Dynamics of the condensate in the reversible inclusion process

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joint work with Sander Dommers & Cristian Giardinà

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Condensation phenomena in stochastic systems,
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Outline

Inclusion process

Interacting particles system with N particles moving on a (finite) set S following a given Markovian dynamics.

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- **Markovian dynamics:**

$$\mathcal{L}f(\eta) = \sum_{x,y \in S} r(x,y)\eta_x(d_N + \eta_y)\left(f(\eta^{x,y}) - f(\eta)\right) \qquad \text{generator}$$

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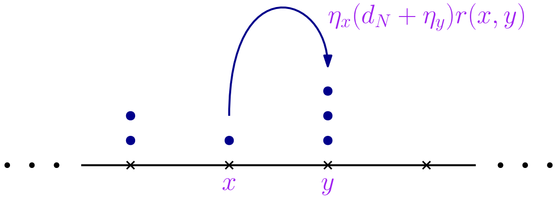
  - $r(x,y) \geq 0$ transition rates of a irreducible RW on S
  - $d_N > 0$ constant tuning the rates of the underlying RW

Example:



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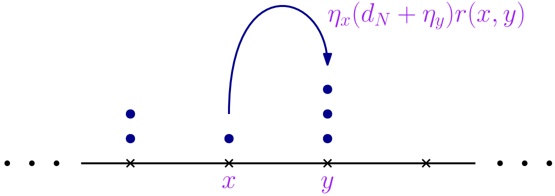

Remark:
Particle jump rates $r(x,y)\eta_x(d_N + \eta_y)$ can be split into

- $r(x,y)\eta_x d_N \longrightarrow$ independent RWs diffusion
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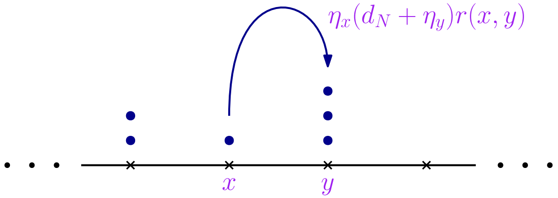

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Comparison with other processes:

- $r(x,y)\eta_x(1-\eta_y) \longrightarrow$ exclusion process
- $r(x,y)g(\eta_x) \longrightarrow$ zero-range process

Motivations

- The SIP on $S \subset \mathbb{Z}$ is **dual** of a heat conduction stochastic model (Brownian momentum process)
  $\longrightarrow$ *infer information from one model to the other one*
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- Under suitable hypotheses (e.g. $d = d_N \longrightarrow 0$; ASIP on $S \subset \mathbb{Z}$), one has
  - condensation (*particles concentrated on a single site*)
  - metastability (*condensate moves btw sites of S*)

Stationary measure

Assume the underlying RW is reversible w.r.t. a measure m

$$m(x)r(x,y) = m(y)r(y,x) \qquad \forall x, y \in S$$

normalized such that $\max_{x \in S} m(x) = 1$

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Then also IP has reversible probability measure μ_N

[Grosskinsky, Redig, Vafayi (2011)]

$$\mu_N(\eta) = \frac{1}{Z_N} \prod_{x \in S} m(x)^{\eta_x} w_N(\eta_x)$$

where Z_N is a normalizing constant and

$$w_N(k) = \frac{\Gamma(k + d_N)}{k!\Gamma(d_N)}, \qquad k \in \mathbb{N}$$

Condensation

Let $\eta^{x,N}$ the configuration with $\eta^{x,N}_x = N$ (condensate at x)

Proposition 1 (SIP - Grosskinsky, Redig, Vafayi '11).

Assume that $r(x,y) = r(y,x)$. If d_N is such that $1/N \ll d_N \ll 1$, then

$$\lim_{N\to\infty} \mu_N(\eta^{x,N}) = \frac{1}{|S|}$$

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Proposition 2 (ASIP - Grosskinsky, Redig, Vafayi '11).

