Natural and Confluent Cut Elimination in Classical Logic

Proof, Computation, Complexity 2015

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May 23, 2015

Cut Elimination: Intuitionistic vs. Classical

- Intuitionistic logic - well-behaved cut elimination procedure gives us nice proof theoretic properties: confluence, orderly denotational semantics, Curry-Howard etc.

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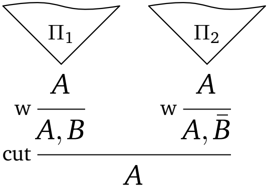
- Classical logic - considered an unwieldy and *ad hoc* procedure, due to critical pairs such as the following (the "Lafont Counterexample"[1]):



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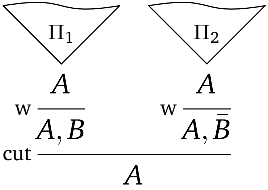

$$\text{cut}\,\dfrac{\text{w}\,\dfrac{\Pi_1 \atop A}{A, B} \qquad \text{w}\,\dfrac{\Pi_2 \atop A}{A, \bar{B}}}{A}$$

- Wouldn’t it be useful if we had more means by which to compose proofs?

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Open Deduction

Open deduction is a deep inference proof system that gives us more ways of composing derivations.

Given derivations $\begin{array}{c} A \\ \Phi \| \\ B \end{array}$ and $\begin{array}{c} C \\ \Psi \| \\ D \end{array}$ we can compose them with:

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3. (In development is Formalism B, which adds the further composition method of substitution)

SKS

- The *structural* rules (all atomic):

$$\mathsf{ai}{\downarrow}\,\frac{\mathsf{t}}{a \vee \bar{a}} \qquad \mathsf{ac}{\downarrow}\,\frac{a \vee a}{a} \qquad \mathsf{aw}{\downarrow}\,\frac{\mathsf{f}}{a}$$

identity *contraction* *weakening*

$$\mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\mathsf{f}} \qquad \mathsf{ac}{\uparrow}\,\frac{a}{a \wedge a} \qquad \mathsf{aw}{\uparrow}\,\frac{a}{\mathsf{t}}$$

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- The *logical* rules (all linear):

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$$A \wedge \mathsf{t} = A \qquad A \vee \mathsf{f} = A$$

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SKS proof example

$$
\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\dfrac{\boxed{\dfrac{\boxed{\mathsf{ac}{\downarrow}\,\dfrac{\mathsf{t}}{\bar a \vee a}} \vee \boxed{\mathsf{aw}{\downarrow}\,\dfrac{\mathsf{f}}{a}}}{\bar a \vee \boxed{\mathsf{ac}{\downarrow}\,\dfrac{a \vee a}{a}}}} \wedge \boxed{\mathsf{ac}{\downarrow}\,\dfrac{\mathsf{t}}{\boxed{\mathsf{ac}{\uparrow}\,\dfrac{\bar a}{\bar a \wedge \boxed{\mathsf{aw}{\uparrow}\,\dfrac{\bar a}{\mathsf{t}}}}} \vee a}}}{[\bar a \vee a] \wedge [a \vee \bar a]}}{([\bar a \vee a] \wedge a) \vee \bar a}}{\bar a \vee \boxed{\mathsf{ai}{\uparrow}\,\dfrac{a \wedge \bar a}{\mathsf{f}}} \vee a}
$$

SKS proof example - no boxes

$$
\text{s}\,\dfrac{\text{s}\,\dfrac{\dfrac{\dfrac{\text{ac}{\downarrow}\,\dfrac{\mathsf{t}}{\bar{a} \vee a} \vee \text{aw}{\downarrow}\,\dfrac{\mathsf{f}}{a}}{\bar{a} \vee \text{ac}{\downarrow}\,\dfrac{a \vee a}{a}} \wedge \text{ac}{\downarrow}\,\dfrac{\mathsf{t}}{\text{ac}{\uparrow}\,\dfrac{\bar{a}}{\bar{a} \wedge \text{aw}{\uparrow}\,\dfrac{\bar{a}}{\mathsf{t}}} \vee a}}{[\bar{a} \vee a] \wedge [a \vee \bar{a}]}}{([\bar{a} \vee a] \wedge a) \vee \bar{a}}}{\bar{a} \vee \text{ai}{\uparrow}\,\dfrac{a \wedge \bar{a}}{\mathsf{f}} \vee a}
$$

SKS proof example - less rule labels

$$\dfrac{\dfrac{\dfrac{\mathsf{t}}{\bar{a} \vee a} \vee \dfrac{\mathsf{f}}{a}}{\bar{a} \vee \dfrac{a \vee a}{a}} \wedge \dfrac{\mathsf{t}}{\dfrac{\bar{a}}{\bar{a} \wedge \dfrac{\bar{a}}{\mathsf{t}}} \vee a}}{\mathsf{s}\,\dfrac{[\bar{a} \vee a] \wedge [a \vee \bar{a}]}{\mathsf{s}\,\dfrac{([\bar{a} \vee a] \wedge a) \vee \bar{a}}{\bar{a} \vee {\color{red}\mathsf{ai}\!\uparrow}\dfrac{a \wedge \bar{a}}{\mathsf{f}} \vee a}}}$$

