Experiments and Epsilons

or

What I've been doing for the last six months or so

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Aims

1. (Modest) Cut elimination for predicate logic in a deep inference system:
   1.1 Experiments Method;
   1.2 Herbrand's Theorem.
2. (Speculative) Finding a more natural syntax for predicate logic:
   2.1 "Skeletal" approach to open deduction;
   2.2 æ-calculus.

Overview

$$\begin{array}{c}\Phi\\ \Big\Vert\,\mathsf{SKS\text{æ}}\\ A\end{array} \xrightarrow{\text{Her.}} \begin{array}{c}H(\Phi,p)\\ \Big\Vert\,\mathsf{SKS}\\ H(A,p)\end{array} \xrightarrow{\text{Exp.}} \begin{array}{c}H(\Phi,p)'\\ \Big\Vert\,\mathsf{KS}\\ H(A,p)\end{array} \xrightarrow{\text{æ}\downarrow,\mathsf{ac}\downarrow} \begin{array}{c}\Phi'\\ \Big\Vert\,\mathsf{KS\text{æ}}\\ A\end{array}$$

$$H(\Phi,p) \approx \bigvee_{\tau_i \in D(\pi_\alpha(\Phi),p)} \pi_\alpha(\Phi)[\tau_1/x_1, \ldots, \tau_n/x_n]$$

Propositional Logic

The signature of classical propositional logic, Σ_0, that we will use consists of:

- Two distinct units: t (true) and f (false), as well as an inexhaustible supply of *atoms*, $a, b, c\ldots$, and their *duals*, $\bar{a}, \bar{b}, \bar{c}\ldots$;
- One unary connective $\bar{A}$ (negation);
- Two binary connectives $\wedge$ (conjunction) and $\vee$ (disjunction).

We consider propositional prederivations equivalent modulo double negation elimination, associativity and commutativity of each of the binary connectives, and the propositional De Morgan laws.

SKS

- The *structural* rules:

$$\mathsf{ai}{\downarrow}\,\frac{\mathsf{t}}{a \vee \bar{a}} \qquad \mathsf{ac}{\downarrow}\,\frac{a \vee a}{a} \qquad \mathsf{aw}{\downarrow}\,\frac{\mathsf{f}}{a}$$

identity *contraction* *weakening*

$$\mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\mathsf{f}} \qquad \mathsf{ac}{\uparrow}\,\frac{a}{a \wedge a} \qquad \mathsf{aw}{\uparrow}\,\frac{a}{\mathsf{t}}$$

cut *cocontraction* *coweakening*

- The *logical* rules:

$$\mathsf{s}\,\frac{A \wedge [B \vee C]}{(A \wedge B) \vee C} \qquad \mathsf{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]}$$

switch *medial*

- The *unit* rules:

$$A \wedge \mathsf{t} = A \qquad A \vee \mathsf{f} = A$$

Coweakening elimination

Theorem

Atomic coweakening is admissible from $\mathsf{SKS}\setminus\{\mathsf{aw}{\uparrow}\}$

$$\mathsf{aw}{\uparrow}\,\dfrac{\mathsf{aw}{\downarrow}\,\dfrac{\mathsf{f}}{a}}{\mathsf{t}} \quad\longrightarrow\quad \mathsf{s}\,\dfrac{\mathsf{f}}{\mathsf{t}} \qquad\qquad \mathsf{aw}{\uparrow}\,\dfrac{\mathsf{ac}{\downarrow}\,\dfrac{a \vee a}{a}}{\mathsf{t}} \quad\longrightarrow\quad =\dfrac{\dfrac{a}{\mathsf{t}} \vee \dfrac{a}{\mathsf{t}}}{\mathsf{t}}$$

$$\mathsf{ac}{\uparrow}\,\dfrac{a}{\dfrac{a}{\mathsf{t}} \wedge a} \quad\longrightarrow\quad =\dfrac{a}{\mathsf{t} \wedge a} \qquad\qquad \mathsf{ai}{\downarrow}\,\dfrac{\mathsf{t}}{\dfrac{a}{\mathsf{t}} \vee \bar{a}} \quad\longrightarrow\quad =\dfrac{\mathsf{t}}{\mathsf{t} \vee \mathsf{aw}{\downarrow}\,\dfrac{\mathsf{f}}{\bar{a}}}$$

