A Natural Cut-elimination Procedure for Classical First-order Logic

Efficient and Natural Proof Systems Workshop

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Motivation

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	Propositional	
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	First-Order	
??	$\longleftarrow$	Herbrand's Theorem

Open Deduction

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1. An inference rule $\sigma : B/C$:

$$\begin{array}{c} A \\ \Phi \| \\ \sigma \dfrac{B}{C} \\ \Psi \| \\ D \end{array}$$

2. A binary (or n-ary) logical relation $\star$:

$$\begin{array}{ccc} A & & C \\ \Phi \| & \star & \| \Psi \\ B & & D \end{array}$$

SKS

- The *structural* rules (all atomic):

$$\mathsf{ai}{\downarrow}\,\frac{\mathsf{t}}{a \vee \bar{a}} \qquad \mathsf{ac}{\downarrow}\,\frac{a \vee a}{a} \qquad \mathsf{aw}{\downarrow}\,\frac{\mathsf{f}}{a}$$

identity — *contraction* — *weakening*

$$\mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\mathsf{f}} \qquad \mathsf{ac}{\uparrow}\,\frac{a}{a \wedge a} \qquad \mathsf{aw}{\uparrow}\,\frac{a}{\mathsf{t}}$$

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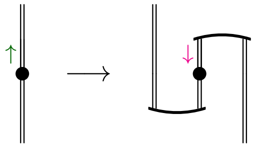
$$
\rho\!\uparrow\frac{A}{B}
\quad\longrightarrow\quad
\cfrac{A}{\mathsf{s}\,\cfrac{A \wedge \left(\mathsf{i}\!\downarrow\cfrac{\mathsf{t}}{\rho\!\downarrow\dfrac{\bar{B}}{\bar{A}} \vee B}\right)}{\cfrac{\mathsf{i}\!\uparrow\dfrac{A \wedge \bar{A}}{\mathsf{f}} \vee B}{B}}}
$$



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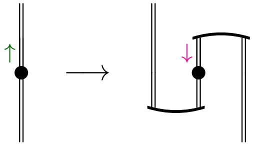

- In propositional logic admissibility of atomic cut, $\mathsf{ai}\!\uparrow$, suffices.

Proof of Cut Elimination

Theorem

If A is true, then there is an KS *proof* $\begin{array}{c}\Psi \Vert \\ A\end{array}$. *A fortiori, if there is a* SKS *proof of a formula A, then there is a proof without* ai$\uparrow$ *(a cut-free proof), i.e.* ai$\uparrow$ *is admissible for* SKS.

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1. Splitting: First in a predecessor formalism to open deduction, a technique similar to Gentzen's. [2]
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3. The experiments method: a confluent and natural procedure for ai↑ elimination, also using atomic flows.

□

Atomic Flows

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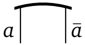
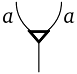
$$\mathsf{ai}\!\downarrow \frac{\mathsf{t}}{a \vee \bar{a}}$$

identity



$$\mathsf{ac}\!\downarrow \frac{a \vee a}{a}$$

contraction

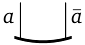


$$\mathsf{aw}\!\downarrow \frac{\mathsf{f}}{a}$$

weakening

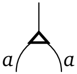
$$\mathsf{ai}\!\uparrow \frac{a \wedge \bar{a}}{\mathsf{f}}$$

cut



$$\mathsf{ac}\!\uparrow \frac{a}{a \wedge a}$$

cocontraction



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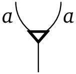
Atomic flows [5] are a geometric invariant of proofs in open deduction. Only structural information about the proof is conserved.

$$\mathsf{ai}{\downarrow}\,\frac{\mathsf{t}}{a \vee \bar{a}}$$ *identity*

$$\mathsf{ac}{\downarrow}\,\frac{a \vee a}{a}$$ *contraction*



$$\mathsf{aw}{\downarrow}\,\frac{\mathsf{f}}{a}$$ *weakening*

$$\mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\mathsf{f}}$$ *cut*

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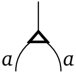
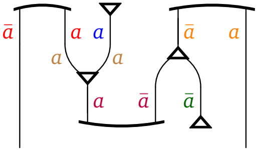


$$\mathsf{aw}{\uparrow}\,\frac{a}{\mathsf{t}}$$ *coweakening*

Composition of proofs naturally corresponds to composition of flows.

