A Ciesielski-Taylor type identity for positive self-similar Markov processes

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on $(0, \infty)$ for $f \in C^2(0, \infty)$ with instantaneous reflection at 0 when $\nu \in (0, 2)$ (i.e. $f'(0^+) = 0$) and when $\nu \geq 2$ the origin is an entrance-non-exit boundary point.

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- Ciesielski and Taylor (1962) and later Getoor and Sharp (1979) show: For $a > 0$ and any $\nu > 0$,

$$\left(T_a, Q^{(\nu)}\right) \stackrel{(d)}{=} \left(\int_0^\infty \mathbb{I}_{\{X_s \leq a\}} ds, Q^{(\nu+2)}\right) \tag{1}$$

  where

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One begins to get the whiff of the possibility of a general result for positive self-similar Markov processes.

Positive self-similar Markov process

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- α-pssMp: A $[0,\infty)$-valued Feller process which enjoys the following α-self-similarity property, where $\alpha > 0$. For any $x > 0$, and $c > 0$,

$$((X_t)_{t\geq 0}, \mathrm{P}_{cx}) \overset{(d)}{=} ((cX_{c^{-\alpha}t})_{t\geq 0}, \mathrm{P}_x)\,.$$

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- Lamperti (1972) showed that, for any $x \in \mathbb{R}$, there exists a one to one mapping between $\mathbb{P}_x$, the law of a generic Lévy process (possibly killed at an independent and exponentially distributed time), say $\xi = (\xi_t : t \geq 0)$, starting from x, and the law P_{e^x} via the relation

$$X_t = e^{\xi_{A_t}},\, 0 \leq t < \zeta,$$

where $\zeta = \inf\{t > 0 : X_t = 0\}$ and

$$A_t = \inf\{s \geq 0; \int_0^s e^{\alpha \xi_u}\, du > t\}.$$

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- When $\mathbb{E}(\xi_1) \geq 0$ and ξ is not killed, one may extend the definition of X to include the case that it is issued from the origin by establishing its entrance law P_0 as the weak limit with respect to the Skorohod topology of P_x as $x \downarrow 0$. Bertoin and Yor (2002).

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- When $\mathbb{E}(\xi_1) < 0$ (resp. ξ is killed) then the boundary state 0 is reached continuously (resp. by a jump). In these two cases, one cannot construct an entrance law, however, Rivero (2005) and Fitzsimmons (2006), show that it is possible instead to construct a unique recurrent extension on $[0, \infty)$ such that paths leave 0 continuously, thereby giving a meaning to P_0, if and only if there exists a $\theta \in (0, \alpha)$ such that $\mathbb{E}(e^{\theta \xi_1}) = 1$.

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- Objective of this talk: Fix $\alpha > 0$ and show that for a given spectrally negative Lévy process fitting the previous two categories, and hence given the associated law P_0,

$$(T_a, \mathrm{P}_0) \stackrel{(d)}{=} \left(\int_0^\infty \mathbb{I}_{\{X_s \leq a\}} \, ds, \mathrm{P}_0^* \right),$$

where $T_a = \inf\{t > 0 : X_t = a\}$ and P_0^* is to be identified.

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- For any spectrally negative Lévy process, henceforth denoted by $\xi = (\xi_t, t \geq 0)$, we define simultaneously the Laplace exponent and its law $\mathbb{P}^\psi$ by

$$\psi(u) = \log \mathbb{E}^\psi(\exp\{u\xi_1\}), \ \ u \geq 0.$$

Warning: included in the definition of ξ is the possibility of exponential killing!

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- Theorem: $\mathcal{T}_\beta\psi$ is the Laplace exponent of another spectrally negative Lévy process with no exponential killing.
- Easy proof when ψ has no killing and $\psi'(0+) \geq 0$:
Algebra: $\mathcal{T}_\beta\psi(u) = \mathcal{E}_\beta\psi(u) - \beta\mathcal{E}_\beta\phi(u)$

$\mathcal{E}_\beta f(u) = f(u+\beta) - f(\beta)$ (Esscher transform)

$\phi(u) = \psi(u)/u$ (Laplace exponent of descending ladder height subordinator).