$M_w(\Phi) = (m_c, m_w)$, $m_c = \#\mathsf{ac}{\uparrow}$, $m_w = \#\mathsf{aw}{\uparrow}$

Theorem

Atomic cocontraction is admissible from $\mathsf{SKS}\setminus\{\mathsf{ac}\uparrow\}$

$$\mathsf{ac}\uparrow\,\dfrac{\mathsf{aw}\downarrow\,\dfrac{\mathsf{f}}{a}}{a\wedge a} \longrightarrow \dfrac{\mathsf{f}}{\dfrac{\mathsf{f}}{a}\wedge\dfrac{\mathsf{f}}{a}} \qquad \mathsf{ac}\uparrow\,\dfrac{\mathsf{ac}\downarrow\,\dfrac{a\vee a}{a}}{a\wedge a} \longrightarrow \mathsf{m}\,\dfrac{\dfrac{a}{a\wedge a}\vee\dfrac{a}{a\wedge a}}{\dfrac{a\vee a}{a}\wedge\dfrac{a\vee a}{a}}$$

$$\mathsf{ai}\downarrow\,\dfrac{\mathsf{t}}{\mathsf{ac}\uparrow\,\dfrac{a}{a\wedge a}\vee\bar{a}} \longrightarrow \dfrac{\mathsf{t}}{\boxed{\mathsf{s}\,\dfrac{\dfrac{a}{a\wedge\boxed{\dfrac{\mathsf{t}}{a\vee\bar{a}}}}}{(a\wedge a)\vee\bar{a}}}\vee\bar{a}}{(a\wedge a)\vee\boxed{\dfrac{\bar{a}\vee\bar{a}}{\bar{a}}}}$$

$M_c(\Phi) = (m_0, m_1, \ldots), m_i = \#\mathsf{ac}\uparrow$ of height i.

Cut Elimination

Theorem

Atomic cut is admissible from $\mathsf{SKS} \setminus \{\mathsf{ai}\uparrow\}$

Proof.

(The Experiments method)

1. For each cut, we create two derivations from the proof:
   $$\begin{array}{c} a \\ \Phi_{i+} \| \\ A \end{array} \text{ and } \begin{array}{c} \bar{a} \\ \Phi_{i-} \| \\ A \end{array}.$$
2. Intuition: each cut $\mathsf{ai}\uparrow \dfrac{a \wedge \bar{a}}{\mathsf{f}}$, either a or $\bar{a}$ is 'true' (in the Tarskian sense), and each version corresponds to one of these possibilities.
3. We then use an identity (at the top) and a contraction (at the bottom) to disjunct the two proofs, creating a proof with one fewer cut instance.

More detail from old presentation

Standard Predicate Logic

Terms of *standard predicate logic* (SPL), $\mathcal{T}_S$:

- *Variables*, written: x_i, y_i, z_i;
- *Functions*, written f_i^n, of *arity* n, with n terms as arguments, e.g. $f_i^n(\tau_1, \tau_2, \ldots, \tau_n)$. Often the arity indicator is omitted. *Constants* are functions of arity 0, written c_i.

SPL-Atoms, $\mathcal{A}_{SPL}$: *predicate symbols*, written p_i^n, and their duals $\bar{p}_i^n$, of arity n, with n terms as arguments, e.g. $p_i^n(\tau_1, \tau_2, \ldots, \tau_n)$.
The signature of SPL, Σ_{SPL} consists of:

- Two distinct units: t (true) and f (false), as well $\mathcal{A}_{SPL}$;
- One distinct unary connective $\bar{A}$ (negation), as well $\forall x$ and $\exists x$ for all variables;
- Two binary connectives $\wedge$ (conjunction) and $\vee$ (disjunction).

