A Natural Proof System for Herbrand's Theorem

Benjamin Ralph(✉)

University of Bath, Bath, UK
b.d.ralph@bath.ac.uk
http://people.bath.ac.uk/bdr25

Abstract. The reduction of undecidable first-order logic to decidable propositional logic via Herbrand's theorem has long been of interest to theoretical computer science, with the notion of a Herbrand proof motivating the definition of expansion proofs. The problem of building a natural proof system around expansion proofs, with composition of proofs and cut-free completeness, has been approached from a variety of different angles. In this paper we construct a simple deep inference system for first-order logic, KSh2, based around the notion of expansion proofs, as a starting point to developing a rich proof theory around this foundation. Translations between proofs in this system and expansion proofs are given, retaining much of the structure in each direction.

Keywords: Structural proof theory · First-order logic · Deep inference
Herbrand's theorem · Expansion proofs

1 Introduction

A focus on the existential witnesses created in proofs has long been central to first-order proof theory. If one ignores all other information about a first-order proof except for the details of existential introduction rules, one still has an important kernel of the proof, in some sense the part of the proof that is inherently first-order, as opposed to merely propositional. Herbrand, in [13], innovated an approach to first-order proof theory that isolates this first-order content of the proof, and today the notion of a *Herbrand proof* is common, a proof-theoretic object that shows the carrying out of the following four steps, usually but not always in this order:

1. Expansion of existential subformulae.
2. Prenexification/elimination of universal quantifiers.
3. Term assignment.
4. Propositional tautology check.

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For example, we have the following theorem from [8] which exactly follows this scheme:

Theorem 1 (Herbrand's theorem). *A first-order formula A is valid if and only if A has a Herbrand proof. A Herbrand proof of A consists of a prenexification $A^\star$ of a strong $\vee$-expansion of A plus a witnessing substitution σ for $A^\star$.*

Or take the presentation of Herbrand's theorem in a deep inference system in [7]:

Theorem 2 (Herbrand's theorem). *For each proof of a formula S in system* SKSgr *there is a substitution σ, a propositional formula P, a context $Q\{\ \}$ consisting only of quantifiers and a* Herbrand proof*:*

$$\begin{array}{c} \| \mathsf{KS}\cup\{\mathsf{ai}\uparrow\} \\ \forall \boldsymbol{x} P\sigma \\ \| \{\mathsf{n}\downarrow\} \\ Q\{P\} \\ \| \{\mathsf{gr}\downarrow\} \\ S' \\ \| \{\mathsf{qc}\downarrow\} \\ S \end{array}$$

One obvious difference between the two formulations is that while the first definition of a Herbrand proof does not involve a proof in any commonly used proof system, the second definition is based around a factorisation of a proof in deep inference. Thus, the second definition gives us more opportunities to manipulate, compose and identify Herbrand proofs as proof theoretic objects.

The basic observation of this paper is that defining Herbrand proofs in a deep inference setting is easier and more natural than doing so in Gentzen-style systems (in particular the Sequent Calculus and Natural Deduction). This is because the steps (1), (2) and (3) as defined above are standard inference rules in first-order deep inference proof systems, and, while it is obviously possible to include them as *ad hoc* rules, they are not natural for Gentzen-style systems, especially carried out in this order.

To put it another way: if we want to build a proof theory around Herbrand's theorem, in which the propositional and first-order content of a cut-free proof is separated in a natural way, then deep inference is a superior setting to the sequent calculus, in some concrete senses. To substantiate this claim, we define two inter-translatable classes of deep inference proofs. The first class comprises analytic Herbrand proofs, defined similarly to those in Theorem 2 above, and we borrow a result from [7] to show that the class is complete for FOL. We show a tight correspondence between the second class and expansion proofs, a minimalistic formalism for first-order (and higher-order) proofs that ignores all but the most essential first-order structure. This correspondence suggests the second class as a good candidate for canonical first-order proofs. Therefore, the translation between the two classes enables us to see Herbrand proofs as canonical first-order proofs.

It should be noted that, while a translation between expansion proofs and first-order deep inference proofs has not previously been shown, Straßburger has developed a notion of expansion proofs for MLL2, and provided a similar translation between these structures and a deep inference proof system for that logic [18,19].

2 Expansion Proofs

In [17], Miller generalises the concept of the Herbrand expansion to higher order logic, representing the witness information in a tree structure, and explicit transformations between these 'expansion proofs' and cut-free sequent proofs are provided. Miller's presentation of expansion proofs lacked some of the usual features of a formal proof system, crucially composition by an eliminable cut, but extensions in this direction have been carried out by multiple authors. In [12], Heijltjes presents a system of 'proof forests', a graphical formalism of expansion proofs with cut and a local rewrite relation that performs cut elimination. Similar work has been carried out by McKinley [16] and more recently by Hetzl and Weller [14] and Alcolei et al. [1]. As expansion proofs and the related formalisms only represent the first-order content of a proof, we will first define expansion proofs in order to guide the definition of the proof systems.

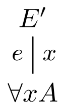
Remark 1. Throughout the paper, we use $\star$ in place of $\wedge$ and $\vee$, and Q in place of $\forall$ and $\exists$ if both cases can be combined into one. For clarity, we will sometimes distinguish between connectives in expansion trees, $\star_E$, and in formulae/derivations, $\star_F$.

Definition 1. *We define* expansion trees*, the two functions Sh (shallow) and Dp (deep) from expansion trees to formulae, a set of* eigenvariables $EV(E)$ *for each expansion tree, and a partial function Lab from edges to terms, following [9,12,17]:*

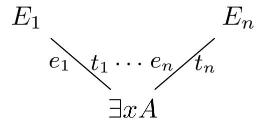
1. *Every literal A (including the units* t *and* f*) is an expansion tree. $Sh(A) := A$, $Dp(A) := A$, and $EV(A) = \emptyset$.*
2. *If E_1 and E_2 are expansion trees with $EV(E_1) \cap EV(E_2) = \emptyset$, then $E_1 \star E_2$ is an expansion tree, with $Sh(E_1 \star_E E_2) := Sh(E_1) \star_F Sh(E_2)$, $Dp(E_1 \star_E E_2) := Dp(E_1) \star_F Dp(E_2)$, and $EV(E_1 \star E_2) = EV(E_1) \cup EV(E_2)$. We call $\star$ a $\star$-*node *and each unlabelled edge e_i connecting the $\star$-node to E_i a $\star$-*edge*. We represent $E_1 \star E_2$ as:*

$$\begin{array}{ccc} E_1 & & E_2 \\ e_1\diagdown & & \diagup e_2 \\ & \star & \end{array}$$

3. *If E' is an expansion tree s.t. $Sh(E') = A$ and $x \notin EV(E')$, then $E = \forall x A +^x E'$ is an expansion tree with $Sh(E) := \forall x A$, $Dp(E) := Dp(E')$, and $EV(E) := EV(E') \cup \{x\}$. We call $\forall x A$ a $\forall$-*node *and the edge e connecting the $\forall$-node and E' a $\forall$-*edge*, with $Lab(e) = x$. We represent E as:*



4. If $t_1, \ldots, t_n$ *are terms* $(n \geq 0)$*, and* $E_1, \ldots, E_n$ *are expansion trees s.t.* $x \notin EV(E_i)$ *and* $EV(E_i) \cap EV(E_j) = \emptyset$ *for all* $1 \leq i < j \leq n$*, and* $Sh(E_i) = A\{x \Leftarrow t_i\}$*, then* $E = \exists xA +^{t_1} E_1 +^{t_2} \cdots +^{t_n} E_n$ *is an expansion tree, where* $Sh(E) := \exists xA$*,* $Dp(E) := Dp(E_1) \vee \cdots \vee Dp(E_n)$*, and* $EV(E) = \bigcup_1^n EV(E_i)$*. We call* $\exists xA$ *an* ∃-node *and each edge* e_i *connecting the* ∃*-node with* E_i *an* ∃-edge*, with* $Lab(e_i) = t_i$*. We represent* E *as:*



Remark 2. Let ρ be a permutation of $[1 \ldots n]$. We consider the expansion trees $\exists xA +^{t_1} E_1 +^{t_2} \cdots +^{t_n} E_n$ and $\exists xA +^{t_{\rho(1)}} E_{\rho(1)} \cdots +^{t_{\rho(n)}} E_{\rho(n)}$ equal. Our trees are also presented the other way up to usual, e.g. [12]. This is so that they are the same way up as the deep inference proofs we will translate them to below.