Let $S = \{0, 1, \ldots, L\}$ and $p = r(x, x+1)$, $q = r(x, x-1)$ with $p > q > 0$. Then

$$\lim_{N\to\infty} \frac{\eta^{L,N}}{N} = 1\,, \qquad \mu_N - a.s.$$

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Remark: Taking independent RWs , η_x diverges $\forall x \in S$.

Condensation in reversible dynamics

Let $r(x, y)$ be reversibile w.r.t m, and $S^* = \{x \in S \,:\, m(x) = 1\}$.

Proposition 3 (Condensation- B., Dommers, Giardinà '16).

If d_N is such that $d_N \ll 1/\log N$, then

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Remark: This generalize the result for the SIP [Grosskinsky, Redig, Vafayi '11] but in a different regime of vanishing d_N.

Assumption on d_N is such that

$$\mu_N(\eta \,:\, \eta \neq \eta^{x,N},\ \text{for some } x \in S^*) \underset{N\to\infty}{\to} 0$$

Main related questions

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- How can we characterize the limiting dynamics of the condensate?

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Let $1/N \ll d_N \ll 1$ and $\eta_x(0) = N$ for some $x \in S$. Then

$$X_N(t \cdot 1/d_N) \text{ converges weakly to } x(t) \qquad \text{as } N \to \infty$$

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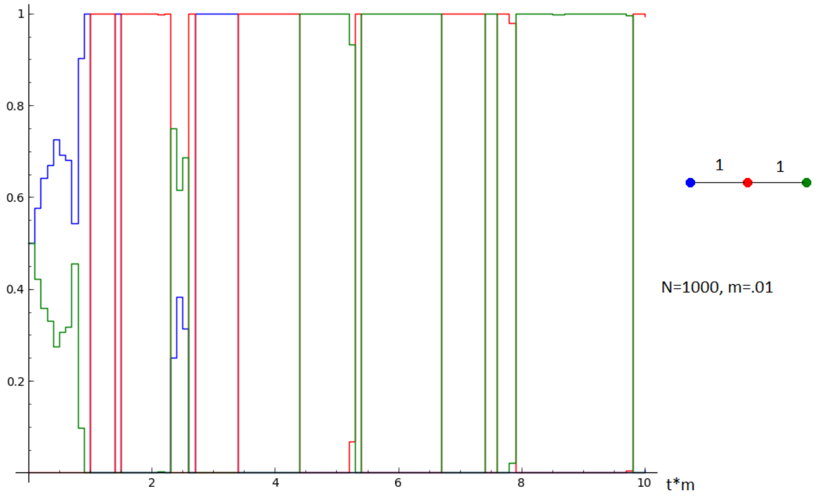
Remark. In [Grosskinsky, Redig, Vafayi '13] is also shown that the condensation time is of order $1/d_N$.

Symmetric case



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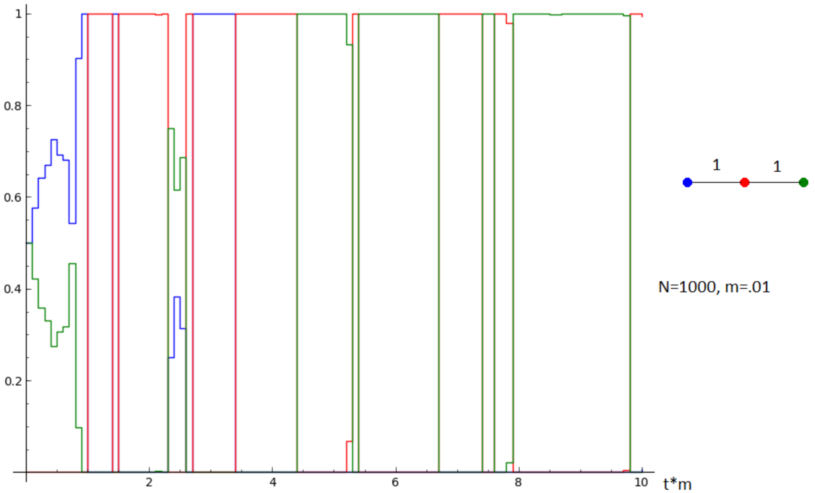


Goal: What happens in the reversible (generally non-symmetric) case?