Ciesielski-Taylor identity for spectrally negative pssMp

Theorem

Fix $\alpha > 0$. Suppose that ψ is the Laplace exponent of a spectrally negative Lévy process. Assume that θ, the largest root in $[0, \infty)$ of the equation $\psi(\theta) = 0$, satisfies $\theta < \alpha$. Let P_0^{ψ} be the law of the pssMp associated with $\mathbb{P}^{\psi}$ and issued from 0. Then for any $a > 0$, the following Ciesielski-Taylor type identity in law

$$\left(T_a, \mathrm{P}_0^{\psi}\right) \stackrel{(d)}{=} \left(\int_0^{\infty} \mathbb{I}_{\{X_s \leq a\}} ds, \mathrm{P}_0^{\mathcal{T}_\alpha \psi}\right)$$

holds.

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- Define the positive and entire function $\mathcal{I}_{\psi,\alpha}(z)$ which admits the series representation

$$\mathcal{I}_{\psi,\alpha}(z) = \sum_{n=0}^{\infty} a_n(\psi;\alpha) z^n, \quad z \in \mathbb{C}.$$

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- Theorem (Patie): For $0 \leq x \leq a$ and $q \geq 0$, we have

$$\mathrm{E}_x^{\psi}\left[e^{-qT_a}\right] = \frac{\mathcal{I}_{\psi,\alpha}(qx^{\alpha})}{\mathcal{I}_{\psi,\alpha}(qa^{\alpha})}.$$

In particular

$$\mathrm{E}_0^{\psi}\left[e^{-qT_a}\right] = \frac{1}{\mathcal{I}_{\psi,\alpha}(qa^{\alpha})}.$$

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- Define

$$O_q^{\mathcal{T}_\alpha \psi}(x;a) = \mathrm{E}_x^{\mathcal{T}_\alpha \psi}\left[e^{-q\int_0^\infty \mathbb{I}_{\{X_s \leq a\}}\, ds}\right]$$

and try to show that $O_q^{\mathcal{T}_\alpha \psi}(0;a) = \mathrm{E}_0^{\psi}\left[e^{-qT_a}\right] = 1/\mathcal{I}_{\psi,\alpha}(qa^\alpha)$ for all $q \geq 0$.

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- Self-similarlty of X means that

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so enough to establish the suggested equality when $a = 1$.

- Next, note using spectral negativity,

$$O_q^{\mathcal{T}_\alpha\psi}(0;1) = \mathrm{E}_0^{\mathcal{T}_\alpha\psi}(e^{-qT_1})\,O_q^{\mathcal{T}_\alpha\psi}(1;1)$$

so it is enough to show that

$$O_q^{\mathcal{T}_\alpha\psi}(1;1) = \frac{\mathcal{I}_{\mathcal{T}_\alpha\psi,\alpha}(q)}{\mathcal{I}_{\psi,\alpha}(q)}.$$

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- Fix $y > 1$.
- Use Strong Markov Property, $(\mathcal{T}_\alpha\psi)'(0^+) = \psi(\alpha)/\alpha > 0$ and spectral negativity:

$$
\begin{aligned}
& O_q^{\mathcal{T}_\alpha\psi}(1;1) \\
= {} & \mathrm{E}_1^{\mathcal{T}_\alpha\psi}\left[e^{-q\int_0^{T_y}\mathbb{I}_{\{X_s\le 1\}}\,ds}\right]\left(\mathrm{E}_y^{\mathcal{T}_\alpha\psi}\left[\mathbb{I}_{\{\tau_1<\infty\}}\mathrm{E}_{X_{\tau_1}}^{\mathcal{T}_\alpha\psi}\left[e^{-qT_1}\right]\right]O_q^{\mathcal{T}_\alpha\psi}(1;1)+\mathrm{E}_y^{\mathcal{T}_\alpha\psi}\left[\mathbb{I}_{\{\tau_1=\infty\}}\right]\right)
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$$O_q^{\mathcal{T}_\alpha\psi}(1;1)$$
$$= \mathrm{E}_1^{\mathcal{T}_\alpha\psi}\left[e^{-q\int_0^{T_y}\mathbb{I}_{\{X_s\le 1\}}ds}\right]\left(\mathrm{E}_y^{\mathcal{T}_\alpha\psi}\left[\mathbb{I}_{\{\tau_1<\infty\}}\mathrm{E}_{X_{\tau_1}}^{\mathcal{T}_\alpha\psi}\left[e^{-qT_1}\right]\right]O_q^{\mathcal{T}_\alpha\psi}(1;1) + \mathrm{E}_y^{\mathcal{T}_\alpha\psi}\left[\mathbb{I}_{\{\tau_1=\infty\}}\right]\right)$$