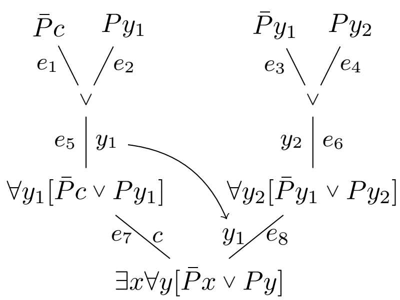
Definition 2. *Let* E *be an expansion tree and let* $<_E^-$ *be the relation on the edges in* E *defined by:*

- $e <_E^- e'$ *if the node directly below* e *is the node directly above* e'*.*
- $e <_E^- e'$ *if* e *is an* ∃*-edge with* $Lab(e) = t$*, there is an* x *which is free in* t*,* e' *is a* ∀*-edge and* $Lab(e') = x$*. In this case, we say* e' points to e*.*

The dependency relation *of* E*,* $<_E$*, is the transitive closure of* $<_E^-$*.*

Definition 3. *An expansion tree* E *is* correct *if* $<_E$ *is acyclic and* $Dp(E)$ *is a tautology. We can then call* E *an* expansion proof *of* $Sh(E)$*.*

Example 1. Below is an expansion tree E, with $Sh(E) = \exists x \forall y[\bar{P}x \vee Py]$ and $Dp(E) = [\bar{P}c \vee Py_1] \vee [\bar{P}y_1 \vee Py_2]$. The tree is presented with all edges explicitly named, to define the dependency relation below, as well as the labels for the ∃-edges and ∀-edges.



The dependency relation is generated by the following inequalities: $e_3, e_4 < e_6 < e_8$ and $e_1, e_2 < e_5 < e_7$ and $e_8 < e_5$. e_5 points to e_8. As this dependency relation is acyclic and $[\bar{P}c \vee Py_1] \vee [\bar{P}y_1 \vee Py_2]$ is a tautology, E is correct, and thus an expansion proof.

3 Proof Systems

3.1 Motivation for the Proof Systems

What features would a proof system, PS, designed around Expansion Proofs, EP, have? Say we have a translation $\pi : EP \to PS$.

Firstly, we might want that composition of proofs in PS matches closely to composition of expansion proofs, that something close to functoriality of π holds:

$$\pi(E_1 \star_E E_2) \approx \pi(E_1) \star_F \pi(E_2)$$

For Gentzen-style systems this will prove difficult, as there is no natural way to compose two proofs by disjunction.

A second attractive feature would be that we could isolate a part of the proof system that is relevant to Herbrand's theorem, stating and proving it as a factorisation of proofs, where the first order content of the proof is isolated from the propositional content:

$$\pi(E) = \begin{array}{c} {\scriptstyle \pi^{Up}(E)} \Big\Vert {\scriptstyle Prop} \\ Dp(E) \\ {\scriptstyle \pi^{Lo}(E)} \Big\Vert {\scriptstyle FO} \\ Sh(E) \end{array}$$

Interestingly, this is impossible in the usual sequent calculus systems, which we can see by considering Brünnler's second restriction on contraction. Consider the following property of proof systems:

"Proofs can be separated into two phases (seen bottom-up): The lower phase only contains instances of contraction. The upper phase contains instances of the other rules, but no contraction. No formulae are duplicated in the upper phase" [6].

Brünnler shows that a standard sequent calculus proof system with multiplicative rules cannot satisfy this property. The suggested way round this restriction is to use systems with *deep contraction*. In fact, this restriction on sequent calculus systems is shown by McKinley in [15] to create a gap in Buss's proof of Herbrand's theorem in [8]. The faulty proof assumes that if one restricts contraction to only existential formulae, one retains completeness (assuming a multiplicative $\wedge R$ rule). That this is false can be seen by considering the sequent below, where the application of any multiplicative $\wedge R$ rule leads to an invalid sequent:

$$\vdash \forall x A \wedge \forall x B, \left[\exists x \bar{A} \vee \exists x \bar{B}\right] \wedge \left[\exists x \bar{A} \vee \exists x \bar{B}\right]$$

Thus to achieve the desired factorisation property, we either need to add unrestricted contraction by the back-door, say by using additive $\wedge R$ rules, or use a system with deep contraction.

Both the above considerations suggest the formalism of *open deduction* for proofs of Herbrand's theorem [10]. It is a deep inference formalism which allows for free composition of proofs by $\wedge$ and $\vee$ at the propositional level, satisfying the first desideratum. Also, the deep contraction rule is certainly a natural rule for a deep inference system, allowing us to satisfy the second desideratum.

3.2 Open Deduction

As discussed above, we will work in the open deduction formalism. Open deduction differs from the sequent calculus in that we build up complex derivations with connectives and quantifiers in the same way that we build up formulae [10]. We can compose two derivations horizontally with $\star$, quantify over derivations, and compose derivations vertically with an inference rule.

Definition 4. *An open deduction derivation is inductively defined in the following way:*

- *Every atom $Pt_1 \dots t_n$ is a derivation, where P is an n-ary predicate, and t_i are terms. The units* t *and* f *are also derivations.*

If $\begin{array}{c} A \\ \phi \Vert \\ B \end{array}$ *and* $\begin{array}{c} C \\ \psi \Vert \\ D \end{array}$ *are derivations, then:*

- $\begin{array}{c} A \star C \\ \phi\star\psi \Vert \\ B \star D \end{array} = \begin{array}{c} A \\ \phi \Vert \\ B \end{array} \star \begin{array}{c} C \\ \psi \Vert \\ D \end{array}$ *and* $\begin{array}{c} QxA \\ Qx\phi \Vert \\ QxB \end{array} = Qx \left[\begin{array}{c} A \\ \phi \Vert \\ B \end{array} \right]$ *are derivations.*
- $\begin{array}{c} A \\ \chi \Vert \\ D \end{array} = \begin{array}{c} A \\ \phi \Vert \\ \rho \dfrac{B}{C} \\ \psi \Vert \\ D \end{array}$ *is a derivation, if* $\rho \dfrac{B}{C}$ *is an instance of ρ.*

When we write $\begin{array}{c} A \\ \phi \Vert S \\ B \end{array}$*, it means that every inference rule in ϕ is an element of the set S or an equality rule.*

Remark 3. Formulae are just derivations built up with no vertical composition. Open deduction and the *calculus of structures* (the better known deep inference formalism) polynomially simulate each other [11].