æ-Predicate Logic

The terms of *æ-predicate logic*, (æPL) $\mathcal{T}_{æ}$ are either:

- Variables, x_i, y_i, z_i;
- Functions, f_i^n, including constants, c_i.
- *Alpha terms*, written $\alpha_x(y_1, ...y_n)$, with $n \geq 0$.
- *Epsilon terms*, written $\epsilon_x(y_1, ...y_n)$, with $n \geq 0$.

Atoms of æPL, $\mathcal{A}_{æ}$ defined as in $\mathcal{A}_S$, except predicates range over $\mathcal{T}_{æ}$. The prederivations of æPL, $\mathcal{F}'_{æ}$, are just propositional prederivations with $\mathcal{A}_{æ}$ as atoms.

From SPL to æPL

Only the semantically important information about quantification is preserved:

- $A[x \rightsquigarrow \alpha_x] = A[\alpha_x/x][\epsilon_y(\vec{z}, x)/\epsilon_y(\vec{z})]$ for all ϵ-terms $\epsilon_y(\vec{z})$.
- $A[x \rightsquigarrow \epsilon_x] = A[\epsilon_x/x][\alpha_y(\vec{z}, x)/\alpha_y(\vec{z})]$ for all α-terms $\alpha_y(\vec{z})$.

$\theta : \mathcal{F}_S \to \mathcal{F}'_{æ}$:

- $\theta(\mathsf{t}) = \mathsf{t}$, $\theta(\mathsf{f}) = \mathsf{f}$;
- $\theta(\forall x A) = \theta(A)[x \rightsquigarrow \alpha_x]$;
- $\theta(\exists x A) = \theta(A)[x \rightsquigarrow \epsilon_x]$;
- $\theta(A \wedge B) = \theta(A) \wedge \theta(B)$
- $\theta(A \vee B) = \theta(A) \vee \theta(B)$

$\theta^*(A) := \theta(A^*)$ (where $*$ is scope-reduction) and the set of formulas of αPL, $\mathcal{F}_{æ}$, is the image of θ^*.

Examples

$$\forall x \forall y \forall z (Pxy \wedge Qxz \wedge Ryz) \quad \longrightarrow \quad P\alpha_x\alpha_y \wedge Q\alpha_x\alpha_z \wedge R\alpha_y\alpha_z$$

$$\forall x \exists y [Pxy \vee Qy] \wedge \exists z Rz \quad \longrightarrow \quad [P\alpha_x\epsilon_y(x) \vee Q\epsilon_y(x)] \wedge R\epsilon_z$$

$$\forall x \exists y [Px \vee Qy] \wedge \exists z Rz \quad \longrightarrow \quad [P\alpha_x\epsilon_y \vee Q\epsilon_y] \wedge R\epsilon_z$$

$$\forall x \exists y \forall z \exists w [(Pw \wedge Qwz) \vee (Rzy \wedge Ryx)]$$
$$\downarrow$$
$$[(P\epsilon_w(z,x) \wedge Q\epsilon_w(z,x)\alpha_z(y)) \vee (R\alpha_z(y)\epsilon_y(x) \wedge R\epsilon_y(x)\alpha_x)]$$

SKSæ

$$\mathsf{SKS} +$$

$$\text{æ}\!\downarrow \frac{A[a/x]}{A[x \rightsquigarrow \epsilon_x(\vec{y})]} \qquad \text{æ}\!\uparrow \frac{A[x \rightsquigarrow \alpha_x(\vec{y})]}{A[a/x]}$$

generalization *instantiation*

æ-squeezing (– cut)

$$\dfrac{\dfrac{p[x \rightsquigarrow \alpha_x]}{\dfrac{p[x \rightsquigarrow \alpha_x]}{p[a/x]}\,\text{æ}\uparrow \;\wedge\; \dfrac{p[x \rightsquigarrow \alpha_x]}{p[a/x]}\,\text{æ}\uparrow}}{}\;\mathsf{ac}\uparrow$$

$$\mathsf{ac}\uparrow\,\dfrac{p[x \rightsquigarrow \alpha_x]}{\text{æ}\uparrow\,\dfrac{p[x \rightsquigarrow \alpha_x]}{p[a/x]} \;\wedge\; \text{æ}\uparrow\,\dfrac{p[x \rightsquigarrow \alpha_x]}{p[a/x]}} \quad\rightarrow\quad \mathsf{ac}\uparrow\,\dfrac{\text{æ}\uparrow\,\dfrac{p[x \rightsquigarrow \alpha_x]}{p[a/x]}}{p[a/x] \wedge p[a/x]}$$