Dynamics of the condensate: reversible case

As before, let $X_N(t) = \sum_{z\in S^*} z\mathbb{1}_{\{\eta_z(t) = N\}}$ with $S^* = \{x \in S \,:\, m(x) = 1\}$

Theorem 2 (B., Dommers, Giardinà '16).

Let $d_N \ll 1/\log N$ and $\eta_x(0) = N$ for some $z \in S^*$. Then

$$(1) \qquad X_N(t\cdot 1/d_N) \text{ converges weakly to } x(t) \qquad \text{as } N \to \infty$$

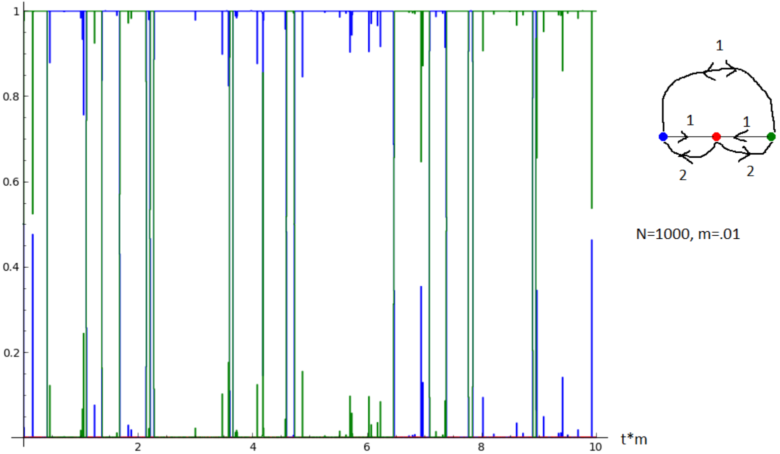
where $x(t)$ is a MP on S^* with rates $p(x,y) = r(x,y)$ and $x(0) = x$.

$$(2) \qquad \lim_{N\to\infty} d_N \cdot \mathbb{E}_{\eta^{x,N}}[\tau_{\mathcal{M}^{\setminus x}}] = \Big(\sum_{\substack{y\in S^*\\ y\neq x}} r(x,y)\Big)^{-1}$$

where $\tau_{\mathcal{M}^{\setminus x}}$ is the hitting time on the set $\mathcal{M}^{\setminus x} = \bigcup_{y\neq x}\eta^{y,N}$.

Simulations

First example



On the timescale $1/d_N$, the condensate moves between sites **maximizing the measure** $\boldsymbol{m}$.

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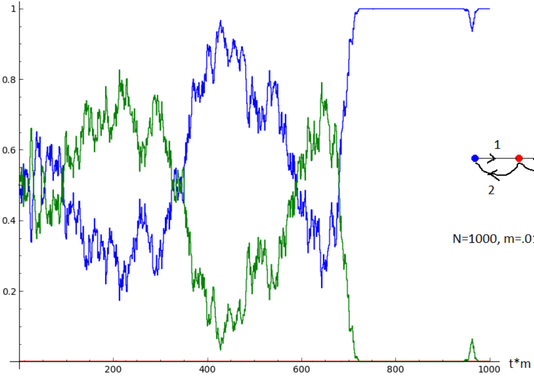
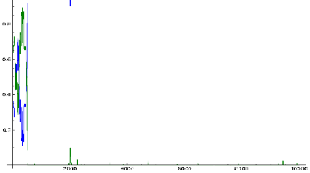
Second example