- Solving for $O_q^{\mathcal{T}_\alpha\psi}(1;1)$ we get

$$O_q^{\mathcal{T}_\alpha\psi}(1;1) = \frac{\mathrm{E}_y^{\mathcal{T}_\alpha\psi}\left[\mathbb{I}_{\{\tau_1=\infty\}}\right]}{\left\{\mathrm{E}_1^{\mathcal{T}_\alpha\psi}\left[e^{-q\int_0^{T_y}\mathbb{I}_{\{X_s\le 1\}}ds}\right]\right\}^{-1} - \mathrm{E}_y^{\mathcal{T}_\alpha\psi}\left[\mathbb{I}_{\{\tau_1<\infty\}}\mathrm{E}_{X_{\tau_1}}^{\mathcal{T}_\alpha\psi}\left[e^{-qT_1}\right]\right]}.$$

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- Proof is formalised by taking limits as $y \downarrow 1$.

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- In which case, $\mathrm{E}_1^{\mathcal{T}_\alpha \psi}\left[e^{-q \int_0^{T_1} \mathbb{I}_{\{X_s \leq 1\}} ds}\right] = 1$.

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- Hence $\mathrm{P}_1^{\mathcal{T}_\alpha\psi}(\tau_1 > 0) = 1$. Hence can just set $y = 0$ in the formula for $O_q^{\mathcal{T}_\alpha\psi}(1;1)$.
- In which case, $\mathrm{E}_1^{\mathcal{T}_\alpha\psi}\left[e^{-q\int_0^{T_1}\mathbb{I}_{\{X_s\le 1\}}ds}\right] = 1$.
- On the one hand, recalling that $(\mathcal{T}_\alpha\psi)'(0^+) = \psi(\alpha)/\alpha > 0$, we observe that

$$\mathrm{E}_1^{\mathcal{T}_\alpha\psi}\left[\mathbb{I}_{\{\tau_1=\infty\}}\right] = \mathbb{P}_0^{\mathcal{T}_\alpha\psi}\left(\tau_0^\xi = \infty\right) = \frac{1}{(\mathcal{T}_\alpha\psi)'(0^+)}W_{\mathcal{T}_\alpha\psi}(0+) = \frac{\psi(\alpha)}{\alpha}W_{\mathcal{T}_\alpha\psi}(0+) > 0$$

Here $W_{\mathcal{T}_\alpha\psi}$ is the scale function under $\mathbb{P}^{\mathcal{T}_\alpha\psi}$. In other words it is the unique continuous function on $[0,\infty)$ whose Laplace transform satisfies

$$\int_0^\infty e^{-ux}W_{\mathcal{T}_\alpha\psi}(x)\,dx = \frac{1}{\mathcal{T}_\alpha\psi(u)}.$$

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- For the remaining term, by Fubini's theorem (recalling the positivity of coefficients in the definition of $\mathcal{I}_{\mathcal{T}_\alpha\psi,\alpha}$), we have with the help of Patie's identity,