Atomic Flow Example

$$
\cfrac{
  \cfrac{\cfrac{\mathsf{t}}{\bar{a} \vee a} \vee \cfrac{\mathsf{f}}{a}}{\bar{a} \vee \cfrac{a \vee a}{a}}
  \wedge
  \cfrac{\mathsf{t}}{\cfrac{\bar{a}}{\bar{a} \wedge \cfrac{\bar{a}}{\mathsf{t}}} \vee a}
}{
  \mathsf{s}^2 \, \cfrac{[\bar{a} \vee a] \wedge [a \vee \bar{a}]}{\bar{a} \vee \mathsf{ai}\!\uparrow \cfrac{a \wedge \bar{a}}{\mathsf{f}} \vee a}
}
$$



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Atomic cut is admissible from SKS

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1. Create $\begin{array}{c} a \\ \Phi_{i+} \Vert \\ A \end{array}$ and $\begin{array}{c} \bar{a} \\ \Phi_{i-} \Vert \\ A \end{array}$ from Φ. These are essentially projections, informed by the atomic flow.

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2. Combine Φ_{i+} and Φ_{i-} : $$\frac{\mathsf{t}}{\dfrac{\boxed{\begin{array}{c} a \\ \Phi_{i+} \| \\ A \end{array}} \vee \boxed{\begin{array}{c} \bar{a} \\ \Phi_{i-} \| \\ A \end{array}}}{A}}$$

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$$\begin{array}{c} \Phi \, \| \\ A \end{array} \quad \to \quad \begin{array}{c} a_1 \wedge \ldots \wedge a_n \\ \Phi_1 \, \| \\ A \end{array} \quad , \ldots , \quad \begin{array}{c} \bar{a}_1 \wedge \ldots \wedge \bar{a}_n \\ \Phi_{2^n} \, \| \\ A \end{array}$$

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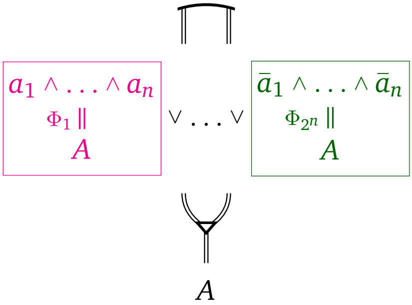

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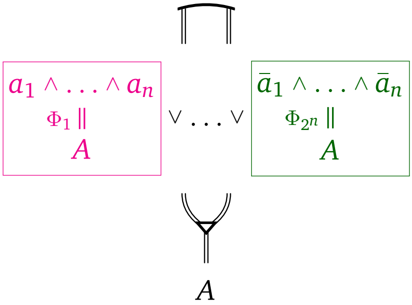

Thus we have, to a certain extent, a straight-forward, high-level understanding of *why* this procedure works.

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- We can use Herbrand's Theorem to help understand the first mechanism. We can use cut reduction to deal with the second.

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- An introduction and elimination rule for quantifiers:

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$$\mathsf{m}_1\!\downarrow \frac{\exists x A \vee \exists x B}{\exists x[A \vee B]} \qquad \mathsf{m}_2\!\downarrow \frac{\forall x A \vee \forall x B}{\forall x[A \vee B]} \qquad \mathsf{m}_1\!\uparrow \frac{\forall x(A \wedge B)}{(\forall x A \wedge \forall x B)} \qquad \mathsf{m}_2\!\uparrow \frac{\exists x(A \wedge B)}{\exists x A \wedge \exists x B}$$

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- Equality rules for vacuous quantification and change of variables.