SKS proof example - rule compression

$$\dfrac{\dfrac{\dfrac{\mathsf{t}}{\bar{a} \vee a} \vee \dfrac{\mathsf{f}}{a}}{\bar{a} \vee \dfrac{a \vee a}{a}} \wedge \dfrac{\mathsf{t}}{\dfrac{\bar{a}}{\bar{a} \wedge \dfrac{\bar{a}}{\mathsf{t}}} \vee a}}{\mathsf{s}^2\,\dfrac{[\bar{a} \vee a] \wedge [a \vee \bar{a}]}{\bar{a} \vee \mathsf{ai}{\uparrow}\dfrac{a \wedge \bar{a}}{\mathsf{f}} \vee a}}$$

Atomic Flows

Atomic flows[2] are a geometric invariant of proofs in open deduction. Only structural information about the proof is conserved.

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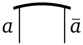
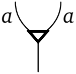


identity

$$\mathsf{ac}{\downarrow}\,\frac{a \vee a}{a}$$

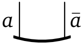


contraction

$$\mathsf{aw}{\downarrow}\,\frac{\mathsf{f}}{a}$$

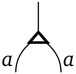
weakening

$$\mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\mathsf{f}}$$



cut

$$\mathsf{ac}{\uparrow}\,\frac{a}{a \wedge a}$$



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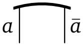

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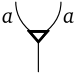

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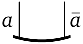

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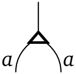

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Composition of proofs naturally corresponds to composition of flows.

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Atomic Flow Example

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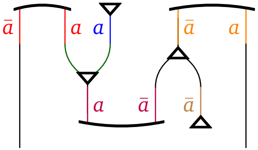

Basic Theorems

Theorem

If A implies B then there is an SKS *derivation* $\begin{array}{c} A \\ \Phi \Vert \\ B \end{array}$.

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By comparison with Gentzen systems (many other ways). □

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Theorem

If A is true, then there is an KS *proof* $\begin{array}{c} \Psi \overline{\Vert} \\ A \end{array}$. *A fortiori, if there is a* SKS *proof of a formula A, then there is a proof without* ai↑ *(a cut-free proof), i.e.* ai↑ *is admissible for* SKS.

Cut Elimination

Proof.

Numerous:

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2. As a quasipolynomial procedure using atomic flows and threshold formulae.[4]
3. We give an exponential, but confluent and natural procedure for $\mathsf{ai}\!\uparrow$ elimination, also using atomic flows.

□

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Is ai↑ a cut?

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- $\mathsf{ai}\uparrow$ admissiblility shows consistency of SKS.

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Experiments Method

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Atomic cut is admissible from SKS

Proof.

(The Experiments method) Given a proof Φ of A:

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1. For each cut, we create two derivations from the proof:

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2. We then use an identity (at the top) and a contraction (at the bottom) to disjunct the two proofs, creating a proof with one fewer cut instance.

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2. We then use an identity (at the top) and a contraction (at the bottom) to disjunct the two proofs, creating a proof with one fewer cut instance.
3. Intuition: in each cut $\mathsf{ai}{\uparrow}\,\dfrac{a \wedge \bar{a}}{\mathsf{f}}$, either a or $\bar{a}$ is ‘true’, and each version corresponds to one of these possibilities.

□

Experiments Method - Many Cuts

When there are multiple cuts in a proof, a “truth-table” approach is used.

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$$\begin{array}{c} \Phi \, ||| \\ A \end{array} \quad \rightarrow \quad \begin{array}{c} a_1 \wedge \ldots \wedge a_n \\ \Phi_1 \, \| \\ A \end{array} \;, \ldots , \; \begin{array}{c} \bar{a}_1 \wedge \ldots \wedge \bar{a}_n \\ \Phi_{2^n} \, \| \\ A \end{array}$$

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$$\begin{array}{c} \Phi \,|\!|\!| \\ A \end{array} \quad \to \quad \begin{array}{c} a_1 \wedge \ldots \wedge a_n \\ \Phi_1 \,\| \\ A \end{array} \quad , \ldots , \quad \begin{array}{c} \bar{a}_1 \wedge \ldots \wedge \bar{a}_n \\ \Phi_{2n} \,\| \\ A \end{array}$$

$$\begin{array}{c} |\!|\!| \, \{\mathsf{s},\mathsf{ai}\uparrow\} \\ \boxed{\begin{array}{c} a_1 \wedge \ldots \wedge a_n \\ \Phi_1 \,\| \\ A \end{array}} \vee \ldots \vee \boxed{\begin{array}{c} \bar{a}_1 \wedge \ldots \wedge \bar{a}_n \\ \Phi_{2n} \,\| \\ A \end{array}} \\ \| \, \{\mathsf{ac}\downarrow,\mathsf{m}\} \\ A \end{array}$$

It should be clear that this procedure is confluent, modulo reasonable geometric equivalences.