Definition 5. *We define a* section *of a derivation in the following way:*

– *Every atom a has one section, a.*
– *If A is a section of ϕ, and B is a section of ψ, then $A \star B$ is a section of $\phi \star \psi$, and QxA is a section of $Qx\phi$.*
– *If A is a section of* $\begin{array}{c} B \\ \phi_1 \| \\ C \end{array}$ *or* $\begin{array}{c} D \\ \phi_2 \| \\ E \end{array}$ *and* $\phi = \begin{array}{c} B \\ \phi_1 \| \\ \rho\,\dfrac{C}{D} \\ \phi_2 \| \\ E \end{array}$ *then A is a section of ϕ.*

The premise and conclusion of a derivation are, respectively, the uppermost and lowermost section of the derivation. A proof *of A is a derivation with premise* t *and conclusion A, sometimes written* $\begin{array}{c}\phi \,\Pi \\ A\end{array}$.

Definition 6. *We define the rewriting system* Seq *as containing the following two rewrites S_l and S_r:*

$$
= \dfrac{\begin{array}{c} A \\ \| \\ K\left\{\rho_1\,\dfrac{A_1}{B_1}\right\}\{A_2\} \end{array}}{\begin{array}{c} K\{B_1\}\left\{\rho_2\,\dfrac{A_2}{B_2}\right\} \\ \| \\ B \end{array}}
\quad \xleftarrow{S_l} \quad
\begin{array}{c} A \\ \| \\ K\left\{\rho_1\,\dfrac{A_1}{B_1}\right\}\left\{\rho_2\,\dfrac{A_2}{B_2}\right\} \\ \| \\ B \end{array}
\quad \xrightarrow{S_r} \quad
= \dfrac{\begin{array}{c} A \\ \| \\ K\{A_1\}\left\{\rho_2\,\dfrac{A_2}{B_2}\right\} \end{array}}{\begin{array}{c} K\left\{\rho_1\,\dfrac{A_1}{B_1}\right\}\{B_2\} \\ \| \\ B \end{array}}
$$

If ϕ is in normal form w.r.t. Seq*, we say ϕ is in* sequential form. *If $\phi \longrightarrow^*_{\mathsf{Seq}} \psi$ and ψ is in sequential form, we say that ψ is a* sequentialisation *of ϕ.*

Proposition 1. *A derivation ϕ is in sequential form iff. It is in the following form, where ρ_i are all the non-equality rules:*

$$
= \dfrac{A}{= \dfrac{K_1\left\{\rho_1\,\dfrac{A_1}{B_1}\right\}}{\begin{array}{c} \vdots \\ = \dfrac{K_n\left\{\rho_n\,\dfrac{A_n}{B_n}\right\}}{B} \end{array}}}
$$

Definition 7. *A* closed *derivation is one where every section of the derivation is a sentence (i.e. a formula with no free variables).*

3.3 KSh1 and Herbrand Proofs

We can now define an open deduction proof system for Herbrand proofs. We will extend the propositional system KS [5]:

$$\mathsf{KS} = \boxed{\begin{array}{ccc} \mathsf{ai}\!\downarrow \dfrac{\mathsf{t}}{a \vee \bar{a}} & \mathsf{ac}\!\downarrow \dfrac{a \vee a}{a} & \mathsf{aw}\!\downarrow \dfrac{\mathsf{f}}{a} \\[2ex] \mathsf{s}\,\dfrac{A \wedge [B \vee C]}{(A \wedge B) \vee C} & \mathsf{m}\,\dfrac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]} & \end{array}} + \boxed{\begin{array}{ccccccc} A \wedge \mathsf{t} & = & A & & A \vee \mathsf{f} & = & A \\ \mathsf{t} \vee \mathsf{t} & = & \mathsf{t} & & \mathsf{f} \wedge \mathsf{f} & = & \mathsf{f} \\ A \wedge (B \wedge C) & = & (A \wedge B) \wedge C & & A \vee [B \vee C] & = & [A \vee B] \vee C \\ A \wedge B & = & B \wedge A & & A \vee B & = & B \vee A \end{array}}$$

We will write $=\dfrac{A}{B}$ if B can be obtained from A by equality rules (treating multiple instances of different equality rules as one instance of a general equality rule).

We now introduce rules for the first three steps of a Herbrand Proof:

1. For expansion of existential subformulae, we have the rule:

$$\mathsf{qc}\!\downarrow \frac{\exists x A \vee \exists x A}{\exists x A}$$

   For technical reasons, we insist that all three instances of $\exists x A$ in $\mathsf{qc}\!\downarrow$ are α-equivalent with unique bound variables, but as above we will sometimes refer to them all as $\exists x A$ for simplicity.
2. For prenexification, we have four rules, where B is free for x:

$$\mathsf{r1}\!\downarrow \frac{\forall x [A \vee B]}{\forall x A \vee B} \quad \mathsf{r2}\!\downarrow \frac{\forall x (A \wedge B)}{\forall x A \wedge B} \quad \mathsf{r3}\!\downarrow \frac{\exists x [A \vee B]}{\exists x A \vee B} \quad \mathsf{r4}\!\downarrow \frac{\exists x (A \wedge B)}{\exists x A \wedge B}$$

   We consider commutative variants of these rules valid instances, e.g. $\mathsf{r1}\!\downarrow \dfrac{\forall x [B \vee A]}{B \vee \forall x A}$.
3. For term assignment, we have the rule:

$$\mathsf{n}\!\downarrow \frac{A\{x \Leftarrow t\}}{\exists x A}$$

Definition 8. *We define a proof system for FOL,* KSh1*:*

$$
\mathsf{KSh1} = \mathsf{KS} + \boxed{\begin{array}{ccc}
\mathsf{r1}\!\downarrow \dfrac{\forall x[A \vee B]}{[\forall x A \vee B]} & \mathsf{r2}\!\downarrow \dfrac{\forall x(A \wedge B)}{(\forall x A \wedge B)} & \mathsf{n}\!\downarrow \dfrac{A\{x \Leftarrow t\}}{\exists x A} \\[2ex]
\mathsf{r3}\!\downarrow \dfrac{\exists x[A \vee B]}{[\exists x A \vee B]} & \mathsf{r4}\!\downarrow \dfrac{\exists x(A \wedge B)}{(\exists x A \wedge B)} & \mathsf{qc}\!\downarrow \dfrac{\exists x A \vee \exists x A}{\exists x A}
\end{array}}
+
\boxed{\begin{array}{rclrcl}
\forall x A & = & \forall z A\{x \Leftarrow z\} & \exists z A & = & \exists z A\{x \Leftarrow z\} \\
\forall x \forall y A & = & \forall y \forall x A & \exists x \exists y A & = & \exists y \exists x A \\
\forall x \mathsf{t} & = & \mathsf{t} = \exists x \mathsf{t} & \forall x \mathsf{f} & = & \mathsf{f} = \exists x \mathsf{f}
\end{array}}
$$

Where z does not occur in A for the top two equalities.