$$\mathsf{ac}\downarrow\,\dfrac{p[x \rightsquigarrow \alpha_x] \vee p[x \rightsquigarrow \alpha_x]}{\text{æ}\uparrow\,\dfrac{p[x \rightsquigarrow \alpha_x]}{p[a/x]}} \quad\rightarrow\quad \text{æ}\uparrow\,\dfrac{p[x \rightsquigarrow \alpha_x] \vee p[x \rightsquigarrow \alpha_x]}{\mathsf{ac}\downarrow\,\dfrac{p[a/x] \vee p[a/x]}{p[a/x]}}$$

$$\mathsf{aw}\downarrow\,\dfrac{\mathsf{f}}{\text{æ}\uparrow\,\dfrac{p[x \rightsquigarrow \alpha_x]}{p[a/x]}} \quad\rightarrow\quad \mathsf{aw}\downarrow\,\dfrac{\mathsf{f}}{p[a/x]}$$

$$\mathsf{ai}\uparrow\,\dfrac{\mathsf{t}}{\text{æ}\uparrow\,\dfrac{p[x \rightsquigarrow \alpha_x]}{p[a/x]} \vee \bar{p}[x \rightsquigarrow \epsilon_x]} \quad\rightarrow\quad \mathsf{ai}\uparrow\,\dfrac{\mathsf{t}}{p[a/x] \vee \text{æ}\downarrow\,\dfrac{\bar{p}[a/x]}{\bar{p}[x \rightsquigarrow \epsilon_x]}}$$

æ-squeezing example (1)

$$
\mathsf{s}\,\dfrac{
\dfrac{\text{æ}{\downarrow}\,\dfrac{Pa}{P\epsilon_x}}{
\boxed{\mathsf{ac}{\uparrow}\,\dfrac{P\epsilon_x}{P\epsilon_x \wedge P\epsilon_x}} \wedge \boxed{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{t}}{\bar{P}\alpha_z \vee P\epsilon_y}}
}
\over
P\epsilon_x \wedge \left(P\epsilon_x \wedge \left[\bar{P}\alpha_z \vee P\epsilon_y\right]\right)
}{
\dfrac{
\boxed{\mathsf{aw}{\uparrow}\,\dfrac{P\epsilon_x}{\mathsf{t}}} \wedge \boxed{\dfrac{\boxed{\mathsf{ai}{\uparrow}\,\dfrac{P\epsilon_x \wedge \bar{P}\alpha_z}{\mathsf{f}}} \vee P\epsilon_y}{P\epsilon_y}}
}{P\epsilon_y}
}
$$

æ-squeezing example (2)

$$
\dfrac{
\dfrac{Pa}{
\boxed{\mathsf{ac}\!\uparrow\,\dfrac{\dfrac{Pa}{Pa \wedge Pa}}{} \; \mathsf{æ}\!\downarrow\dfrac{}{P\epsilon_x \wedge P\epsilon_x}}
\wedge
\boxed{\mathsf{ai}\!\downarrow\,\dfrac{\mathsf{t}}{\bar{P}\alpha_z \vee P\epsilon_y}}
}
\\
P\epsilon_x \wedge \left(P\epsilon_x \wedge \left[\bar{P}\alpha_z \vee P\epsilon_y\right]\right)
}{
\boxed{\mathsf{aw}\!\uparrow\,\dfrac{P\epsilon_x}{\mathsf{t}}}
\wedge
\boxed{\dfrac{\boxed{\mathsf{ai}\!\uparrow\,\dfrac{P\epsilon_x \wedge \bar{P}\alpha_z}{\mathsf{f}}} \vee P\epsilon_y}{P\epsilon_y}}
}\;\mathsf{s}
\\
\overline{\qquad P\epsilon_y \qquad}
$$

