Herbrand's Theorem, Expansion Proofs and Deep Inference

Twenty Years of Deep Inference, 2018

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What is Herbrand's theorem about? Why is it of interest?

- A "reduction" of first-order logic to propositional logic.
- A "reduction" of undecidable first-order logic to decidable propositional logic.
- Proof theory: we can obtain a separation of a first-order proof into first-order and propositional parts, joined by a Herbrand disjunction.

An example

There exist two irrational numbers a and b such that a^b is rational.

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$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

Prop: $\overline{\mathbb{Q}}(\sqrt{2}) \wedge \overline{\mathbb{Q}}(\sqrt{2}) \wedge \mathbb{Q}(\sqrt{2}^{\sqrt{2}}) \quad \vee \quad \overline{\mathbb{Q}}(\sqrt{2}^{\sqrt{2}}) \wedge \overline{\mathbb{Q}}(\sqrt{2}) \wedge \mathbb{Q}(2)$

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& \swarrow & & \searrow \\
& \exists a, b(\overline{\mathbb{Q}}(a) \wedge \overline{\mathbb{Q}}(b) \wedge \mathbb{Q}(a^b)) & \vee & \exists a, b(\overline{\mathbb{Q}}(a) \wedge \overline{\mathbb{Q}}(b) \wedge \mathbb{Q}(a^b)) \\
& \downarrow & & \downarrow \\
\textbf{Prop:} & \overline{\mathbb{Q}}(\sqrt{2}) \wedge \overline{\mathbb{Q}}(\sqrt{2}) \wedge \mathbb{Q}(\sqrt{2}^{\sqrt{2}}) & \vee & \overline{\mathbb{Q}}(\sqrt{2}^{\sqrt{2}}) \wedge \overline{\mathbb{Q}}(\sqrt{2}) \wedge \mathbb{Q}(2) \\
& \downarrow & & \downarrow \\
\textbf{Taut:} & \mathbb{Q}(\sqrt{2}^{\sqrt{2}}) & \vee & \overline{\mathbb{Q}}(\sqrt{2}^{\sqrt{2}})
\end{array}
$$

Herbrand Proof (Buss 1991)

Theorem (Herbrand's theorem)

A first-order formula A is valid if and only if A has a Herbrand proof. A Herbrand proof of A consists of a prenexification $A^\star$ of a strong $\vee$-expansion of A plus a witnessing substitution σ for $A^\star$.

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4. Propositional tautology check.

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Sleeper's Formula: There is someone in this room such that, if they are asleep, then everyone in the room is asleep.

$$\exists x[\bar{P}x \vee \forall y Py]$$

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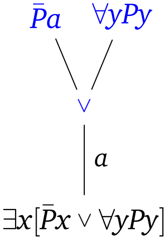

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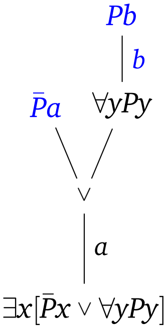

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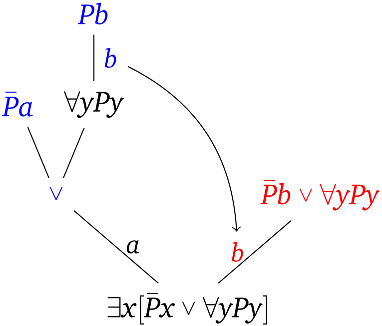

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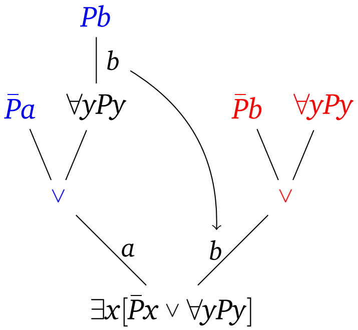

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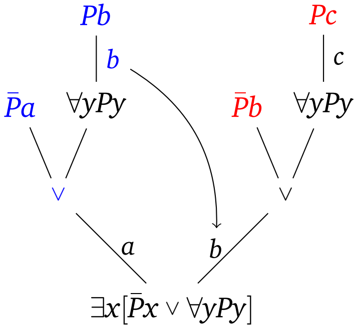

Expansion Proofs (Miller 1987)

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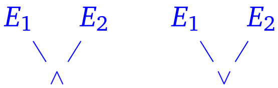

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Expansion trees are recursive structures produced from literal leaves and the following nodes:

- $\wedge$ and $\vee$ nodes:
$$\begin{array}{ccc} E_1 & & E_2 \\ & \diagdown \; \diagup & \\ & \wedge & \end{array} \qquad \begin{array}{ccc} E_1 & & E_2 \\ & \diagdown \; \diagup & \\ & \vee & \end{array}$$
- $\forall$ nodes:
$$\begin{array}{c} E' \\ \big| \, x \\ \forall x A \end{array}$$

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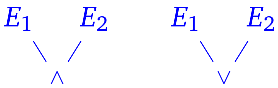
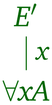
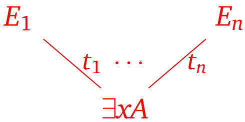
- $\wedge$ and $\vee$ nodes: 

- $\forall$ nodes: 

- $\exists$ nodes: 

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\qquad\qquad
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$$

- $\forall$ nodes:

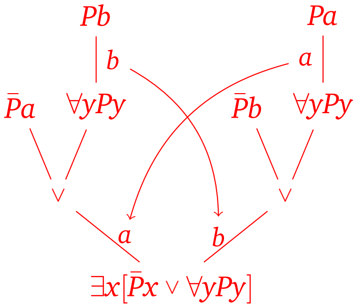
$$
\begin{array}{c} E' \\ \big| \, x \\ \forall x A \end{array}
$$

- $\exists$ nodes:

$$
\begin{array}{ccccc} E_1 & & & & E_n \\ & \diagdown \, t_1 & \cdots & \diagup \, t_n & \\ & & \exists x A & & \end{array}
$$

An expansion tree is correct if the Deep formula is a tautology and the dependency relation on the edges is acyclic.

Incorrect Expansion Tree



Proof Systems

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1. Herbrand Proofs are proofs in the formalism.

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Proof Systems

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1. Herbrand Proofs are proofs in the formalism.

$$\begin{array}{c} \Big\| \, \mathit{Prop} \\ H(A) \\ \Big\| \, \mathit{FO} \\ A \end{array}$$

2. Construction of proofs in the formalism reflects construction of expansion proofs.

$$E_1 \approx \begin{array}{c} \phi_1 \, \| \\ A_1 \end{array} \;\&\; E_2 \approx \begin{array}{c} \phi_2 \, \| \\ A_2 \end{array} \implies \begin{array}{ccc} E_1 & & E_2 \\ e_1 \backslash & & / e_2 \\ & \star & \end{array} \approx \begin{array}{c} \phi_1 \, \| \\ A_1 \end{array} \star \begin{array}{c} \phi_2 \, \| \\ A_2 \end{array}$$

Problems

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Problem: In (Brünnler (2003) it is shown that the following property is impossible to obtain in a sequent calculus system with multiplicative rules.

“Proofs can be separated into two phases (seen bottom-up): The lower phase only contains instances of contraction. The upper phase contains instances of the other rules, but no contraction. No formulae are duplicated in the upper phase.”

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$$\begin{array}{ccc} E_1 & & E_2 \\ e_1 \diagdown & & \diagup e_2 \\ & \vee & \end{array} \quad \approx \quad \vee\,\dfrac{\text{cut}\,\dfrac{\text{w}\,\dfrac{\overset{\Pi_1}{\vdash A_1}}{\vdash A_1, B} \qquad \text{w}\,\dfrac{\overset{\Pi_2}{\vdash A_2}}{\vdash A_2, \bar{B}}}{\vdash A_1, A_2}}{\vdash A_1 \vee A_2}$$

Open Deduction

$$\begin{array}{c} A \\ \Phi \Vert \\ B \end{array} \qquad\qquad \begin{array}{c} C \\ \Psi \Vert \\ D \end{array}$$

Open Deduction

1. Inference Rule $\sigma \in \mathcal{S}$:

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1. Inference Rule $\sigma \in \mathcal{S}$:

$$\begin{array}{c} A \\ \Phi \, \Vert \\ \sigma \, \dfrac{B}{C} \\ \Psi \, \Vert \\ D \end{array}$$

2. Binary Connective $\star \in \{\wedge, \vee\}$:

$$\begin{array}{c} A \\ \Phi \, \Vert \\ B \end{array} \star \begin{array}{c} C \\ \Psi \, \Vert \\ D \end{array} = \begin{array}{c} A \star C \\ \Phi \star \Psi \, \Vert \\ B \star D \end{array}$$

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3. Quantifier $Qx, Q \in \{\forall, \exists\}$:

$$Qx \left[\begin{array}{c} A \\ \phi \, \| \\ B \end{array} \right] = \begin{array}{c} QxA \\ Qx\phi \, \| \\ QxB \end{array}$$