For an example of a KSh1 proof, see (Fig. 1).

$$
\mathsf{qc}\!\downarrow \dfrac{\mathsf{r1}\!\downarrow \dfrac{\mathsf{r1}\!\downarrow \dfrac{= \dfrac{\mathsf{t}}{\forall y_1 \forall y_2 \left[= \dfrac{\mathsf{ai}\!\downarrow \dfrac{\mathsf{t}}{Py_1 \vee \bar{P}y_1} \vee \left[\mathsf{aw}\!\downarrow \dfrac{\mathsf{f}}{\bar{P}c} \vee \mathsf{aw}\!\downarrow \dfrac{\mathsf{f}}{Py_2} \right]}{[\bar{P}c \vee Py_1] \vee [\bar{P}y_1 \vee Py_2]} \right]}}{\forall y_1 \left[[\bar{P}c \vee Py_1] \vee \mathsf{n}\!\downarrow \dfrac{\forall y_2 [\bar{P}y_1 \vee Py_2]}{\exists x_2 \forall y_2 [\bar{P}x_2 \vee Py_2]} \right]}}{\mathsf{n}\!\downarrow \dfrac{\forall y_1 [\bar{P}c \vee Py_1]}{\exists x_1 \forall y_1 [\bar{P}x_1 \vee Py_1]} \vee \exists x_2 \forall y_2 [\bar{P}x_2 \vee Py_2]}}{\exists x \forall y [\bar{P}x \vee Py]}
$$

Fig. 1. A KSh1 proof of a variant of the "drinking principle', $\exists x \forall y[\bar{P}x \vee Py]$, popularised by Smullyan: "There is someone in the pub such that, if he is drinking, then everyone in the pub is drinking."

Proposition 2. KSh1 *is sound and complete.*

Proof. Soundness is trivial. Completeness follows from the proof of Herbrand's theorem in [7] (as stated in the introduction) and the observation that with the four rules $\{\mathsf{r1}\!\downarrow, \mathsf{r2}\!\downarrow, \mathsf{r3}\!\downarrow, \mathsf{r4}\!\downarrow\}$ we can simulate the general retract rule:

$$
\mathsf{gr}\!\downarrow \dfrac{Q\{P\{A\}\}}{P\{Q\{A\}\}},
$$

where $Q\{\ \}$ is a sequence of quantifiers and $P\{\ \}$ is a propositional context with no variables bound by any quantifier in $Q\{\ \}$.

Following [7], we define a Herbrand proof in the context of KSh1 in the following way.

Definition 9. *A closed* KSh1 *proof is a* Herbrand proof *if it is in the following form:*

$$
\begin{array}{c}
\Vert\ \mathsf{KS} \\
\forall \boldsymbol{x} B\{\boldsymbol{y} \Leftarrow \boldsymbol{t}\} \\
\Vert\ \{\mathsf{n}\!\downarrow\} \\
Q\{B\} \\
\Vert\ \{\mathsf{r1}\!\downarrow, \mathsf{r2}\!\downarrow, \mathsf{r3}\!\downarrow, \mathsf{r4}\!\downarrow\} \\
A' \\
\Vert\ \{\mathsf{qc}\!\downarrow\} \\
A
\end{array}
$$

where $Q\{\ \}$ is a context consisting only of quantifiers and B is quantifier-free.

For an example of a Herbrand proof, see (Fig. 2).

Proposition 3. *Every proof in* KSh1 *can be converted to a Herbrand Proof.*

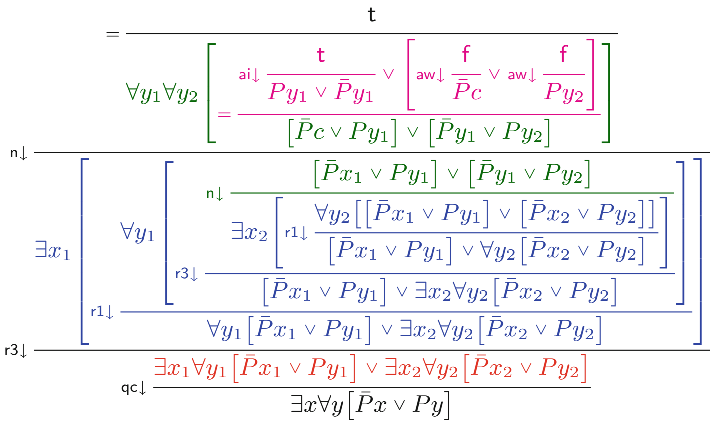


Fig. 2. A Herbrand proof of the drinking principle

Proof. [7, Theorem 4.2]

3.4 KSh2 and Herbrand Normal Form

To aid the translation between open deduction proofs and expansion proofs, we introduce a slightly different proof system to KSh1. It involves two new rules.

Definition 10. *We define the rule* $\mathsf{h}\downarrow$*, which we call a* Herbrand expander *and the rule* $\exists\mathsf{w}\downarrow$*, which we call* existential weakening*:*

$$\mathsf{h}\downarrow\frac{\exists x A \vee A\{x \Leftarrow t\}}{\exists x A} \qquad \exists\mathsf{w}\downarrow\frac{\mathsf{f}}{\exists x A}$$

For technical reasons again, we insist that $A\{x \Leftarrow t\}$ *is in fact* $A'\{x \Leftarrow t\}$*, where* A' *is an* α*-equivalent formula to* A *with fresh variables for all quantifiers, but for simplicity we will usually denote it* A*.*

Definition 11

$$\mathsf{KSh2} = \mathsf{KS} + \boxed{\begin{array}{cc} \mathsf{r1}\downarrow\dfrac{\forall x[A \vee B]}{[\forall x A \vee B]} & \mathsf{h}\downarrow\dfrac{\exists x A \vee A\{x \Leftarrow t\}}{\exists x A} \\ \mathsf{r2}\downarrow\dfrac{\forall x(A \wedge B)}{(\forall x A \wedge B)} & \exists\mathsf{w}\downarrow\dfrac{\mathsf{f}}{\exists x A} \end{array}} + \boxed{\begin{array}{rclrcl} \forall x A &=& \forall z A\{x \Leftarrow z\} & \exists z A &=& \exists z A\{x \Leftarrow z\} \\ \forall x \forall y A &=& \forall y \forall x A & \exists x \exists y A &=& \exists y \exists x A \\ \forall x \mathsf{t} &=& \mathsf{t} = \exists x \mathsf{t} & \forall x \mathsf{f} &=& \mathsf{f} = \exists x \mathsf{f} \end{array}}$$

Where z *does not occur in* A *for the top two equalities.*

Remark 4. The $\exists\mathsf{w}\downarrow$ rule is derivable for $\mathsf{KSh2}\backslash\{\exists\mathsf{w}\downarrow\}$, but we explicitly include it so that we can restrict weakening instances in certain parts of proofs.