Experiments Example

$$\dfrac{\boxed{\dfrac{\boxed{\dfrac{\mathsf{t}}{\bar{a}\vee a}}\vee\boxed{\dfrac{\mathsf{f}}{a}}}{\bar{a}\vee\boxed{\dfrac{a\vee a}{a}}}}\wedge\boxed{\dfrac{\mathsf{t}}{\boxed{\dfrac{\bar{a}}{\bar{a}\wedge\boxed{\dfrac{\bar{a}}{\mathsf{t}}}}}\vee a}}}{\mathsf{s}^2\,\dfrac{[\bar{a}\vee a]\wedge[a\vee\bar{a}]}{\bar{a}\vee\boxed{\mathsf{ai}{\uparrow}\,\dfrac{a\wedge\bar{a}}{\mathsf{f}}}\vee a}}$$

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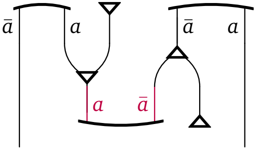
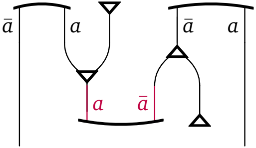
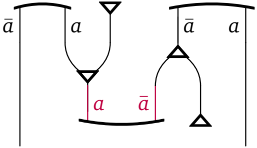


Experiments Example - Duplication

$$\cfrac{\boxed{\cfrac{\boxed{\cfrac{\mathsf{t}}{\bar{a}\vee a}}\vee\boxed{\cfrac{\mathsf{f}}{a}}}{\bar{a}\vee\boxed{\cfrac{a\vee a}{a}}}}\wedge\boxed{\cfrac{\mathsf{t}}{\boxed{\cfrac{\bar{a}}{\bar{a}\wedge\boxed{\cfrac{\bar{a}}{\mathsf{t}}}}}\vee a}}}{\mathsf{s}^2\,\cfrac{[\bar{a}\vee a]\wedge[a\vee\bar{a}]}{\bar{a}\vee\boxed{\mathsf{ai}\!\uparrow\cfrac{a\wedge\bar{a}}{\mathsf{f}}}\vee a}}$$

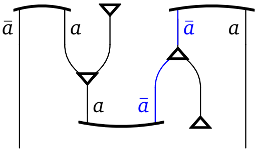
$$\cfrac{\boxed{\cfrac{\boxed{\cfrac{\mathsf{t}}{\bar{a}\vee a}}\vee\boxed{\cfrac{\mathsf{f}}{a}}}{\bar{a}\vee\boxed{\cfrac{a\vee a}{a}}}}\wedge\boxed{\cfrac{\mathsf{t}}{\boxed{\cfrac{\bar{a}}{\bar{a}\wedge\boxed{\cfrac{\bar{a}}{\mathsf{t}}}}}\vee a}}}{\mathsf{s}^2\,\cfrac{[\bar{a}\vee a]\wedge[a\vee\bar{a}]}{\bar{a}\vee\boxed{\mathsf{ai}\!\uparrow\cfrac{a\wedge\bar{a}}{\mathsf{f}}}\vee a}}$$

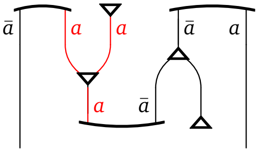




Experiments Example - Tracing the atoms up

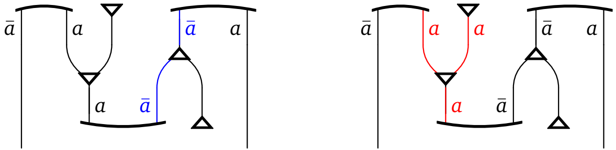
$$\mathsf{s}^2\;\cfrac{\cfrac{\boxed{\cfrac{\boxed{\cfrac{\mathsf{t}}{\bar a \vee a}} \vee \boxed{\cfrac{\mathsf{f}}{a}}}{\bar a \vee \boxed{\cfrac{a \vee a}{a}}}} \wedge \boxed{\cfrac{\textcolor{blue}{\mathsf{t}}}{\boxed{\cfrac{\textcolor{blue}{\bar a}}{\textcolor{blue}{\bar a} \wedge \boxed{\cfrac{\bar a}{\mathsf{t}}}}} \vee a}}}{[\bar a \vee a] \wedge [a \vee \textcolor{blue}{\bar a}]}}{\bar a \vee \boxed{\textcolor{blue}{\mathsf{ai}\!\uparrow}\,\cfrac{a \wedge \textcolor{blue}{\bar a}}{\mathsf{f}}} \vee a}
\qquad
\mathsf{s}^2\;\cfrac{\cfrac{\boxed{\cfrac{\boxed{\cfrac{\textcolor{red}{\mathsf{t}}}{\bar a \vee \textcolor{red}{a}}} \vee \boxed{\cfrac{\mathsf{f}}{\textcolor{red}{a}}}}{\bar a \vee \boxed{\cfrac{\textcolor{red}{a \vee a}}{\textcolor{red}{a}}}}} \wedge \boxed{\cfrac{\mathsf{t}}{\boxed{\cfrac{\bar a}{\bar a \wedge \boxed{\cfrac{\bar a}{\mathsf{t}}}}} \vee a}}}{[\bar a \vee \textcolor{red}{a}] \wedge [a \vee \bar a]}}{\bar a \vee \boxed{\textcolor{red}{\mathsf{ai}\!\uparrow}\,\cfrac{\textcolor{red}{a} \wedge \bar a}{\mathsf{f}}} \vee a}$$