æ-squeezing example (3)

$$
\cfrac{\cfrac{\cfrac{Pa}{\boxed{\mathsf{ac}\!\uparrow\,\cfrac{Pa}{\color{blue}{Pa \wedge Pa}}} \wedge \boxed{\mathsf{ai}\!\downarrow\,\cfrac{\mathsf{t}}{\bar{P}\alpha_z \vee P\epsilon_y}}}}{\mathsf{\color{blue}{s}}\,\cfrac{\color{blue}{Pa \wedge \left(Pa \wedge \left[\color{black}{\bar{P}\alpha_z \vee P\epsilon_y}\right]\right)}}{\color{blue}{Pa \wedge \left[\left(Pa \wedge \color{black}{\bar{P}\alpha_z}\right) \vee \color{black}{P\epsilon_y}\right]}}}\ \text{æ}\!\downarrow}{\boxed{\mathsf{aw}\!\uparrow\,\cfrac{P\epsilon_x}{\mathsf{t}}} \wedge \boxed{\cfrac{\color{red}{\boxed{\mathsf{ai}\!\uparrow\,\cfrac{P\epsilon_x \wedge P\alpha_x}{\mathsf{f}}}} \vee P\epsilon_y}{P\epsilon_y}}}}{P\epsilon_y}
$$

æ-squeezing example (4)

$$
\begin{array}{c}
Pa \\
\hline
\boxed{\mathsf{ac}{\uparrow}\,\dfrac{Pa}{Pa \wedge Pa}} \wedge \boxed{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{t}}{\bar{P}\alpha_z \vee P\epsilon_y}} \\
\hline
Pa \wedge \left(Pa \wedge \left[{\color{red}\bar{P}\alpha_z} \vee P\epsilon_y\right]\right) \\
\hline
{\color{red}\mathsf{s}} \\
{\color{red}\boxed{\begin{array}{r}\mathsf{æ}{\downarrow}\,\dfrac{Pa}{P\epsilon_x} \\ \mathsf{aw}{\uparrow}\,\dfrac{}{\mathsf{t}}\end{array}}} \wedge \boxed{\dfrac{\boxed{{\color{blue}\mathsf{ai}{\uparrow}}\,\dfrac{{\color{blue}Pa} \wedge \boxed{{\color{red}\mathsf{æ}{\uparrow}}\,\dfrac{{\color{red}\bar{P}\alpha_z}}{{\color{blue}\bar{P}a}}}}{{\color{blue}\mathsf{f}}}} \vee P\epsilon_y}{P\epsilon_y}} \\
\hline
P\epsilon_y
\end{array}
$$

æ-squeezing example (5)

$$
\begin{array}{c}
Pa \\
\hline
\boxed{\mathsf{ac}\!\uparrow\dfrac{Pa}{Pa \wedge Pa}} \wedge \boxed{\mathsf{ai}\!\downarrow\dfrac{\mathsf{t}}{\boxed{\text{æ}\!\uparrow\dfrac{\bar{P}\alpha_z}{\bar{P}a}} \vee P\epsilon_y}} \\
\hline
\mathsf{s}\,\dfrac{Pa \wedge \left(Pa \wedge \left[\bar{P}a \vee P\epsilon_y\right]\right)}{\boxed{\mathsf{aw}\!\uparrow\dfrac{Pa}{\mathsf{t}}} \wedge \boxed{\dfrac{\boxed{\mathsf{ai}\!\uparrow\dfrac{Pa \wedge \bar{P}a}{\mathsf{f}}} \vee P\epsilon_y}{P\epsilon_y}}} \\
\hline
P\epsilon_y
\end{array}
$$

æ-squeezing example (6)

$$
\begin{array}{lc}
 & Pa \\ \hline
 & \boxed{\mathsf{ac}\!\uparrow\,\dfrac{Pa}{Pa \wedge Pa}} \wedge \boxed{\mathsf{ai}\!\downarrow\,\dfrac{\mathsf{t}}{\bar{P}a \vee \boxed{\text{æ}\!\downarrow\,\dfrac{Pa}{P\epsilon_y}}}} \\ \hline
\mathsf{s} & Pa \wedge \left(Pa \wedge \left[\bar{P}a \vee P\epsilon_y\right]\right) \\ \hline
 & \boxed{\mathsf{aw}\!\uparrow\,\dfrac{Pa}{\mathsf{t}}} \wedge \boxed{\dfrac{\boxed{\mathsf{ai}\!\uparrow\,\dfrac{Pa \wedge \bar{P}a}{\mathsf{f}}} \vee P\epsilon_y}{P\epsilon_y}} \\ \hline
 & P\epsilon_y
\end{array}
$$