Example

$$
\mathrm{c}\!\downarrow\dfrac{\mathrm{i}\!\uparrow\dfrac{\mathsf{t}}{\mathrm{n}\!\downarrow\dfrac{\forall y_1\left[\mathrm{w}\!\downarrow\dfrac{Py_1}{\bar{P}a \vee Py_1}\right]}{\exists x_1 \forall y_1\left[\bar{P}x_1 \vee Py_1\right]} \vee \exists x_2\left[\mathrm{r3}\!\uparrow\dfrac{\mathrm{w}\!\downarrow\dfrac{\bar{P}x_2}{\bar{P}x_2 \vee \forall y_2 Py_2}}{\forall y_2\left[\bar{P}x_2 \vee Py_2\right]}\right]}}{\exists x \forall y\left[\bar{P}x \vee Py\right]}
$$

Herbrand Proof

1. Expansion of existential subformulae.
2. Prenexification
3. Term assignment.
4. Propositional tautology check.

Inference Rules

1. For expansion of existential subformulae we have the existential contraction rule:

$$\mathsf{qc}{\downarrow}\ \frac{\exists x A \vee \exists x A}{\exists x A}$$

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2. For prenexification we have four rules:

$$\mathsf{r1}\!\downarrow \frac{\forall x[A \vee B]}{\forall x A \vee B} \qquad \mathsf{r2}\!\downarrow \frac{\forall x(A \wedge B)}{(\forall x A \wedge B)} \qquad \mathsf{r3}\!\downarrow \frac{\exists x[A \vee B]}{\exists x A \vee B} \qquad \mathsf{r4}\!\downarrow \frac{\exists x(A \wedge B)}{\exists x A \wedge B}$$

(where B is free for x)

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(where B is free for x)

3. For term assignment we have the rule:

$$\mathsf{n}{\downarrow}\ \frac{A[t/x]}{\exists x A}$$

KS

For propositional tautology check we have the propositional open deduction proof system KS.

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Structural rules

$$\mathsf{ai}{\downarrow}\,\frac{\mathsf{t}}{a \vee \bar{a}} \qquad \mathsf{ac}{\downarrow}\,\frac{a \vee a}{a} \qquad \mathsf{aw}{\downarrow}\,\frac{\mathsf{f}}{a}$$

identity *contraction* *weakening*

Logical rules

$$\mathsf{s}\,\frac{A \wedge [B \vee C]}{(A \wedge B) \vee C} \qquad \mathsf{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]}$$

switch *medial*

KSh1

$$\mathsf{KSh1} = \boxed{\mathsf{KS}\quad \begin{array}{ccc} \mathsf{ai}{\downarrow}\,\dfrac{\mathsf{t}}{a \vee \bar{a}} & \mathsf{ac}{\downarrow}\,\dfrac{a \vee a}{a} & \mathsf{aw}{\downarrow}\,\dfrac{\mathsf{f}}{a} \\[2ex] \mathsf{s}\,\dfrac{A \wedge [B \vee C]}{(A \wedge B) \vee C} & \mathsf{m}\,\dfrac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]} & \end{array}} + \boxed{\begin{array}{ccc} \mathsf{r1}{\downarrow}\,\dfrac{\forall x[A \vee B]}{[\forall x A \vee B]} & \mathsf{r2}{\downarrow}\,\dfrac{\forall x(A \wedge B)}{(\forall x A \wedge B)} & \mathsf{n}{\downarrow}\,\dfrac{A[t/x]}{\exists x A} \\[2ex] \mathsf{r3}{\downarrow}\,\dfrac{\exists x[A \vee B]}{[\exists x A \vee B]} & \mathsf{r4}{\downarrow}\,\dfrac{\exists x(A \wedge B)}{(\exists x A \wedge B)} & \mathsf{qc}{\downarrow}\,\dfrac{\exists x A \vee \exists x A}{\exists x A} \end{array}}$$

Herbrand Proof in KSh1

A KSh1 proof is a *Herbrand proof* if it is in the following form:

$$
\begin{array}{c}
\overset{}{\underset{}{\big\Vert}}\,\mathsf{KS} \\
\forall\vec{x}B[\vec{t}/\vec{y}] \\
\Vert\,\{\mathsf{n}\!\downarrow\} \\
Q\{B\} \\
\Vert\,\{\mathsf{r1}\!\downarrow,\mathsf{r2}\!\downarrow,\mathsf{r3}\!\downarrow,\mathsf{r4}\!\downarrow\} \\
A' \\
\Vert\,\{\mathsf{qc}\!\downarrow\} \\
A
\end{array}
$$

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Q\{B\} \\
\quad \Vert\, \{\mathsf{r1}\downarrow, \mathsf{r2}\downarrow, \mathsf{r3}\downarrow, \mathsf{r4}\downarrow\} \\
A' \\
\quad \Vert\, \{\mathsf{qc}\downarrow\} \\
A
\end{array}
$$

Theorem (Brünnler, 2006)

Every proof in KSh1 *can be converted to a Herbrand proof.*

Proof.