Experiments Example - Removal of boxes

$$\frac{\dfrac{\dfrac{\mathsf{t}}{\bar a \vee a} \vee \dfrac{\mathsf{f}}{a}}{\bar a \vee \dfrac{a \vee a}{a}} \wedge \dfrac{\dfrac{\color{blue}{\mathsf{t}}}{\color{blue}{\bar a}}}{\color{blue}{\bar a}\color{black}{} \wedge \dfrac{\bar a}{\mathsf{t}} \vee a}}{\mathsf{s}^2\,\dfrac{[\bar a \vee a] \wedge [a \vee \color{blue}{\bar a}\color{black}{}]}{\bar a \vee \color{blue}{\mathsf{ai}\!\uparrow}\color{black}{}\,\dfrac{a \wedge \color{blue}{\bar a}}{\mathsf{f}} \vee a}} \qquad \frac{\dfrac{\dfrac{\color{red}{\mathsf{t}}}{\bar a \vee \color{red}{a}} \vee \dfrac{\mathsf{f}}{\color{red}{a}}}{\bar a \vee \dfrac{\color{red}{a \vee a}}{\color{red}{a}}} \wedge \dfrac{\dfrac{\mathsf{t}}{\bar a}}{\bar a \wedge \dfrac{\bar a}{\mathsf{t}} \vee a}}{\mathsf{s}^2\,\dfrac{[\bar a \vee \color{red}{a}\color{black}{}] \wedge [a \vee \bar a]}{\bar a \vee \color{red}{\mathsf{ai}\!\uparrow}\color{black}{}\,\dfrac{\color{red}{a}\color{black}{} \wedge \bar a}{\mathsf{f}} \vee a}}$$

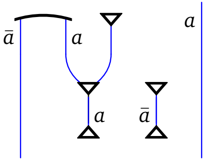


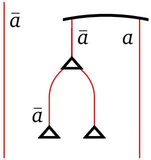
Experiments Example - Formation of each ‘experiment’

$$\mathsf{s}^2\,\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{t}}{\bar a \vee a} \vee \dfrac{\mathsf{f}}{a}}{\bar a \vee \dfrac{a \vee a}{a}} \wedge \dfrac{a}{\dfrac{\mathsf{f}}{\mathsf{f} \wedge \dfrac{\dfrac{\mathsf{f}}{\bar a}}{\mathsf{t}}} \vee a}}{[\bar a \vee a] \wedge [a \vee \mathsf{f}]}}{\bar a \vee \cancel{\mathsf{cut}}\,\dfrac{\dfrac{a}{\mathsf{t}} \wedge \mathsf{f}}{\mathsf{f}} \vee a}$$

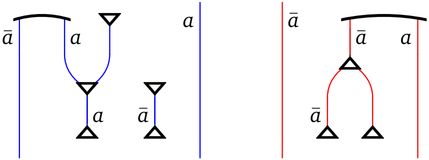
or

$$\mathsf{s}^2\,\dfrac{\dfrac{\dfrac{\dfrac{\bar a}{\bar a \vee \mathsf{f}} \vee \dfrac{\mathsf{f}}{\mathsf{f}}}{\bar a \vee \dfrac{\mathsf{f} \vee \mathsf{f}}{\mathsf{f}}} \wedge \dfrac{\mathsf{t}}{\dfrac{\bar a}{\bar a \wedge \dfrac{\bar a}{\mathsf{t}}} \vee a}}{[\bar a \vee \mathsf{f}] \wedge [a \vee \bar a]}}{\bar a \vee \cancel{\mathsf{cut}}\,\dfrac{\mathsf{f} \wedge \dfrac{\bar a}{\mathsf{t}}}{\mathsf{f}} \vee a}$$





Experiments Example - Combination of ‘experiments’



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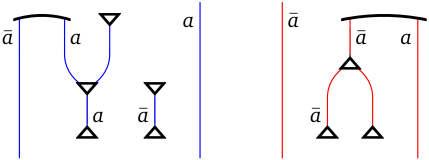