æ-squeezing example (7)

$$
\mathsf{æ}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{\dfrac{Pa}{\boxed{\mathsf{ac}{\uparrow}\,\dfrac{Pa}{Pa \wedge Pa}} \wedge \boxed{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{t}}{\bar{P}a \vee Pa}}}}{Pa \wedge \left(Pa \wedge \left[\bar{P}a \vee Pa\right]\right)}}{\boxed{\mathsf{aw}{\uparrow}\,\dfrac{Pa}{\mathsf{t}}} \wedge \boxed{\dfrac{\boxed{\mathsf{ai}{\uparrow}\,\dfrac{Pa \wedge \bar{P}a}{\mathsf{f}}} \vee Pa}{Pa}}}}{P\epsilon_y}
$$

First æ-theorem

Theorem

If Φ is a æ-derivation from A to B, both containing no α or ϵ symbols, then we can transform Φ into Φ', an SKS *derivation.*

Proof.

1. Perform æ-squeezing.
2. Deal with the following case by moving contraction up:

$$\mathsf{ac}\!\uparrow \frac{p[x \rightsquigarrow \alpha_x]}{\boxed{\text{æ}\!\uparrow \dfrac{p[x \rightsquigarrow \alpha_x]}{p[a/x]}} \wedge p[x \rightsquigarrow \alpha_x]}$$

3. æ $\uparrow$ rules and æ $\downarrow$ pass through each other when they meet.
4. The æ $\uparrow$ and æ $\downarrow$ rules must disappear.
5. Streamline the derivation, such that no æ-terms are created and then destroyed.

Herbrand Expansion

Definition

The *Herbrand Base* of a prederivation Φ is the set of constants that appear in Φ (with c_0). The set of Φ-*terms*, $\mathcal{T}_\Phi \subset \mathcal{T}_æ$, is all æ-terms that contain only constants in the Herbrand Base of Φ and function symbols that occur in Φ, including α and ϵ symbols.

Definition

Herbrand domain of degree p: $D(A,p) = \{\tau \in \mathcal{T}(A) \mid h(\tau) \leq p\}$.

Definition

Herbrand expansion of degree p:

$$H(A,p) = \bigvee_{\tau_i \in D(\pi_\alpha(\Phi),p)} \pi_\alpha(A)[\tau_1/x_1, \ldots, \tau_n/x_n]$$

This concept is to be extended to proofs: $H(\Phi,p)$.

Statement of the Theorems

Theorem

(Herbrand's Theorem on formulas of æPL) There is some p such that $H(A, p)$ *is a propositional tautology if and only if A is* SKSæ*-provable.*

Theorem

(Herbrand's Theoreom on proofs of SKSæ*) If* Φ *is an* SKSæ*-proof of A, then there is some p (probably the same p as for A) such that* $H(\Phi, p)$ *is an* SKS *proof of* $H(A, p)$.

The End of the Road (hopefully)

Theorem

Every proof in SKSæ *is reducible to a (unique? modulo what?) proof in* KSæ*.*

$$\begin{array}{c} \Phi \Big\| \mathsf{SKSæ} \\ A \end{array} \xrightarrow{\text{Her.}} \begin{array}{c} H(\Phi,p) \Big\| \mathsf{SKS} \\ H(A,p) \end{array} \xrightarrow{\text{Exp.}} \begin{array}{c} H(\Phi,p)' \Big\| \mathsf{KS} \\ H(A,p) \end{array} \xrightarrow{\text{æ}\downarrow,\mathsf{ac}\downarrow} \begin{array}{c} \Phi' \Big\| \mathsf{KSæ} \\ A \end{array}$$