N=1000, m=.01



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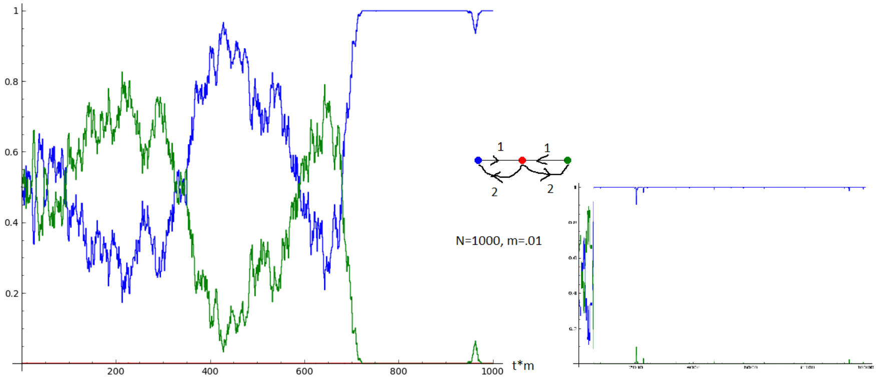

On the timescale $1/d_N$, condensation takes place *(though at a long scaled time)*, while once created, the condensate remains trapped for very long time on a vertex of S^*.

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Open problem: Characterization of further metastable timescales, and motion of the condensate between traps

Martingale approach

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Successfully applied to zero range process.

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To prove the theorem we need to check the following hypotheses:

(H0) $$\lim_{N\to\infty} \frac{1}{d_N} r(\eta^{x,N}, \eta^{y,N}) = p(x,y) \equiv r(x,y)$$

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(H2) $$\lim_{N\to\infty} \frac{\mu_N(\eta \,:\, \eta \neq \eta^{x,N} \text{ for some } x \in S^*)}{\mu_N(\eta^{x,N})} = 0 \qquad \forall x \in S^*$$

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Easy

Hypothesis H0

By [Beltrán Landim '10], the rate $r(\eta^{x,N}, \eta^{y,N})$ may be computed as a combination of capacities as

$$\mu_N(\eta^{x,N}) r(\eta^{x,N}, \eta^{y,N})$$
$$= \mathrm{Cap}\left(\eta^{x,N}, \bigcup_{\substack{z \in S^* \\ z \neq x}} \eta^{z,N}\right) + \mathrm{Cap}\left(\eta^{y,N}, \bigcup_{\substack{z \in S^* \\ z \neq y}} \eta^{z,N}\right)$$
$$- \mathrm{Cap}\left(\eta^{x,N} \cup \eta^{y,N}, \bigcup_{\substack{z \in S^* \\ z \neq x,y}} \eta^{z,N}\right)$$

Capacity versus Metastability

Capacity is a key quantity in the analysis of metastable systems

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If $A, B \subset \Omega$ disjoint, let $h_{A,B}$ the equilibrium potential:

$$\textbf{Dirichlet problem} \quad \begin{cases} \mathcal{L}h_{A,B}(x) = 0 & \text{if } x \notin A \cup B \\ h_{A,B}(x) = 1 & \text{if } x \in A \\ h_{A,B}(x) = 0 & \text{if } x \in B \end{cases}$$

Probabilistic interpretation:

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Along this direction, one can define capacities

$$\mathrm{Cap}(A, B) := \sum_{x \in A} \mu(\eta) \mathbb{P}_x[\tau_B < \tau_A^+]$$

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or in other terms

$$\mathrm{Cap}(A,B) = \mathcal{D}(h_{A,B}) = \frac{1}{2} \sum_{x,y \in \Omega} c(x,y) \left(h_{A,B}(x) - h_{A,B}(y)\right)^2$$

Advantages:

I Fact. If A e B are disjoint sets and $h_{A,B}(x) = \mathbb{P}_x(\tau_A < \tau_B)$, then

$$\text{(MT)} \qquad \mathbb{E}_{\nu_A}[\tau_B] = \frac{\mu(h_{A,B})}{\text{Cap(A,B)}} \qquad .$$

[Bovier, Eckhoff, Gayrard, Klein ’01-’04]

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$\longrightarrow$ look for a reduction to a **lower dimensional space**.