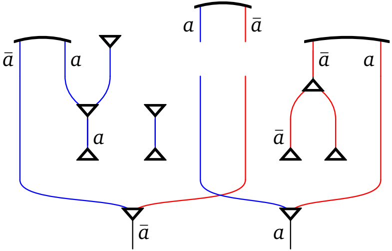

Experiments Example - Cut free proof

$$
\dfrac{
\dfrac{\mathsf{t}}{a \vee \bar{a}}
}{
\boxed{
\dfrac{
\dfrac{\dfrac{\mathsf{t}}{\bar{a} \vee a} \vee \dfrac{\mathsf{f}}{a}}{\bar{a} \vee \dfrac{a \vee a}{a}}
\wedge
\dfrac{a}{\dfrac{\mathsf{f}}{\mathsf{f} \wedge \dfrac{\dfrac{\mathsf{f}}{\bar{a}}}{\mathsf{t}}} \vee a}
}{
\mathsf{s}^2\,\dfrac{[\bar{a} \vee a] \wedge [a \vee \mathsf{f}]}{\bar{a} \vee \dfrac{\dfrac{a}{\mathsf{t}} \wedge \mathsf{f}}{\mathsf{f}} \vee a}
}
}
\vee
\boxed{
\dfrac{
\dfrac{\dfrac{\bar{a}}{\bar{a} \vee \mathsf{f}} \vee \dfrac{\mathsf{f}}{\mathsf{f}}}{\bar{a} \vee \dfrac{\mathsf{f} \vee \mathsf{f}}{\mathsf{f}}}
\wedge
\dfrac{\mathsf{t}}{\dfrac{\bar{a}}{\bar{a} \wedge \dfrac{\bar{a}}{\mathsf{t}}} \vee a}
}{
\mathsf{s}^2\,\dfrac{[\bar{a} \vee \mathsf{f}] \wedge [a \vee \bar{a}]}{\bar{a} \vee \dfrac{\mathsf{f} \wedge \dfrac{\bar{a}}{\mathsf{t}}}{\mathsf{f}} \vee a}
}
}
}{
\dfrac{\bar{a} \vee \bar{a}}{\bar{a}} \vee \dfrac{a \vee a}{a}
}
$$

Predicate Calculus

Can this method be extended to the predicate calculus?

---

[6] Samuel R Buss. “On Herbrand’s theorem”. In: *Logic and Computational Complexity*. Springer, 1995.

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- We can then perform the experiments method on this isolated section.

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KSgr

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$$\mathsf{n}{\downarrow}\,\frac{A[\tau/x]}{\exists x A x} \qquad\qquad \mathsf{gr}{\downarrow}\,\frac{Q\{P\{R\}\}}{P\{Q\{R\}\}}$$

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- Contraction is broken down into four rules.

$$\mathsf{ac}\!\downarrow \frac{a \vee a}{a} \qquad \mathsf{m}\, \frac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]}$$

$$\mathsf{qc}\!\downarrow \frac{\exists x A \vee \exists x A}{\exists x A} \qquad \mathsf{m}_2\!\downarrow \frac{\forall x A \vee \forall x B}{\forall x[A \vee B]}$$

Herbrand Stratification

As noted by Brünnler, there is a simple stratification process for KSgr.

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$$\begin{array}{l} \overline{\Vert}\,\mathsf{KSgr} \\ A \end{array} \quad \xrightarrow{1} \quad \begin{array}{l} \overline{\Vert}\,\mathsf{KS}\cup\{\mathsf{n}\downarrow,\mathsf{gr}\downarrow,\mathsf{m_2}\downarrow,\mathsf{ai}\uparrow\} \\ A' \\ \Vert\,\mathsf{qc}\downarrow \\ A \end{array}$$

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$$
\begin{array}{l}
\prod \mathsf{KSgr} \\
A
\end{array}
\xrightarrow{1}
\begin{array}{l}
\prod \mathsf{KS}\cup\{\mathsf{n}\downarrow,\mathsf{gr}\downarrow,\mathsf{m_2}\downarrow,\mathsf{ai}\uparrow\} \\
A' \\
\Vert \mathsf{qc}\downarrow \\
A
\end{array}
\xrightarrow{2}
\begin{array}{l}
\prod \mathsf{KS}\cup\{\mathsf{n}\downarrow,\mathsf{ai}\uparrow\} \\
Q\{B\} \\
\Vert \mathsf{gr}\downarrow \\
A' \\
\Vert \mathsf{qc}\downarrow \\
A
\end{array}
$$

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$$\begin{array}{c} \Pi \, \mathsf{KSgr} \\ A \end{array} \xrightarrow{1} \begin{array}{c} \Pi \, \mathsf{KS} \cup \{\mathsf{n}\downarrow, \mathsf{gr}\downarrow, \mathsf{m_2}\downarrow, \mathsf{ai}\uparrow\} \\ A' \\ \| \, \mathsf{qc}\downarrow \\ A \end{array} \xrightarrow{2} \begin{array}{c} \Pi \, \mathsf{KS} \cup \{\mathsf{n}\downarrow, \mathsf{ai}\uparrow\} \\ Q\{B\} \\ \| \, \mathsf{gr}\downarrow \\ A' \\ \| \, \mathsf{qc}\downarrow \\ A \end{array} \xrightarrow{3} \begin{array}{c} \Pi \, \mathsf{KS} \cup \{\mathsf{ai}\uparrow\} \\ \forall \vec{x} \, W(B) \\ \| \, \mathsf{n}\downarrow \\ Q\{B\} \\ \| \, \mathsf{gr}\downarrow \\ A' \\ \| \, \mathsf{qc}\downarrow \\ A \end{array}$$

$\forall \vec{x} \, W(B)$ is a “Herbrand Proof” for A.

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$\forall\vec{x}\,W(B)$ is a “Herbrand Proof” for A.