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$$\begin{array}{c}\Phi_0 \\ \Vert \,\mathsf{SKSq} \\ A\end{array} \xrightarrow{(1)} \begin{array}{c}\Phi_1 \\ \Vert \,\mathsf{KSq}\cup\{\mathsf{i}\uparrow\} \\ A\end{array} \xrightarrow{(2)} \begin{array}{c}\Phi_2 \\ \Vert \,\mathsf{KSq}\cup\{\mathsf{ai}\uparrow,\mathsf{u}\uparrow\} \\ A\end{array} \xrightarrow{(3)} \begin{array}{c}\Phi_3 \\ \Vert \,\mathsf{KSq}\cup\{\mathsf{ai}\uparrow\} \\ A\end{array}$$

$$\begin{array}{c}\Phi_3 \\ \Vert \,\mathsf{KSq}\cup\{\mathsf{ai}\uparrow\} \\ A\end{array} \xrightarrow{(4)} \begin{array}{c}\Phi_4 \\ \Vert \,\mathsf{strKSr}\cup\{\mathsf{ai}\uparrow\} \\ A\end{array} \xrightarrow{(5)} \begin{array}{c}\Phi_5 \\ \Vert \,\mathsf{KSq} \\ A\end{array}$$

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3. u↑ elimination

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1. i↑-elimination sufficient for elimination of up-rules
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1. $\mathsf{i}\!\uparrow$-elimination sufficient for elimination of up-rules
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Herbrand's Theorem

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These ideas have a natural affinity with deep inference: deep contraction of existential formulae is an important (and often overlooked) tool for proving Herbrand's Theorem. [4, 7]

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- Previously this stratification has been seen as a way to prove Herbrand's Theorem as a corollary to cut elimination.
- We now use it as a tool to help us understand cut elimination for first-order logic.

Links to other work

- The study of the relationship between cut elimination and Herbrand's Theorem is not novel.

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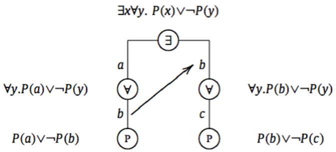
- Heijltjes [6] and McKinley [8] both use graphical methods to expand Miller's approach to include an eliminable cut.



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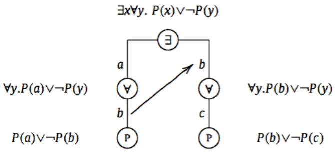

- The compositional properties of open deduction mean that little innovation is needed: we can use these other techniques to design attractive cut-elimination procedures that require no additional syntax.

u$\uparrow$-elimination

- Our predicate logic cut elimination procedure contains three significant steps.

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- The first of these is u↑-elimination: the intuition is that we are converting cuts on quantified formulae to cuts on propositional formulae:

$$\begin{array}{c}\Phi_2 \Big\Vert \mathsf{KSq}\cup\{\mathsf{ai}\uparrow,\mathsf{u}\uparrow\} \\ A\end{array} \xrightarrow{\text{u}\uparrow\text{-elim}} \begin{array}{c}\Phi_3 \Big\Vert \mathsf{KSq}\cup\{\mathsf{ai}\uparrow\} \\ A\end{array}$$

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- The procedure has much in common with certain steps of the cut elimination procedure for Heijltjes's Proof Forests. [6]
- We shall show two key steps of our elimination procedure.

u↑-elimination: ∃-contraction

$$\mathsf{u}\!\uparrow \dfrac{\mathsf{m_1}\!\downarrow \dfrac{\exists x A \vee \exists x B}{\exists x[A \vee B]} \wedge \forall x C}{\exists x([A \vee B] \wedge C)}$$

u↑-elimination: ∃-contraction

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\mathsf{u}{\uparrow}\,\frac{\mathsf{m}_1{\downarrow}\,\dfrac{\exists x A \vee \exists x B}{\exists x [A \vee B]} \wedge \forall x C}{\exists x ([A \vee B] \wedge C)}
\quad\longrightarrow\quad
\mathsf{m}_1{\downarrow}\,\frac{\dfrac{[\exists x A \vee \exists x B] \wedge \mathsf{c}{\uparrow}\,\dfrac{\forall x C}{(\forall x C \wedge \forall x C)}}{\left[\mathsf{u}{\uparrow}\,\dfrac{\exists x A \wedge \forall x C}{\exists x (A \wedge C)} \vee \mathsf{u}{\uparrow}\,\dfrac{\exists x B \wedge \forall x C}{\exists x (B \wedge C)}\right]}}{\exists x \left[\dfrac{(A \wedge C) \vee (B \wedge C)}{[A \vee B] \wedge \dfrac{C \vee C}{C}}\right]}
$$