Via cut elimination. □

Herbrand Proof Example

$$
\text{r3}\!\downarrow \dfrac{
\text{n}\!\downarrow \dfrac{
= \dfrac{\mathsf{t}}{\forall y_1 \forall y_2 \left[= \dfrac{\text{ai}\!\downarrow \dfrac{\mathsf{t}}{Py_1 \vee \bar{P}y_1} \vee \left[\text{aw}\!\downarrow \dfrac{\mathsf{f}}{\bar{P}c} \vee \text{aw}\!\downarrow \dfrac{\mathsf{f}}{Py_2} \right]}{[\bar{P}c \vee Py_1] \vee [\bar{P}y_1 \vee Py_2]} \right]}
}{
\exists x_1 \left[\text{r1}\!\downarrow \dfrac{\forall y_1 \left[\text{r3}\!\downarrow \dfrac{\text{n}\!\downarrow \dfrac{[\bar{P}x_1 \vee Py_1] \vee [\bar{P}y_1 \vee Py_2]}{\exists x_2 \left[\text{r1}\!\downarrow \dfrac{\forall y_2 \left[[\bar{P}x_1 \vee Py_1] \vee [\bar{P}x_2 \vee Py_2] \right]}{[\bar{P}x_1 \vee Py_1] \vee \forall y_2 [\bar{P}x_2 \vee Py_2]} \right]}}{[\bar{P}x_1 \vee Py_1] \vee \exists x_2 \forall y_2 [\bar{P}x_2 \vee Py_2]} \right]}{\forall y_1 [\bar{P}x_1 \vee Py_1] \vee \exists x_2 \forall y_2 [\bar{P}x_2 \vee Py_2]} \right]
}
}{
\text{qc}\!\downarrow \dfrac{\exists x_1 \forall y_1 [\bar{P}x_1 \vee Py_1] \vee \exists x_2 \forall y_2 [\bar{P}x_2 \vee Py_2]}{\exists x \forall y [\bar{P}x \vee Py]}
}
$$

More Inference Rules

- $\star$ nodes are simulated by horizontal composition of derivations:

$$\begin{array}{ccc} E_1 & & E_2 \\ & \diagdown \quad \diagup & \\ & \star & \end{array} \qquad \approx \qquad \begin{array}{ccccc} & A & & C & \\ \Phi & \| & \star & \| & \Psi \\ & B & & D & \end{array}$$

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- $\forall$ nodes are simulated by the rules r1$\downarrow$ and r2$\downarrow$:

$$\begin{array}{c} E' \\ \big| \, x \\ \forall x A \end{array} \quad \approx \quad \text{r1}\!\downarrow \frac{\forall x[A \vee B]}{\forall x A \vee B} \qquad \text{r2}\!\downarrow \frac{\forall x(A \wedge B)}{\forall x A \wedge B}$$

More Inference Rules

- $\star$ nodes are simulated by horizontal composition of derivations:

$$
\begin{tikzcd}[column sep=small]
E_1 \arrow[dr, dash] & & E_2 \arrow[dl, dash] \\
& \star &
\end{tikzcd}
\quad \approx \quad
\begin{array}{ccc}
A & & C \\
\Phi \, \Big\| & \star & \Big\| \, \Psi \\
B & & D
\end{array}
$$

- $\forall$ nodes are simulated by the rules $\mathsf{r1}\!\downarrow$ and $\mathsf{r2}\!\downarrow$:

$$
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E' \\
\Big| \, x \\
\forall x A
\end{array}
\quad \approx \quad
\mathsf{r1}\!\downarrow \frac{\forall x[A \vee B]}{\forall x A \vee B}
\qquad
\mathsf{r2}\!\downarrow \frac{\forall x(A \wedge B)}{\forall x A \wedge B}
$$

- $\exists$ nodes are simulated by $\mathsf{h}\!\downarrow$, the *Herbrand expander*:

$$
\begin{array}{c}
E' \\
\Big| \, t \\
\exists x A
\end{array}
\quad \approx \quad
\mathsf{h}\!\downarrow \frac{\exists x A \vee A[t/x]}{\exists x A}
$$