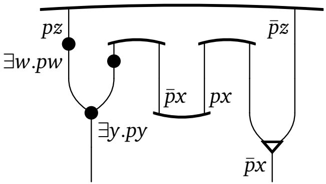
More technically: $Q\{B\}$ is a prenexification of a strong $\vee$-expansion of A (A') plus a witnessing substitution W for $Q\{B\}$

Predicate Logic Example

$$\mathsf{gr}{\downarrow}\,\dfrac{\mathsf{m_2}{\downarrow}\,\dfrac{\forall x\left[\dfrac{\dfrac{\mathsf{t}}{px \vee \bar{p}x} \wedge \dfrac{\mathsf{t}}{px \vee \bar{p}x}}{\mathsf{n}{\downarrow}\,\dfrac{px}{\exists y.py} \vee (\bar{p}x \wedge px) \vee \bar{p}x}\right] \vee \forall z\left[\dfrac{\mathsf{t}}{\mathsf{n}{\downarrow}\,\dfrac{pz}{\exists w.pw} \vee \bar{p}z}\right]}{\forall x\left[\mathsf{qc}{\downarrow}\,\dfrac{\exists w.pw \vee \exists y.py}{\exists y.py} \vee \mathsf{ai}{\uparrow}\,\dfrac{(\bar{p}x \wedge px)}{\mathsf{f}} \vee \bar{p}x \vee \bar{p}x\right]}}{\exists y.py \vee \forall x\left[\mathsf{ac}{\downarrow}\,\dfrac{\bar{p}x \vee \bar{p}x}{\bar{p}x}\right]}$$

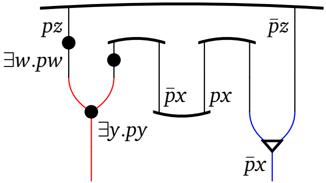
Predicate Logic Example

$$
\mathsf{m_2}\!\downarrow\frac{
\forall x\left[\dfrac{\dfrac{\mathsf{t}}{px \vee \bar{p}x} \wedge \dfrac{\mathsf{t}}{px \vee \bar{p}x}}{\mathsf{n}\!\downarrow\dfrac{px}{\exists y.py} \vee (\bar{p}x \wedge px) \vee \bar{p}x}\right] \vee \forall z\left[\dfrac{\mathsf{t}}{\mathsf{n}\!\downarrow\dfrac{pz}{\exists w.pw} \vee \bar{p}z}\right]
}{
\mathsf{gr}\!\downarrow\dfrac{\forall x\left[\mathsf{qc}\!\downarrow\dfrac{\exists w.pw \vee \exists y.py}{\exists y.py} \vee \mathsf{ai}\!\uparrow\dfrac{(\bar{p}x \wedge px)}{\mathsf{f}} \vee \bar{p}x \vee \bar{p}x\right]}{\exists y.py \vee \forall x\left[\mathsf{ac}\!\downarrow\dfrac{\bar{p}x \vee \bar{p}x}{\bar{p}x}\right]}
}
$$



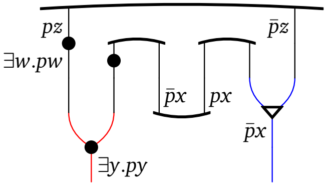
Predicate Logic Example - Step 1

$$\mathsf{gr}{\downarrow}\,\dfrac{\mathsf{m}_2{\downarrow}\,\dfrac{\forall x\left[\dfrac{\dfrac{\mathsf{t}}{px\vee\bar{p}x}\wedge\dfrac{\mathsf{t}}{px\vee\bar{p}x}}{\mathsf{n}{\downarrow}\,\dfrac{px}{\exists y.py}\vee(\bar{p}x\wedge px)\vee\bar{p}x}\right]\vee\forall z\left[\dfrac{\mathsf{t}}{\mathsf{n}{\downarrow}\,\dfrac{pz}{\exists w.pw}\vee\bar{p}z}\right]}{\forall x\left[\mathsf{qc}{\downarrow}\,\dfrac{\exists w.pw\vee\exists y.py}{\exists y.py}\vee\mathsf{ai}{\uparrow}\,\dfrac{(\bar{p}x\wedge px)}{\mathsf{f}}\vee\bar{p}x\vee\bar{p}x\right]}}{\exists y.py\vee\forall x\left[\mathsf{ac}{\downarrow}\,\dfrac{\bar{p}x\vee\bar{p}x}{\bar{p}x}\right]}$$



Predicate Logic Example - Step 1

$$\mathsf{m_2}{\downarrow}\,\dfrac{\forall x\left[\dfrac{\dfrac{\mathsf{t}}{px \vee \bar{p}x} \wedge \dfrac{\mathsf{t}}{px \vee \bar{p}x}}{\mathsf{n}{\downarrow}\,\dfrac{px}{\exists y.py} \vee (\bar{p}x \wedge px) \vee \bar{p}x}\right] \vee \forall z\left[\dfrac{\mathsf{t}}{\mathsf{n}{\downarrow}\,\dfrac{pz}{\exists w.pw} \vee \bar{p}z}\right]}{\mathsf{gr}{\downarrow}\,\dfrac{\forall x\left[{\color{red}\exists w.pw \vee \exists y.py} \vee \mathsf{ai}{\uparrow}\,\dfrac{(\bar{p}x \wedge px)}{\mathsf{f}} \vee {\color{blue}\mathsf{ac}{\downarrow}\,\dfrac{\bar{p}x \vee \bar{p}x}{\bar{p}x}}\right]}{{\color{red}\mathsf{qc}{\downarrow}\,\dfrac{\exists w.pw \vee \exists y.py}{\exists y.py}} \vee \forall x.{\color{blue}\bar{p}x}}}$$