Definition 12. *We say that a proof in* $\mathsf{KSh2}$ *is* regular *if there are no* α*-substitutions in the proof, and no variable is used in two different quantifiers.*

Definition 13. *If* ϕ *is a closed* $\mathsf{KSh2}$ *proof in the following form, where* $\forall \boldsymbol{x}$ *is a list of universal quantifiers with distinct variables, and* $Lo(\phi)$ *is regular and in sequential form, we say* ϕ *is in* Herbrand Normal Form *(HNF):*

$$\begin{array}{c} Up(\phi) \,\prod\, \mathsf{KS} \\ \forall \boldsymbol{x} H_\phi(A) \\ \| \{\exists\mathsf{w}\downarrow\} \\ \forall \boldsymbol{x} H_\phi^+(A) \\ Lo(\phi) \,\|\, \{\mathsf{r1}\downarrow, \mathsf{r2}\downarrow, \mathsf{h}\downarrow\} \\ A \end{array}$$

$H_\phi(A)$*, the* Herbrand disjunction of A according to ϕ*, or just the* Herbrand disjunction of A*, contains no quantifiers, whereas* $H_\phi^+(A)$*, the* expansive Herbrand disjunction of A according to ϕ*, may contain quantifiers.* $Up(\phi)$ *is called the* upper part *of* ϕ*, and* $Lo(\phi)$ *the* lower part *of* ϕ*.*

For an example of a proof in HNF, see (Fig. 3).

$$
\mathsf{r1}\!\downarrow\frac{=\dfrac{=\dfrac{\mathsf{t}}{\forall y_1\forall y_2\left[\mathsf{ai}\!\downarrow\dfrac{\mathsf{t}}{Py_1\vee\bar{P}y_1}\vee\left[\mathsf{aw}\!\downarrow\dfrac{\mathsf{f}}{\bar{P}c}\vee\mathsf{aw}\!\downarrow\dfrac{\mathsf{f}}{Py_2}\right]\right]}}{\forall y_1\left[\mathsf{r1}\!\downarrow\dfrac{\forall y_2\left[\left[\exists\mathsf{w}\!\downarrow\dfrac{\mathsf{f}}{\exists x\forall y[\bar{P}x\vee Py]}\vee[\bar{P}y_1\vee Py_2]\right]\vee[\bar{P}c\vee Py_1]\right]}{\mathsf{h}\!\downarrow\dfrac{\mathsf{r1}\!\downarrow\dfrac{\forall y_2[\exists x\forall y[\bar{P}x\vee Py]\vee[\bar{P}y_1\vee Py_2]]}{\exists x\forall y[\bar{P}x\vee Py]\vee\forall y_2[\bar{P}y_1\vee Py_2]}}{\exists x\forall y[\bar{P}x\vee Py]}\vee[\bar{P}c\vee Py_1]}\right]}}{\mathsf{h}\!\downarrow\dfrac{\exists x\forall y[\bar{P}x\vee Py]\vee\forall y_1[\bar{P}c\vee Py_1]}{\exists x\forall y[\bar{P}x\vee Py]}}
$$

Fig. 3. A proof of the drinking principle in HNF

Proposition 4. *A formula A has a proof in HNF iff. It has a Herbrand proof.*

Proof. Let ϕ be a proof of A in HNF. As $H_\phi(A)$ is the Herbrand expansion of A, it is straightforward to construct a Herbrand proof for A: one can infer the necessary $\mathsf{n}\!\downarrow$ and $\mathsf{qc}\!\downarrow$ rules by comparing $H_\phi(A)$ and A. Now let ϕ be a Herbrand Proof. The order of the quantifiers in $Q\{\ \}$ (as in Definition 9) is used to build the HNF proof. Thus, we proceed by induction on the number of quantifiers in $Q\{\ \}$. If there are none, it is obviously trivial. We split the inductive step into two cases.

First, consider ϕ_1 of the form shown, where P is a quantifier-free context and $Q\{\ \}=\forall zQ'\{\ \}$. Clearly ϕ_2 is also a Herbrand proof, so by the IH the proof ϕ_3 in HNF is constructible, from which we can construct ϕ_4.

$$
\begin{array}{cccc}
\begin{array}{c}
\Vert\,\mathsf{KS}\\
\forall z\forall\boldsymbol{x}B\{\boldsymbol{y}\Leftarrow\boldsymbol{t}\}\\
\Vert\,\{\mathsf{n}\!\downarrow\}\\
\forall zQ'\{B\}\\
\Vert\,\{\mathsf{r1}\!\downarrow,\mathsf{r2}\!\downarrow,\mathsf{r3}\!\downarrow,\mathsf{r4}\!\downarrow\}\\
P\{\forall zC'\}\\
\Vert\,\{\mathsf{qc}\!\downarrow\}\\
P\{\forall zC\}
\end{array}
&
\begin{array}{c}
\Vert\,\mathsf{KS}\\
\forall\boldsymbol{x}B\{\boldsymbol{y}\Leftarrow\boldsymbol{t}\}\\
\Vert\,\{\mathsf{n}\!\downarrow\}\\
Q'\{B\}\\
\Vert\,\{\mathsf{r1}\!\downarrow,\mathsf{r2}\!\downarrow,\mathsf{r3}\!\downarrow,\mathsf{r4}\!\downarrow\}\\
P\{C'\}\\
\Vert\,\{\mathsf{qc}\!\downarrow\}\\
P\{C\}
\end{array}
&
\begin{array}{c}
Up\phi_3\,\Vert\,\mathsf{KS}\\
\forall\boldsymbol{x}H_{\phi_3}P\{C\}\\
\Vert\,\{\exists\mathsf{w}\!\downarrow\}\\
\forall\boldsymbol{x}H^+_{\phi_3}P\{C\}\\
Lo(\phi_3)\,\Vert\,\{\mathsf{r1}\!\downarrow,\mathsf{r2}\!\downarrow,\mathsf{h}\!\downarrow\}\\
P\{C\}
\end{array}
&
\begin{array}{c}
\forall z\,Up\phi_3\,\Vert\,\mathsf{KS}\\
\forall z\forall\boldsymbol{x}H_{\phi_3}P\{C\}\\
\Vert\,\{\exists\mathsf{w}\!\downarrow\}\\
\forall z\forall\boldsymbol{x}H^+_{\phi_3}P\{C\}\\
\forall z Lo\phi_3\,\Vert\,\{\mathsf{r1}\!\downarrow,\mathsf{r2}\!\downarrow,\mathsf{h}\!\downarrow\}\\
\forall zP\{C\}\\
\Vert\,\{\mathsf{r1}\!\downarrow,\mathsf{r2}\!\downarrow\}\\
P\{\forall zC\}
\end{array}
\\
\phi_1 & \phi_2 & \phi_3 & \phi_4
\end{array}
$$

In the same way, we consider the case where $Q\{\ \}=\exists zQ'\{\ \}$. Below we only show the case where there is no contraction acting on $\exists zC$, but the case with such a contraction is similar.