$$\begin{aligned}
&\mathrm{E}_1^{\mathcal{T}_\alpha\psi}\left[\mathbb{I}_{\{\tau_1<\infty\}}\mathrm{E}_{X_{\tau_1}}^{\mathcal{T}_\alpha\psi}\left[e^{-qT_1}\right]\right] \\
&= \mathrm{E}_1^{\mathcal{T}_\alpha\psi}\left[\frac{\mathcal{I}_{\mathcal{T}_\alpha\psi,\alpha}(qX_{\tau_1}^{\alpha})\mathbb{I}_{\{\tau_1<\infty\}}}{\mathcal{I}_{\mathcal{T}_\alpha\psi,\alpha}(q)}\right] \\
&= \frac{1}{\mathcal{I}_{\mathcal{T}_\alpha\psi,\alpha}(q)}\mathrm{E}_1^{\mathcal{T}_\alpha\psi}\left[\sum_{n=0}^{\infty} a_n(\mathcal{T}_\alpha\psi;\alpha)q^n X_{\tau_1}^{\alpha n}\mathbb{I}_{\{\tau_1<\infty\}}\right] \\
&= \frac{1}{\mathcal{I}_{\mathcal{T}_\alpha\psi,\alpha}(q)}\sum_{n=0}^{\infty} a_n(\mathcal{T}_\alpha\psi;\alpha)q^n \mathbb{E}_0^{\mathcal{T}_\alpha\psi}\left[e^{\alpha n\xi_{\tau_0^\xi}}\mathbb{I}_{\{\tau_0^\xi<\infty\}}\right].
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&\quad = \frac{1}{\mathcal{I}_{\mathcal{T}_\alpha\psi,\alpha}(q)}\sum_{n=0}^{\infty} a_n(\mathcal{T}_\alpha\psi;\alpha)q^n \mathbb{E}_0^{\mathcal{T}_\alpha\psi}\left[e^{\alpha n\xi_{\tau_0^\xi}}\mathbb{I}_{\{\tau_0^\xi<\infty\}}\right].
\end{aligned}$$

- Classical fluctuation theory for spectrally negative Lévy processes gives (in the bounded variation case):

$$\mathbb{E}_0^{\mathcal{T}_\alpha\psi}(e^{u\xi_{\tau_0^\xi}}\mathbb{I}_{\{\tau_0^\xi<\infty\}}) = 1 - \frac{\mathcal{T}_\alpha\psi(u)}{u}W_{\mathcal{T}_\alpha\psi}(0+).$$

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- Putting the bits together

$$
\begin{aligned}
&O_q^{\mathcal{T}_\alpha \psi}(1;1)\\
&= \frac{\psi(\alpha)\, W_{\mathcal{T}_\alpha\psi}(0+)/\alpha}{1 - \frac{\sum_{n=0}^{\infty} a_n(\mathcal{T}_\alpha\psi;\alpha)\, q^n \left\{1 - \frac{\mathcal{T}_\alpha\psi(\alpha n)}{\alpha n}\, y^{-\alpha n}\, W_{\mathcal{T}_\alpha\psi}(0+)\right\}}{\mathcal{I}_{\mathcal{T}_\alpha\psi,\alpha}(q)}}\\
&= \frac{\psi(\alpha)\mathcal{I}_{\mathcal{T}_\alpha\psi,\alpha}(q)}{\alpha\sum_{n=0}^{\infty} a_n(\mathcal{T}_\alpha\psi;\alpha)\, q^n\, \frac{\mathcal{T}_\alpha\psi(\alpha n)}{\alpha n}}.
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$$= \frac{\psi(\alpha)\mathcal{I}_{\mathcal{T}_\alpha\psi,\alpha}(q)}{\alpha\sum_{n=0}^{\infty} a_n(\mathcal{T}_\alpha\psi;\alpha)\, q^n\, \frac{\mathcal{T}_\alpha\psi(\alpha n)}{\alpha n}}.$$

- Next, observing that, for any $n \geq 1$,

$$\frac{\psi(\alpha)\alpha n}{\alpha\mathcal{T}_\alpha\psi(\alpha n)}\, a_n(\mathcal{T}_\alpha\psi;\alpha)^{-1} = \prod_{k=1}^{n} \psi(\alpha k) \quad (= 1 \text{ when } n = 0).$$

we deduce, as required, the identity

$$O_q^{\mathcal{T}_\alpha\psi}(1;1) = \frac{\mathcal{I}_{\mathcal{T}_\alpha\psi,\alpha}(q)}{\sum_{n=0}^{\infty} a_n(\psi;\alpha)\, q^n} = \frac{\mathcal{I}_{\mathcal{T}_\alpha\psi,\alpha}(q)}{\mathcal{I}_{\psi,\alpha}(q)}.$$

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- Note that the root θ is zero for $\nu \geq 2$ and when $\nu \in (0, 2)$ we have $\theta = 2 - \nu < 2 = \alpha$ thereby fulfilling the required condition of the main result.
- The transformation $\mathcal{T}_2$ gives us the new Laplace exponent

$$\mathcal{T}_2\psi_\nu(u) = \frac{1}{2}u^2 + \frac{\nu}{2}u = \psi_{\nu+2}(u).$$