $0, \ldots N$ with

$$b(k, k+1) = \frac{N-k}{L},\, k \leq N-1, \quad d(k, k-1) = k\, e^{-2(k-1)}\, \frac{L-N+k}{L},\, k \geq 1$$

and invariant measure

$$\nu_{L,N} = \frac{1}{Z} \binom{L-1}{N-k} \frac{e^{k^2-k}}{k!}$$

With probability 1 as $L \to \infty$, the trajectory between two condensate η^x and η^y, $x \neq y$, passes through $\mathcal{F}$.

The end