KSh2

$$\mathsf{KSh2} = \begin{array}{c} \mathsf{KS}\ \boxed{\begin{array}{ccc} \mathsf{ai}\!\downarrow \dfrac{\mathsf{t}}{a \vee \bar{a}} & \mathsf{ac}\!\downarrow \dfrac{a \vee a}{a} & \mathsf{aw}\!\downarrow \dfrac{\mathsf{f}}{a} \\[2em] \multicolumn{3}{c}{\mathsf{s}\,\dfrac{A \wedge [B \vee C]}{(A \wedge B) \vee C} \quad \mathsf{m}\,\dfrac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]}} \end{array}} \\ + \\ \boxed{\begin{array}{cc} \mathsf{r1}\!\downarrow \dfrac{\forall x [A \vee B]}{[\forall x A \vee B]} & \mathsf{r2}\!\downarrow \dfrac{\forall x (A \wedge B)}{(\forall x A \wedge B)} \\[2em] \mathsf{h}\!\downarrow \dfrac{\exists x A \vee A[t/x]}{\exists x A} & \exists\mathsf{w}\!\downarrow \dfrac{\mathsf{f}}{\exists x A} \end{array}} \end{array}$$

Herbrand Normal Form

A KSh2 proof of the form below is said to be in *Herbrand Normal Form*:

$$
\begin{array}{c}
\Big\Vert \, \mathsf{KS} \\
\forall \vec{x} H_{\phi}(A) \\
\Big\Vert \, \{\exists \mathsf{w}{\downarrow}\} \\
\forall \vec{x} H^{+}_{\phi}(A) \\
\Big\Vert \, \{\mathsf{r1}{\downarrow}, \mathsf{r2}{\downarrow}, \mathsf{h}{\downarrow}\} \\
A
\end{array}
$$

HNF Proof Example

$$
\mathsf{r1}{\downarrow}\,\dfrac{=\dfrac{=\dfrac{\mathsf{t}}{\forall y_1 \forall y_2 \left[\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{t}}{Py_1 \vee \bar{P}y_1} \vee \left[\mathsf{aw}{\downarrow}\,\dfrac{\mathsf{f}}{\bar{P}c} \vee \mathsf{aw}{\downarrow}\,\dfrac{\mathsf{f}}{Py_2} \right] \right]}}{\forall y_1 \left[\mathsf{r1}{\downarrow}\,\dfrac{\forall y_2 \left[\left[\left[\exists\mathsf{w}{\downarrow}\,\dfrac{\mathsf{f}}{\exists x \forall y \left[\bar{P}x \vee Py\right]} \vee \left[\bar{P}y_1 \vee Py_2\right] \right] \right] \vee \left[\bar{P}c \vee Py_1\right] \right]}{\mathsf{h}{\downarrow}\,\dfrac{\mathsf{r1}{\downarrow}\,\dfrac{\forall y_2 \left[\exists x \forall y \left[\bar{P}x \vee Py\right] \vee \left[\bar{P}y_1 \vee Py_2\right]\right]}{\exists x \forall y \left[\bar{P}x \vee Py\right] \vee \forall y_2 \left[\bar{P}y_1 \vee Py_2\right]}}{\exists x \forall y \left[\bar{P}x \vee Py\right]} \vee \left[\bar{P}c \vee Py_1\right]} \right]}}{\mathsf{h}{\downarrow}\,\dfrac{\exists x \forall y \left[\bar{P}x \vee Py\right] \vee \forall y_1 \left[\bar{P}c \vee Py_1\right]}{\exists x \forall y \left[\bar{P}x \vee Py\right]}}
$$

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A formula has a proof in HNF iff it has a Herbrand proof.

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$$\begin{array}{c} \mathit{HP}\,\Vert \\ A \end{array} \quad \longleftrightarrow \quad \begin{array}{c} \mathit{HNF}\,\Vert \\ A \end{array} \quad \longleftrightarrow \quad \begin{array}{c} \mathit{EP}\,\Vert \\ A \end{array}$$

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A Natural Proof System for Herbrand's Theorem (2018)

Further Work

- Herbrand's Theorem as decomposition: $\begin{array}{l} \|\,\mathsf{SKSq} \\ A \end{array} \longrightarrow \begin{array}{r} \color{green}{\mathit{HNF}}\,\| \\ A \end{array}$

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- Herbrand's Theorem as decomposition: $\begin{array}{c} \Vert\,\mathsf{SKSq} \\ A \end{array} \longrightarrow \begin{array}{c} \color{green}{HNF}\,\Vert \\ A \end{array}$
- Comparing with the cut elimination procedures for expansion proofs in Heijltjes (2010), Alcolei et al. (2017).
- Situating this work in the context of recent work by Aler Tubella and Guglielmi on a general theory of normalisation for open deduction.

It's coming home!

Football's coming home.