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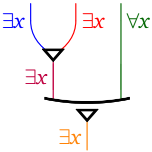
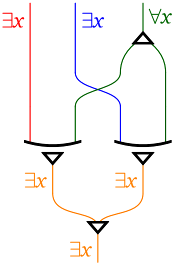


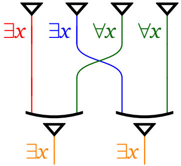
$\longrightarrow$



u↑-elimination: Final Step

Once we have duplicated the universal quantifier, we can permute the u↑ rules up to the introduction rules for the quantifiers, and thus eliminate them.

$$\mathsf{s}^2\,\frac{\left[\mathsf{n}{\downarrow}\,\dfrac{A[\tau/x]}{\exists x A} \vee \mathsf{vq}{=}\,\dfrac{B}{\exists x B}\right] \wedge \left(\mathsf{vq}{=}\,\dfrac{C}{\forall x C} \wedge \mathsf{vq}{=}\,\dfrac{C}{\forall x C}\right)}{\left[\mathsf{u}{\uparrow}\,\dfrac{\exists x A \wedge \forall x C}{\exists x (A \wedge C)} \vee \mathsf{u}{\uparrow}\,\dfrac{\exists x B \wedge \forall x C}{\exists x (B \wedge C)}\right]}$$



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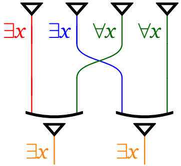

$$\downarrow$$

$$\mathsf{s}^2\ \dfrac{[A[\tau/x] \vee B] \wedge (C \wedge C)}{\left[\mathsf{n}{\downarrow}\,\dfrac{A[\tau/x] \wedge C}{\exists x(A \wedge C)} \vee \mathsf{vq}{=}\,\dfrac{B \wedge C}{\exists x(B \wedge C)}\right]}$$

Herbrand Stratification for KSgr

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\quad \xrightarrow{2} \quad
\begin{array}{l} \Vert \mathsf{KS} \cup \{\mathsf{n}\downarrow, \mathsf{ai}\uparrow\} \\ Q\{B\} \\ \Vert\, \mathsf{gr}\downarrow \\ A' \\ \Vert\, \mathsf{qc}\downarrow \\ A \end{array}
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\overset{2}{\to}
\begin{array}{c} \Pi \, \mathsf{KS} \cup \{\mathsf{n}\downarrow, \mathsf{ai}\uparrow\} \\ Q\{B\} \\ \Vert \, \mathsf{gr}\downarrow \\ A' \\ \Vert \, \mathsf{qc}\downarrow \\ A \end{array}
\overset{3}{\to}
\begin{array}{c} \Pi \, \mathsf{KS} \cup \{\mathsf{ai}\uparrow\} \\ \forall \vec{x}\, W(B) \\ \Vert \, \mathsf{n}\downarrow \\ Q\{B\} \\ \Vert \, \mathsf{gr}\downarrow \\ A' \\ \Vert \, \mathsf{qc}\downarrow \\ A \end{array}
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$$

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More technically: $Q\{B\}$ is a prenexification of a strong $\vee$-expansion of A (A') plus a witnessing substitution W for $Q\{B\}$

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$$

1. $\exists$-contractions to the end of the proof: $\mathsf{qc}\downarrow = \mathsf{qc}\downarrow \dfrac{\exists xA \vee \exists xA}{\exists xA}$
2. $\mathsf{m}_2\downarrow$-elimination, using $\mathsf{gr}\downarrow = \mathsf{gr}\downarrow \dfrac{QxK\{A\}}{K\{QxA\}}$
3. Trivial.

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Conclusions

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Further work

- Develop a natural extension of atomic flows for predicate logic.
- An exploration of further ideas that can be used to design attractive cut elimination procedures for many logics.

Bibliography I

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