Confluent and Natural Cut Elimination in Classical Logic

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Cut elimination in classical logic is widely regarded as intrinsically non-confluent because of the 'Lafont counterexample' [3]: eliminating the cut in

$$\dfrac{\dfrac{\Pi_1}{A}\,\mathsf{w}\dfrac{}{A,B} \qquad \dfrac{\Pi_2}{A}\,\mathsf{w}\dfrac{}{A,\bar{B}}}{A}\,\mathsf{cut}$$

requires choosing between Π_1 and Π_2. Since there is no natural choice between the two, the only way to obtain canonical cut-free proofs in classical logic is by imposing a strategy on the normalisation procedure. The underlying assumption, of course, is that cut elimination has to be performed in a Gentzen system. However, Gentzen systems were not designed for proof semantics and computational interpretations of proofs, so it should not be surprising that they do not possess the desired computational properties.

Deep inference is being developed as a modern alternative to Gentzen systems [4], and it does offer a solution to the canonicity problem, as we show in this work. Deep inference stipulates that proofs can be composed by the same connectives used to compose formulae [5]. For example, if $\phi = \begin{array}{c}A\\ \|\\ B\end{array}$ and $\psi = \begin{array}{c}C\\ \|\\ D\end{array}$ are two proofs whose premisses are A and C and conclusions are B and D, then $\phi \wedge \psi = \begin{array}{c}A \wedge C\\ \|\\ B \wedge D\end{array}$ and $\phi \vee \psi = \begin{array}{c}A \vee C\\ \|\\ B \vee D\end{array}$ are valid proofs with premisses $A \wedge C$ and $A \vee C$, and conclusions $B \wedge D$ and $B \vee D$. It turns out that, as a nontrivial but direct result of this stipulation, every cut instance can be transformed into several atomic cut instances by a local procedure of polynomial-size complexity [1]. Significantly for this work, while $\phi \wedge \psi$ can be represented in Gentzen, $\phi \vee \psi$ cannot. This is very unfortunate and it is the reason behind Lafont's counterexample.

One way to achieve cut elimination in deep inference uses structures called *atomic flows*, which are obtained by tracing all the atom occurrences in a proof [6]. Atomic flows yield a more general normal form than the cut free one, but so far, as in Gentzen theory, we obtained neither a canonical form nor a semantically natural one.

In this work, which is heavily inspired by atomic flows, we show that there is indeed a confluent cut elimination procedure with a natural semantic justification. We proceed in two phases: we first tackle the propositional case with a construction called the *experiments method*, and then we lift it to the predicate calculus, using the notion of a *Herbrand proof* [2].

The experiments method. Take a proof ϕ of the propositional formula A. Trace all the atom occurrences in the proof (atomic flows are convenient for this). For every atom a_1 (connected component in the atomic flow) replace its occurrences with truth values, so producing two 'experiment' derivations: one that proves A from a_1 and one that proves A from $\bar{a}_1$. By fixing the truth value of a all the cuts on a become $(\mathsf{t} \wedge \mathsf{f})/\mathsf{f}$ and vanish. Proceed recursively on $a_2, \ldots, a_n$, producing the derivations $\phi_1, \ldots, \phi_{2^n}$. Build the cut free derivation $\psi = \begin{array}{c}B_1\\ \phi_1\|\\ A\end{array} \vee \cdots \vee \begin{array}{c}B_{2^n}\\ \phi_{2^n}\|\\ A\end{array}$, where the B_is are conjunctions of atoms, each representing one assignment. Since $B_1 \vee \cdots \vee B_{2^n}$ is valid, complete ψ into a proof of A, which is unique modulo associativity and commutativity.

Lifting to the predicate calculus. Given a first order proof of A, permute down certain inference steps, including, crucially, *existential contractions*: $\dfrac{\exists x B \vee \exists x B}{\exists x B}$. This process of stratification is naturally confluent and separates the propositional aspect of the proof from that which is intrinsically first-order. *I.e.*, it performs $\begin{array}{c}\phi\|\\ A\end{array} \rightarrow \begin{array}{c}\|\text{prop. rules}\\ H(A)\\ \|\text{quant. rules}\\ A\end{array}$, where $H(A)$ is a *Herbrand proof* of A. As ϕ is propositional, the experiments method can be used on it to give a cut-free proof of $H(A)$ and thus also of A.

This procedure naturally respects the semantic, disjunctive nature of the existential quantifier–precisely what is not possible with Gentzen methods.

References

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