$$
\begin{array}{c}
\Vert \mathsf{KS} \\
\forall \boldsymbol{x} B\{\boldsymbol{y} \Leftarrow \boldsymbol{t}\}\{z \Leftarrow t\} \\
\Vert \{\mathsf{n}\downarrow\} \\
\exists z Q'\{B\} \\
\Vert \{\mathsf{r1}\downarrow,\mathsf{r2}\downarrow,\mathsf{r3}\downarrow,\mathsf{r4}\downarrow\} \\
P\{\exists z C'\} \\
\Vert \{\mathsf{qc}\downarrow\} \\
P\{\exists z C\} \\
\\
\phi_1
\end{array}
\qquad
\begin{array}{c}
\Vert \mathsf{KS} \\
\forall \boldsymbol{x} B\{\boldsymbol{y} \Leftarrow \boldsymbol{t}\}\{z \Leftarrow t\} \\
\Vert \{\mathsf{n}\downarrow\} \\
Q'\{B\}\{z \Leftarrow t\} \\
\Vert \{\mathsf{r1}\downarrow,\mathsf{r2}\downarrow,\mathsf{r3}\downarrow,\mathsf{r4}\downarrow\} \\
P\{C'\{z \Leftarrow t\}\} \\
\Vert \{\mathsf{qc}\downarrow\} \\
P\{C\{z \Leftarrow t\}\} \\
\\
\phi_2
\end{array}
\qquad
\begin{array}{c}
Up(\phi_3) \Vert \mathsf{KS} \\
\forall \boldsymbol{x} P\{D\{z \Leftarrow t\}\} \\
\Vert \{\exists\mathsf{w}\downarrow\} \\
\forall \boldsymbol{x} P\{D^+\{z \Leftarrow t\}\} \\
Lo(\phi_3) \Vert \{\mathsf{r1}\downarrow,\mathsf{r2}\downarrow,\mathsf{h}\downarrow\} \\
P\{C\{z \Leftarrow t\}\} \\
\\
\phi_3
\end{array}
\qquad
\begin{array}{c}
Up(\phi_3) \Vert \mathsf{KS} \\
\forall \boldsymbol{x} P\{D\{z \Leftarrow t\}\} \\
\Vert \{\exists\mathsf{w}\downarrow\} \\
\forall \boldsymbol{x} P\{\exists z C \vee D^+\{z \Leftarrow t\}\} \\
Lo(\phi_3) \Vert \{\mathsf{r1}\downarrow,\mathsf{r2}\downarrow,\mathsf{h}\downarrow\} \\
P\left\{ \mathsf{h}\downarrow \dfrac{\exists z C \vee C\{z \Leftarrow t\}}{\exists z C} \right\} \\
\\
\phi_4
\end{array}
$$

where
$P\{D\{z \Leftarrow t\}\} = H_{\phi_3}(P\{C\{z \Leftarrow t\}\})$ and $P\{D^+\{z \Leftarrow t\}\} = H^+_{\phi_3}(P\{C\{z \Leftarrow t\}\})$.

4 Translations Between KSh2 and Expansion Proofs

Above, we gave translations between Herbrand proofs in KSh1 and KSh2 proofs in HNF. We will now give a translations between KSh2 proofs in HNF and expansion proofs, thus giving us a link between deep inference Herbrand proofs and expansion proofs.

Remark 5. We extend the notion and syntax of contexts from derivations to expansion trees. For the notion to make sense, a context can only take expansion trees with the same shallow formula.

4.1 KSh2 to Expansion Proofs

Before stating and proving the main theorem, we will define the map π_1 from KS proofs to expansion proofs, and then prove some lemmas to help prove that the dependency relation in all expansion proofs in the range of π_1 is acyclic.

Definition 14. *We define a map π_1' from the lower part of* KSh2 *proofs in HNF to expansion trees in the following way, working from the bottom*

On the conclusion of ϕ, we define π_1' as follows:

- $\pi_1'(B \star C) = \pi_1'(B) \star \pi_1'(C)$
- $\pi_1'(\forall x B) = \forall x B +^x \pi_1'(B)$
- $\pi_1'(\exists x B) = \exists x B$

The r1↓ *and* r2↓ *rules are ignored by expansion trees and each* h↓ *rule adds a branch to a $\exists$-node:*

- *If* $\phi = \begin{array}{c} K\left\{ \mathsf{r1}\downarrow \dfrac{\forall x[B \vee C]}{\forall x B \vee C} \right\} \\ \phi' \Vert \\ A \end{array}$ *then* $\pi_1'(\phi) = \pi_1' \left(\begin{array}{c} K\{\forall x B \vee C\} \\ \phi' \Vert \\ A \end{array} \right)$.
- *If* $\phi = \begin{array}{c} K\left\{ \mathsf{r2}\downarrow \dfrac{\forall x(B \wedge C)}{(\forall x B \wedge C)} \right\} \\ \phi' \Vert \\ A \end{array}$ *then* $\pi_1'(\phi) = \pi_1' \left(\begin{array}{c} K\{\forall x B \wedge C\} \\ \phi' \Vert \\ A \end{array} \right)$.

– *If* $\pi'_1\left(\begin{array}{c} K\{\exists xB\} \\ \phi \| \\ A \end{array}\right) = K_{\pi_1}(\exists xB +^{\tau_1} E_1 + \cdots +^{\tau_n} E_n)$*, then:*

$$\pi'_1\left(\begin{array}{c} K\left\{\mathsf{h}\!\downarrow \dfrac{\exists xB \vee B\{x \Leftarrow \tau_{n+1}\}}{\exists xB}\right\} \\ \phi \| \\ A \end{array}\right) = K_{\pi_1}(\exists xB +^{\tau_1} E_1 + \cdots +^{\tau_{n+1}} E_{n+1})$$

where $E_{n+1} = \pi'_1(B\{x \Leftarrow \tau_{n+1}\})$*.*

We then define the map π_1 *from* KSh2 *proofs in HNF to expansion trees as* $\pi_1(\phi) = \pi'_1(Lo(\phi))$*.*

To show that $\pi_1(\phi)$ is an expansion proof, we need to prove that $\forall \boldsymbol{x} H_\phi(A)$ is a tautology and $<_E$ is acyclic. As $\forall \boldsymbol{x} H_\phi(A)$ has a proof in KS it is a tautology. Thus all that is needed is the acyclicity of $<_E$. To do so, we define the following partial order on variables in the lower part of KSh2 proofs in HNF.

Definition 15. *Let* ϕ *be a proof in HNF. Define the partial order* $<_\phi$ *on the variables of occurring in* $Lo(\phi)$ *to be the minimal partial order such that* $y <_\phi x$ *if* $K_1\{Q_1 x K_2\{Q_2 y B\}\}$ *is a section of* $Lo(\phi)$*.*

Proposition 5. $<_\phi$ *is well-defined for all* KSh2 *proofs in HNF.*

Proof. Let ϕ be a proof of A in HNF, as in Definition 13. As $Lo(\phi)$ only contains $\mathsf{h}\!\downarrow$, $\mathsf{r1}\!\downarrow$ and $\mathsf{r2}\!\downarrow$ rules and no α-substitution, if a variable v occurs in $Lo(\phi)$ then v occurs in $\forall x H^+_\phi(A)$. Notice also that none of $\mathsf{h}\!\downarrow$, $\mathsf{r1}\!\downarrow$ and $\mathsf{r2}\!\downarrow$ can play the role of ρ in the following scheme:

$$\rho\,\frac{K\{Q_1 v_1 A_1\}\{Q_2 v_2 A_2\}}{K'\{Q_1 v_1\{K'' Q_2 v_2 B\}\}}\,.$$

Therefore, we observe that if $K_1\{Q_1 x K_2\{Q_2 y B\}\}$ is a section of $Lo(\phi)$, then $\forall x H^+_\phi(A)$ is of the form $L_1\{Q_1 x L_2\{Q_2 y C\}\}$, i.e. no dependencies can be introduced below $\forall x H^+_\phi(A)$. Thus $x <_\phi y$ iff. $\forall x H^+_\phi(A)$ can be written $L_1\{Q_1 x L_2\{Q_2 y C\}\}$ for some $L_1\{\ \}, L_2\{\ \}$ and C and is therefore a well-defined partial order.