Capacities estimates

Look for matching upper and lower bounds over $\mathrm{Cap}(\eta^{x,N}, \eta^{y,N})$, $x, y \in S^*$.

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- To get a upper bound, we choose a test function which is combination of solutions of the 1D problem over $A^{x,y}$, for $x, y \in S^*$, suitably regularized.

and in conclusion...

Precise asymptotic estimates over capacities

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First metastable timescale $1/d_N$ with exact asymptotics
by formula (MT)

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Asymptotic scaling and value of $r(\eta^{x,N}, \eta^{y,N})$ (H0)

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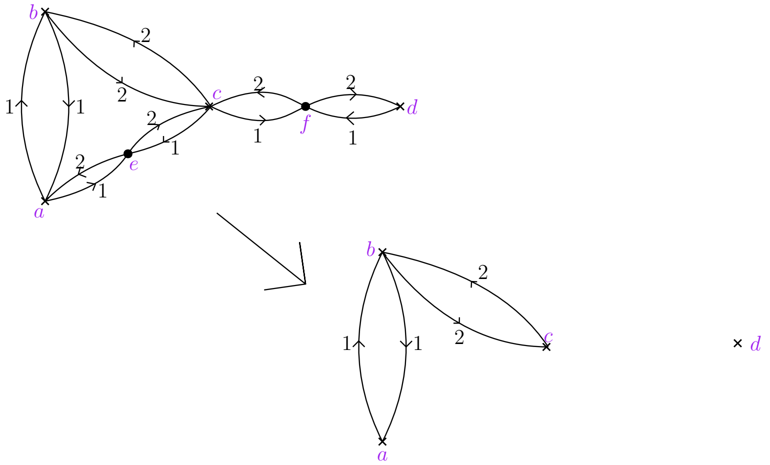
What happens if $(S^*, r_{|_{S^*}})$ is not irreducible?

Example



Metastable timescale(s)

Assume $\{r(x,y)\}_{x,y\in S^*}$ is reducible, and let $C_1,\ldots,C_m$, $m\geq 2$, the connected components of $(S^*, r_{|_{S^*}})$

$$S^* = \bigcup_{j=1}^{m} C_j \ , \ \ C_i \cup C_j = \emptyset, \ \text{for } i \neq j$$

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- Define a new set of metastable sets $\mathcal{E}_1,\dots,\mathcal{E}_m$:

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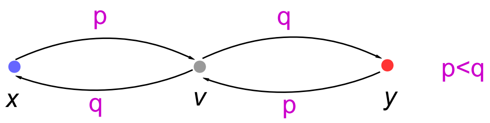
- Verify the hypotheses H_0, H_1 and H_2 of [Beltrán, Landim, 2010] $\longrightarrow$ **compute capacities** $\mathrm{Cap}_N(\mathcal{E}_i,\mathcal{E}_j)$.

Analysis of a 3- sites IP

Consider the IP defined through the underlying RW on $S = \{v, x, y\}$ with transition rates s.t.

$$\begin{cases} r(y,x) = r(x,y) = 0 \\ m(x) = m(y) = 1 > m(v) \end{cases}$$

$\Longrightarrow \eta^{N,x}, \eta^{N,y}$ are disconnected components of $(S^*, r_{|S^*})$



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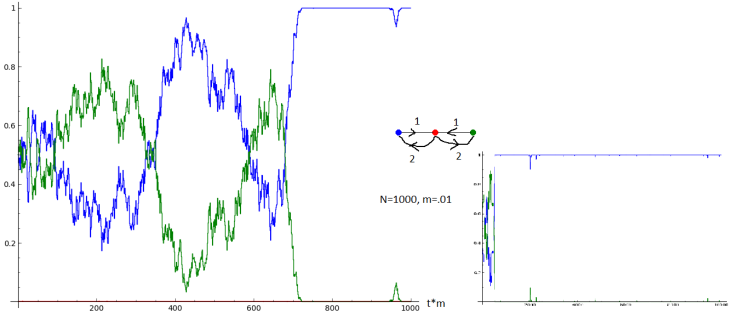

Capacities for the 3-sites IP

Proposition 4.