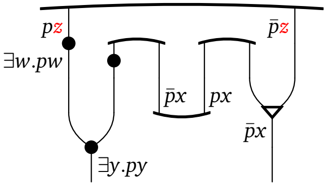


Predicate Logic Example - Step 2

$$
\mathsf{qc}{\downarrow}\,\dfrac{\exists w.pw \vee \exists y.py}{\exists y.py} \vee \forall x.\bar{p}x
$$

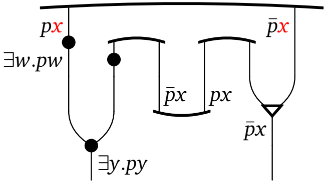
The formula above is the final conclusion. The full derivation is:

$$
\mathsf{gr}{\downarrow}\,\dfrac{\mathsf{m_2}{\downarrow}\,\dfrac{\forall x\left[\dfrac{\dfrac{\mathsf{t}}{px \vee \bar{p}x} \wedge \dfrac{\mathsf{t}}{px \vee \bar{p}x}}{\mathsf{n}{\downarrow}\,\dfrac{px}{\exists y.py} \vee (\bar{p}x \wedge px) \vee \bar{p}x}\right] \vee \forall z\left[\dfrac{\mathsf{t}}{\mathsf{n}{\downarrow}\,\dfrac{pz}{\exists w.pw} \vee \bar{p}z}\right]}{\forall x\left[\exists w.pw \vee \exists y.py \vee \mathsf{ai}{\uparrow}\,\dfrac{(\bar{p}x \wedge px)}{\mathsf{f}} \vee \mathsf{ac}{\downarrow}\,\dfrac{\bar{p}x \vee \bar{p}x}{\bar{p}x}\right]}}{\mathsf{qc}{\downarrow}\,\dfrac{\exists w.pw \vee \exists y.py}{\exists y.py} \vee \forall x.\bar{p}x}
$$



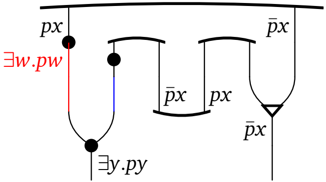
Predicate Logic Example - Step 2

$$\text{gr}\!\downarrow\frac{\forall x\left[\; = \frac{\dfrac{\dfrac{\mathsf{t}}{px\vee\bar{p}x}\wedge\dfrac{\mathsf{t}}{px\vee\bar{p}x}}{\text{n}\!\downarrow\dfrac{px}{\exists y.py}\vee(\bar{p}x\wedge px)\vee\bar{p}x}\vee\dfrac{\mathsf{t}}{\text{n}\!\downarrow\dfrac{px}{\exists w.pw}\vee\bar{p}x}}{\exists w.pw\vee\exists y.py\vee\text{ai}\!\uparrow\dfrac{(\bar{p}x\wedge px)}{\mathsf{f}}\vee\text{ac}\!\downarrow\dfrac{\bar{p}x\vee\bar{p}x}{\bar{p}x}}\right]}{\text{qc}\!\downarrow\dfrac{\exists w.pw\vee\exists y.py}{\exists y.py}\vee\forall x.\bar{p}x}$$



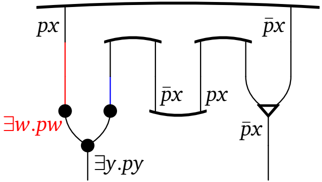
Predicate Logic Example - Step 3

$$\mathsf{gr}{\downarrow}\frac{\forall x\left[\;=\frac{\dfrac{\dfrac{\mathsf{t}}{px \vee \bar{p}x} \wedge \dfrac{\mathsf{t}}{px \vee \bar{p}x}}{\mathsf{n}{\downarrow}\dfrac{px}{\exists y.py} \vee (\bar{p}x \wedge px) \vee \bar{p}x} \vee \dfrac{\mathsf{t}}{\mathsf{n}{\downarrow}\dfrac{px}{\exists w.pw} \vee \bar{p}x}}{\exists w.pw \vee \exists y.py \vee \mathsf{ai}{\uparrow}\dfrac{(\bar{p}x \wedge px)}{\mathsf{f}} \vee \mathsf{ac}{\downarrow}\dfrac{\bar{p}x \vee \bar{p}x}{\bar{p}x}}\right]}{\mathsf{qc}{\downarrow}\dfrac{\exists w.pw \vee \exists y.py}{\exists y.py} \vee \forall x.\bar{p}x}$$