Lemma 1. *Let* ϕ *be an* KSh2 *proof in HNF and* e' *an* $\forall$*-edge in* $\pi_1(\phi)$ *that points to the* $\exists$*-edge* e*. If* $Lab(e') = y$ *and the* $\exists$*-node below* e *is* $\exists xA$*, then* $x <_\phi y$*.*

Proof. Since we have an $\exists$-node $\exists xA$ in $\pi_1(\phi)$ with an edge labelled t below it, there must be the following $\mathsf{h}\!\downarrow$ rule in ϕ:

$$K\left\{\mathsf{h}\!\downarrow \frac{\exists xA \vee A\{x \Leftarrow t\}}{\exists xA}\right\}$$

Since e points to e', y must occur freely in t. As ϕ is closed, y cannot be a free variable in $K\{\exists xA \vee A\{x \Leftarrow t\}\}$. Thus $K\{\ \}$ must be of the form $K_1\{\forall y K_2\{\ \}\}$. Therefore $x <_\phi y$.

Lemma 2. *Let ϕ be an* KSh2 *proof in HNF, e a $\forall$-edge of $\pi_1(\phi)$ labelled by x and e' an $\exists$-edge above an $\exists$-node $\exists yA$. If e is a descendant of e' then $x <_\phi y$.*

Proof. $Sh(\pi_1(\phi)) = K_1\{\exists y K_2 \forall x\{B\}\}$ (for some $K_1\{\ \}, K_2\{\ \}$, and B) is the conclusion of ϕ, so $x <_\phi y$.

Lemma 3. *Let ϕ be an* KSh2 *proof in HNF, $E_\phi = \pi_1(\phi)$ and e and e' be edges in E_ϕ s.t. $e <_{E_\phi} e'$, $Lab(e) = x$ and $Lab(e') = x'$. Then $x <_\phi x'$.*

Proof. As $e <_{E_\phi} e'$, there must be a chain

$$e_{q_0} <^-_{E_\phi} \cdots <^-_{E_\phi} e_{p_1} <^-_{E_\phi} e_{q_1} <^-_{E_\phi} \cdots <^-_{E_\phi} e_{p_m} <^-_{E_\phi} e_{q_m} <^-_{E_\phi} \cdots <^-_{E_\phi} e_{p_n}$$

where $e_{q_0} = e$ and $e_{p_n} = e'$, e_{q_i} points to e_{p_i}, and e_{q_i} is a descendant of $e_{p_{i+1}}$ in the expansion tree. By Lemma 1, we know that if $\exists x_{p_i}$ is the node above p_i and $Lab(e_{q_i}) = x_{q_i}$, then $x_{p_i} <_\phi x_{q_i}$. By Lemma 2, since e_{q_i} is a descendant of $e_{p_{i+1}}$ in the expansion tree, $x_{q_i} <_\phi x_{p_{i+1}}$. Therefore $x <_\phi x'$.

Theorem 3. *If ϕ is an* KSh2 *proof of A in HNF, then we can construct an expansion proof $E_\phi = \pi_1(\phi)$, with $Sh(E_\phi) = A$, and $Dp(E_\phi) = H_\phi(A)$.*

Proof. As described above, we only need to show that the dependency relation of E_ϕ is acyclic. Assume there were a cycle in $<_{E_\phi}$. Clearly, it could not be generated by just by travelling up the expansion tree. Thus, there is some e and e' such that e points to e' and $e <_{E_\phi} e' <_{E_\phi} e$. But then, if $Lab(e) = x$, by Lemma 3, $x <_\phi x$. But this contradicts Proposition 5. Therefore $<_{E_\phi}$ is acyclic.

Expansion Proofs to KSh2: For the translation from expansion proofs to KSh2 proofs in HNF, we show that we can always construct a total order on the edges in an expansion proof that guides the construction of the lower part of a proof in HNF. Unlike the previous translation, there is not necessarily a unique proof corresponding to each expansion proof, but the choice of a total order determines the proof that will be created.

Definition 16. *A* weak *expansion tree is defined in the same way as in Definition 1 except that the first condition is weakened to allow any formula to be a leaf of the tree. A weak expansion tree with an acyclic dependency relation is correct regardless of whether its deep formula is a tautology.*

Definition 17. *We define the* expansive deep formula *$Dp^+(E)$ for (weak) expansion trees, which is defined in the same way as the usual deep formula except that:*

$$Dp^+(\exists xA +^{t_1} E_1 +^{t_2} \cdots +^{t_n} E_n) := \exists xA \vee Dp^+(E_1) \vee \ldots \vee Dp^+(E_n)$$

Definition 18. *A* minimal edge *of a (weak) expansion tree E is an edge that is minimal w.r.t. to $<_E$.*

Definition 19. *Let E be a (weak) expansion proof. Let $<^+_E$ be a total order extending $<_E$ such that the following condition holds: if $\star$ is a node with edges e and e' below it, then e and e' are consecutive elements in the total order. We say $<^+_E$ is a* sequentialisation *of E.*

Lemma 4. *Every (weak) expansion proof has a sequentialisation, often many.*

Proof. We proceed by induction on the number of nodes in a weak expansion proof. The base case is trivial. For the inductive step, we will show that every weak expansion tree has either a minimal edge e below an existential or universal node, or that there are two minimal edges e_1 and e_2 below a $\star$-node. As the rest of the weak expansion proof has a sequentialisation by the inductive hypothesis, we can extend it with the minimal element e or the two minimal elements e_1 and e_2 for a sequentialisation for the full weak expansion proof.

Assume E is a weak expansion proof with no minimal edges below existential or universal nodes. As $<_E$ is a partial order, there must be at least one minimal edge e_0, and by the assumption it must be below a node $\star_0$. Let e'_0 be the other edge below $\star_0$. If e'_0 is minimal, we are done. If not, pick some minimal edge $e_1 < e'_0$, which again, with $e'_1 < e'_0$, must be below some $\star_1$. For each e'_i that is not minimal, we can find $e'_{i+1} < e'_i$. As E is finite, this sequence cannot continue indefinitely, so eventually we will find two minimal edges e_n and e'_n below $\star_n$. Note that e_n and e'_n need not be unique and thus the sequentialisation is not unique.

Proposition 6. *Let $E = K_E\{\forall xA +^x A\}$, with $Dp^+(E) = K\{A\}$, be a correct weak expansion tree with the $\forall$-edge labelled by x (which we will call e) minimal w.r.t. $<_E$. Then there is a derivation* $\begin{array}{c}\forall xK\{A\}\\ \Vert\,\{\mathsf{r1}{\downarrow},\mathsf{r2}{\downarrow}\}\\ K\{\forall xA\}\end{array}$.

Proof. We proceed by induction on the height of the node $\forall xA$ in E. If $\forall xA$ is the bottom node, then $K\{A\} = A$ and we are done. Let E be an expansion tree where $\forall x$ is not the bottom node. There are three possible cases to consider. In each case, $E_1 = K_{E_1}\{\forall xA +^x A\}$ is an expansion tree with $Dp^+(E_1) = K_1\{A\}$ and, by the inductive hypothesis, we have a derivation $\begin{array}{c}\forall xK_1\{A\}\\ \Vert\,\{\mathsf{r1}{\downarrow},\mathsf{r2}{\downarrow}\}\\ K_1\{\forall xA\}\end{array}$.

