Decomposing First-Order Proofs using Deep Inference

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The deep-inference formalism, by allowing for very fine-grained inference steps and freer composition of proofs, has produced important results and innovations in various logics, especially classical propositional logic. A natural progression is to extend these insights to classical first-order logic (FOL) but, although a direct cut-elimination procedure has been provided [2], there has been no work as of yet that incorporates the many perspectives and techniques developed in the last ten years.

In the talk, I will give the outline of a new cut elimination procedure for FOL in deep inference, as well as a decomposition-style presentation of Herbrand's Theorem called a *Herbrand Stratification* that is proved not as a corollary of cut elimination, but in tandem with it. In doing so, I hope to provide a different and perhaps better perspective on FOL normalisation, Herbrand's Theorem, and their relationship. More concretely, there is good reason to believe that, as in propositional logic [1], this research can provide us with new results in proof complexity.

Deep Inference Deep inference differs from the sequent calculus in that composition of proofs is allowed with the same connectives that are used for the composition of formulae [5]. Thus in classical propositional logic, two proofs $\begin{array}{c} A \\ \phi \Vert \\ B \end{array}$ and $\begin{array}{c} C \\ \psi \Vert \\ D \end{array}$ can be composed not only with conjunction, as is possible in the sequent calculus, but also with disjunction:

$$\begin{array}{c} A \\ \phi \Vert \\ B \end{array} \wedge \begin{array}{c} C \\ \psi \Vert \\ D \end{array} = \begin{array}{c} A \wedge C \\ \phi \wedge \psi \Vert \\ B \wedge D \end{array} , \qquad \begin{array}{c} A \\ \phi \Vert \\ B \end{array} \vee \begin{array}{c} C \\ \psi \Vert \\ D \end{array} = \begin{array}{c} A \vee C \\ \phi \vee \psi \Vert \\ B \vee D \end{array}$$

This freedom of composition has enabled many proof-theoretic innovations: the reduction of cut to atomic form by a local procedure of polynomial-time complexity [3], and the development of a quasi-polynomial cut elimination procedure for propositional logic using a geometric invariant of proofs known as the *atomic flow* [5]. In FOL, we also allow quantifiers to be applied to proofs, not only formulae:

$$\exists x \left[\begin{array}{c} A \\ \phi \Vert \\ B \end{array} \right] = \begin{array}{c} \exists x A \\ \exists x \phi \Vert \\ \exists x B \end{array} , \qquad \forall x \left[\begin{array}{c} A \\ \phi \Vert \\ B \end{array} \right] = \begin{array}{c} \forall x A \\ \forall x \phi \Vert \\ \forall x B \end{array}$$

Normalisation in Deep Inference. Recently, study of normalisation in deep-inference proof systems has led to the perspective that the process is a conflation of two mechanisms that operate on two distinct composition methods: contraction and a linear cut. When normalised, the first of these mechanisms increases complexity, whereas the second reduces it. Thus, two-stage cut elimination procedures for proof systems are being developed: those which first *decompose* a proof into a suitable form before linear cut elimination then is performed.

Herbrand's Theorem as Decomposition I will show how for FOL, a certain presentation of Herbrand's theorem I call a *Herbrand Stratification* effectively carries out the decomposition phase of normalisation. This inverts the more common idea of using cut elimination to prove Herbrand's Theorem [4], and fits with the complexity narrative: proving Herbrand's Theorem constructively requires increasing the size of a proof, possibly greatly [6].

The proof of Herbrand's Theorem that I will present, which draws inspiration from normalisation techniques developed for propositional logic that use the atomic flow, proceeds by combining existential instantiation and contraction into a single rule, called a *Herbrand Expander*:

$$\mathsf{cont}\,\frac{\exists x A \vee \mathsf{n}\!\downarrow \dfrac{A[a_1/x]}{\exists x A} \vee \ldots \vee \mathsf{n}\!\downarrow \dfrac{A[a_n/x]}{\exists x A}}{\exists x A} \longrightarrow \mathsf{h}\!\downarrow \frac{\exists x A \vee A[a_i/x]_1^n}{\exists x A}$$

and pushing these rules to the bottom of the proof. The result, what I call a *Herbrand Stratification* of a proof, is a version of Herbrand's Theorem for deep inference and allows for propositional linear cut elimination methods to be used to complete normalisation.

$$\text{Herbrand Stratification}: \begin{array}{c} \phi \Vert \\ A \end{array} \longrightarrow \begin{array}{l} \Vert \text{ Propositional Rules} \\ H(A) \\ \Vert \text{ Herbrand Expanders} \\ A \end{array}$$

References

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