Predicate Logic Example - Step 3

$$
\text{qc}\!\downarrow\dfrac{\text{gr}\!\downarrow\dfrac{\forall x\left[\begin{array}{c}
\dfrac{\dfrac{\mathsf{t}}{px \vee \bar{p}x} \wedge \dfrac{\mathsf{t}}{px \vee \bar{p}x}}{{\color{blue}px} \vee (\bar{p}x \wedge px) \vee \bar{p}x} \vee \dfrac{\mathsf{t}}{{\color{red}px} \vee \bar{p}x} \\ \hline
{\color{red}px} \vee {\color{blue}px} \vee \text{ai}\!\uparrow\dfrac{(\bar{p}x \wedge px)}{\mathsf{f}} \vee \text{ac}\!\downarrow\dfrac{\bar{p}x \vee \bar{p}x}{\bar{p}x} \\ \hline
{\color{red}\text{n}\!\downarrow\dfrac{px}{\exists w.pw}} \vee {\color{blue}\text{n}\!\downarrow\dfrac{px}{\exists y.py}} \vee \bar{p}x
\end{array}\right]}{\exists w.pw \vee \exists y.py \vee \forall x.\bar{p}x}}{\exists y.py \vee \forall x.\bar{p}x}
$$

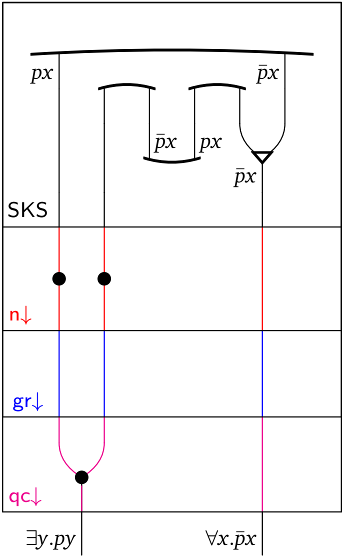


Predicate Logic Example - Stratified Proof

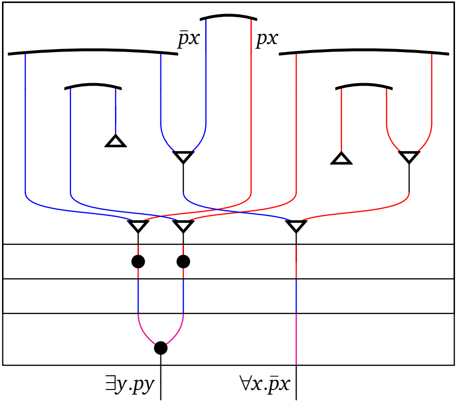
$$\cfrac{\cfrac{\forall x\left[\cfrac{\cfrac{\cfrac{\cfrac{\mathsf{t}}{px\vee\bar{p}x}\wedge\cfrac{\mathsf{t}}{px\vee\bar{p}x}}{px\vee(\bar{p}x\wedge px)\vee\bar{p}x}\vee\cfrac{\mathsf{t}}{px\vee\bar{p}x}}{px\vee px\vee\mathsf{ai}{\uparrow}\cfrac{(\bar{p}x\wedge px)}{\mathsf{f}}\vee\mathsf{ac}{\downarrow}\cfrac{\bar{p}x\vee\bar{p}x}{\bar{p}x}}}{\mathsf{n}{\downarrow}\cfrac{px}{\exists w.pw}\vee\mathsf{n}{\downarrow}\cfrac{px}{\exists y.py}\vee\bar{p}x}\right]}{\mathsf{gr}{\downarrow}\cfrac{\forall x\left[\exists w.pw\vee\exists y.py\vee\bar{p}x\right]}{\exists w.pw\vee\exists y.py\vee\forall x.\bar{p}x}}}{\mathsf{qc}{\downarrow}\cfrac{\exists y.py\vee\exists y.py}{\exists y.py}\vee\forall x.\bar{p}x}}{\exists y.py\vee\forall x.\bar{p}x}$$

Predicate Logic Example - Stratified Proof

$$\dfrac{\forall x\left[\dfrac{\dfrac{\dfrac{\mathsf{t}}{px\vee\bar{p}x}\wedge\dfrac{\mathsf{t}}{px\vee\bar{p}x}}{px\vee(\bar{p}x\wedge px)\vee\bar{p}x}\vee\dfrac{\mathsf{t}}{px\vee\bar{p}x}}{\dfrac{px\vee px\vee\mathsf{ai}{\uparrow}\dfrac{(\bar{p}x\wedge px)}{\mathsf{f}}\vee\mathsf{ac}{\downarrow}\dfrac{\bar{p}x\vee\bar{p}x}{\bar{p}x}}{\mathsf{n}{\downarrow}\dfrac{px}{\exists w.pw}\vee\mathsf{n}{\downarrow}\dfrac{px}{\exists y.py}\vee\bar{p}x}}\right]}{\dfrac{\mathsf{gr}{\downarrow}\dfrac{\forall x\,[\exists w.pw\vee\exists y.py\vee\bar{p}x]}{\exists w.pw\vee\exists y.py\vee\forall x.\bar{p}x}}{\dfrac{\mathsf{qc}{\downarrow}\dfrac{\exists y.py\vee\exists y.py}{\exists y.py}\vee\forall x.\bar{p}x}{\exists y.py\vee\forall x.\bar{p}x}}}$$



Predicate Logic Example - Cut Eliminated



It should hopefully be clear that this process is confluent, modulo reasonable geometric equivalences.

Conclusions

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- These elementary ideas are not available in the sequent calculus, simply due to the limited methods of proof composition available.

Further work

- A natural extension of atomic flows for predicate logic.
- A more ideosyncratic cut elimination procedure for predicate logic.