1. $E = (E_1 \star E_2)$, with $Dp^+(E) = [K_1\{A\} \star Dp^+(E_2)]$. As e is minimal, it cannot point to any edge in E_2. Therefore $B := Dp^+(E_2)$ is free for x. Therefore we can construct the derivations:

$$\mathsf{r1}{\downarrow}\frac{\forall x[K_1\{A\} \vee B]}{\begin{array}{c}\forall xK_1\{A\}\\ \Vert\,\{\mathsf{r1}{\downarrow},\mathsf{r2}{\downarrow}\} \vee B\\ K_1\{\forall xA\}\end{array}} \quad \textit{and} \quad \mathsf{r2}{\downarrow}\frac{\forall x(K_1\{A\} \wedge B)}{\begin{array}{c}\forall xK_1\{A\}\\ \Vert\,\{\mathsf{r1}{\downarrow},\mathsf{r2}{\downarrow}\} \wedge B\\ K_1\{\forall xA\}\end{array}}$$

2. $E = \forall y(Sh(E_1)) +^y E_1$. As $Dp^+(E) = Dp^+(E_1)$, we are already done.

3. $E = \exists y K_0\{A_0\} +^{t_1} E_1 \cdots +^{t_n} E_n$, with $Dp^+(E_i) = B_i := [K_0\{A_0\}]\{y \Leftarrow t_i\}$ and in particular $B_1 = K_1\{A\}$. Thus $Dp^+(E) = \exists y B_0 \vee K_1\{A\} \vee B_2 \vee \ldots \vee B_n$. Again, e cannot point to any edge in any of the E'_i, so we can construct:

$$\mathsf{r1}\!\downarrow \frac{\forall x[\exists y B_0 \vee K_1\{A\} \vee B_2 \vee \ldots \vee B_n]}{\mathsf{r1}\!\downarrow \dfrac{\forall x[\exists y B_0 \vee K_1\{A\}]}{\left[\exists y B_0 \vee \begin{array}{c} \forall x K_1\{A\} \\ \| \{\mathsf{r1}\!\downarrow,\mathsf{r2}\!\downarrow\} \\ K_1\{\forall x A\} \end{array}\right]} \vee [B_2 \vee \ldots \vee B_n]}$$

Definition 20. *We define the map π_2^{Lo} that takes an expansion tree E and a sequentialisation $<_E^+$ to a derivation:*

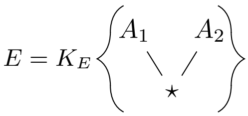
$$\pi_2^{Lo}(E, <_E^+) = \begin{array}{c} \forall \boldsymbol{x} Dp^+(E) \\ \| \{\mathsf{h}\!\downarrow,\mathsf{r1}\!\downarrow,\mathsf{r2}\!\downarrow\} \\ Sh(E) \end{array}$$

In each case $<_{E'}^+$ is $<_E^+$ restricted to E'.

- *If E is just a leaf A, $\pi_2^{Lo}(E, <_E^+) = A$.*
- *If $E = K_E\{A_1 \star_E A_2\}$ is e_1, and the minimal edge w.r.t $<_E^+$ is between $\star_E$ and A_1, then by Definition 19 the next-but-minimal edge is between $\star_E$ and A_2. Then, $E' = K_E\{A_1 \star_F A_2\}$ is a correct weak expansion tree and we can define:*

$$\pi_2^{Lo}(E, <_E^+) = \pi_2^{Lo}(E', <_{E'}^+)$$

Pictorially:



$$E' = K_E\{A_1 \star A_2\}$$

- *If $E = K_E\{\forall x A +^x A\}$ and the minimal edge w.r.t. $<_E^+$ is between $\forall x A$ and A, then, by Proposition 6, $E' = K_E\{\forall x A\}$ is a correct weak expansion tree and we can define:*

$$\pi_2^{Lo}(E, <_E^+) = \; = \frac{\begin{array}{c} \forall x Dp^+(E) \\ \| \{\mathsf{r1}\!\downarrow,\mathsf{r2}\!\downarrow\} \\ Dp^+(E') \end{array}}{\pi_2^{Lo}(E', <_{E'}^+)}$$

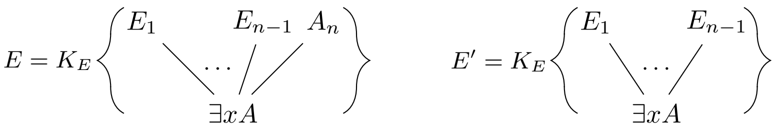
Pictorially:

$$E = K_E\left\{\begin{array}{c} A \\ | \\ \forall x A \end{array}\right\} \qquad\qquad E' = K_E\{\forall x A\}$$

– *If the minimal edge of* $E = K_E\{\exists x A +^{t_1} E_1 \cdots +^{t_n} A_n\}$, *with* $Dp^+(E) = K\{\exists x A \vee A_1 \vee \ldots \vee A_n\}$, *is between* $\exists x A$ *and* A_n, *then* $E' = K_E\{\exists x A +^{t_1} E_1 \cdots +^{t_{n-1}} E_{n-1}\}$ *is a correct weak expansion tree with* $Dp^+(E') = K\{A_1 \vee \ldots \vee A_{n-1}\}$ *and we can define:*

$$\pi_2^{Lo}(E, <_E^+) = \; = \frac{K\left\{ = \dfrac{\exists x A \vee A_1 \vee \ldots \vee A_n}{\mathsf{h}{\downarrow}\,\dfrac{\exists x A \vee A_n}{\exists x A} \vee A_1 \vee \ldots \vee A_{n-1}} \right\}}{\pi_2^{Lo}(E', <_{E'}^+)}$$

Pictorially:



Theorem 4. *If E is an expansion proof with $Sh(E) = A$, then we can construct an* KSh2 *proof ϕ of A in HNF, where $H_\phi(A) = Dp(E)$.*

Proof. As $Dp(E)$ is a tautology, there is a proof $\begin{array}{c} \pi_2^{Up}(E) \,\Vert\, \mathsf{KS} \\ \forall \boldsymbol{x} Dp(E) \end{array}$ and clearly there is a proof $\begin{array}{c} Dp(E) \\ \Vert\, \{\exists \mathsf{w}{\downarrow}\} \\ Dp^+(E) \end{array}$. Thus, choosing an arbitrary sequentialisation $<_E^+$ of E, we can define π_2 from expansion proofs to KSh2 proofs in HNF as:

$$\pi_2(E) = \begin{array}{c} \pi_2^{Up}(E) \,\Vert\, \mathsf{KS} \\ \forall \boldsymbol{x} Dp(E) \\ \Vert\, \{\exists \mathsf{w}{\downarrow}\} \\ \forall \boldsymbol{x} Dp^+(E) \\ \pi_2^{Lo}(E, <_E^+) \,\Vert\, \{\mathsf{r1}{\downarrow}, \mathsf{r2}{\downarrow}, \mathsf{h}{\downarrow}\} \\ Sh(E) \end{array}$$

Remark 6. For all expansion proofs E we have $\pi_2^{Up}(E) = Up(\pi_2(E))$ and $\pi_2^{Lo}(E) = Lo(\pi_2(E))$.

5 Further Work

The translations between deep inference proofs and expansion proofs should be seen as a springboard for further investigations. One obvious next step is to extend KSh2 with cut, and prove cut elimination, so that completeness does not depend on the translation into KSh1 and Brünnler's result. Having done so, we can then make a proper comparison with the cut elimination procedures for expansion proofs described in [1,12,15]. Additionally, it would be interesting to try and situate this work in the context of recent work by Aler Tubella and Guglielmi [2,3], in which they provide a general theory of normalisation for various different propositional logics. In their terminology, a Herbrand proof is close to the notion of a *decomposed* proof, which has two phases: the first contraction-free and the second consisting only of contractions. Extending the procedure, described in [4], to remove identity-cut cycles from SKS proofs to first-order systems is likely to be an important aspect of this research.

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