In the above notation and for $e^{-\delta N} \ll d_N \ll 1/\log N$ *for any* $\delta > 0$*,*

$$\lim_{N\to\infty} \frac{N}{d_N^2} \cdot \mathrm{Cap}_N(\eta^{N,x}, \eta^{N,y}) = \left(\frac{1}{r(v,x)} + \frac{1}{r(v,y)}\right)^{-1} \cdot \frac{m(v)}{1-m(v)}$$

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Following [Beltrán, Landim 2010], hypothesis H_0 is verified:

$$\lim_{N\to\infty} \frac{N}{d_N^2} r(\eta^{N,x}, \eta^{N,y}) = \left(\frac{1}{r(v,x)} + \frac{1}{r(v,y)} \right)^{-1} \cdot \frac{m(v)}{1-m(v)} =: p^{(2)}(x,y)$$

Dynamics of the condensate in the 3-sites IP

As a consequence (hypotheses H_1 and H_2 are easily verified), for

$$X_N(t) = \sum_{z \in S^*} z \mathbb{1}_{\{\eta_z(t) = N\}}$$

Proposition 5.

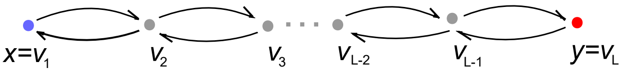
Let $\eta_x(0) = N$ for some $x \in S^*$. Then, for $e^{-\delta N} \ll d_N \ll 1/\log N$ for any $\delta > 0$,

$$X_N(t \cdot N/d_N^2) \text{ converges weakly to } x(t) \qquad \text{as } N \to \infty$$

where $x(t)$ is a Markov process on S^* with symmetric rates $p^{(2)}(x,y)$ and $x(0) = x$.

Analysis of a IP on on $\{1, 2, \ldots, L\}$, $L \geq 4$

Let $S = \{x = v_1, v_2, \ldots, v_L = y\}$ with $L \geq 4$ and consider the IP defined through the following RW



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Capacities for the IP on $\{1, 2, \dots, L\}$

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In the above notation, and for $e^{-\delta N} \ll d_N \ll 1/\log N$ *for any* $\delta > 0$,

$$C_1 \leq \lim_{N\to\infty} \frac{N^2}{d_N^3} \cdot \mathrm{Cap}_N(\eta^{N,x}, \eta^{N,y}) \leq C_2$$

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To prove convergence of the scaled dynamics matching bounds on the capacities are required (to investigate)

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Though multiple metastable timescales have been rigorously obtained only for simple underlying RW (1D RW), we expect that the mechanism highlighted here holds in generality.

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We conjecture the existence of longer metastable timescales

$$T_N^{(2)} \sim N/d_N^2 \qquad \text{and} \qquad T_N^{(3)} \sim N^2/d_N^3$$

such that

- At time $T_N^{(2)}$ the condensate moves between sites $x, y \in S^*$, with $d(x, y) = 2$.
- At time $T_N^{(3)}$ the condensate moves between sites $x, y \in S^*$, with $d(x, y) \geq 3$.

Conclusions and open problems

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We studied the reversible IP on a finite set in the limit $N \to \infty$ and for $e^{-\delta N} \ll d_N \ll 1/\log N$ by martingale approach:

- We derive the dynamics of the condensate at timescale $T_N^{(1)} \sim 1/d_N$;
- We identify longer metastable timescales in simple (1D) IP: $T_N^{(2)} \sim N/d_N^2$ and $T_N^{(3)} \sim N^2/d_N^3$. Derive the dynamics of the condensate on 3 sites.
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Thank you for your attention!