Travelling waves for fragmentation processes.

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- **Super-diffusions:** Markov process $X = \{X_t : t \geq 0\}$ such that X_t is a measure on $\mathbb{R}$, its probabilities denoted by $\mathbb{P}_\mu$ for measures μ on $\mathbb{R}$ where $X_0 = \mu$.

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- **Branching property:** For two initial measures μ_1, μ_2, $\mathbb{P}_{\mu_1+\mu_2} = \mathbb{P}_{\mu_1} \star \mathbb{P}_{\mu_2}$.
- **Non-linear semi-group:** "Infinite divisibility" in the branching property suggests the natural object to describe the semi-group of is the Laplace functional

$$\exp\{-u_f(x,t)\} = \mathbb{E}_{\delta_x}(\exp\{-\langle f, X_t\rangle\})$$

where $f : \mathbb{R} \to [0,\infty)$, $\langle f, X_t\rangle = \int_{\mathbb{R}} f(y) X_t(\mathrm{d}y)$ and one finds

$$\frac{\partial}{\partial t} u_f(x,t) = L u_f(x,t) - \psi(u_f(x,t)) \qquad \text{with} \qquad u_f(x,0) = f(x),$$

where L is the infinitesimal generator of the "underlying motion" and ψ necessarily respects the Lévy-Khintchine formula,

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}})\nu(\mathrm{d}x)$$

for $\lambda \geq 0$ where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and ν is a measure concentrated on $(0,\infty)$ which satisfies $\int_{(0,\infty)} (1 \wedge x^2)\nu(\mathrm{d}x) < \infty$.

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- **Multiplicative martingales:** Look for positive monotone "travelling" solutions with speed $c \in \mathbb{R}$, i.e. $u_f(x,t) = f(x - ct)$ and consequently $Lf + cf' - \psi(f) = 0.$ Let X^c be the super-diffusion with added linear drift c to the support, then the associated motion operator is $L + c\frac{d}{dx}$ and

$$e^{-f(x)} = \mathbb{E}_{\delta_x}(e^{-\langle f, X_t^c\rangle}) \Rightarrow e^{-\langle f, X_t^c\rangle} \text{ is a martingale.}$$

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- **Additive martingales:** Look for "travelling" solutions of the form $v_g(x,t) = g(x-ct)$, i.e. $Lg + cg' - \psi'(0)g = 0.$ Then,

$$g(x) = \mathbb{E}_{\delta_x}(\langle g, X^c_t\rangle) \Rightarrow \langle g, X^c_t\rangle \text{ is a martingale.}$$

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- **Martingale limits:** Positive martingales have limits so what does the relation between $\lim_{t\uparrow\infty}\langle f, X^c_t\rangle$, $\lim_{t\uparrow\infty}\langle g, X^c_t\rangle$ tell us (about f and g)??

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- (McKean/Neveu/Chauvin/Lalley-Sellke/Harris/K./Murillo/Liu/Ren) All this works for branching Brownian motion/super-Brownian motion ($\psi(\lambda) = -\mathtt{a}\lambda + \mathtt{b}\lambda^2$), in which case we see that for $\lambda \in \mathbb{R}$, one may take $g(x) = e^{-\lambda x}$ and $c = c_\lambda = \lambda/2 + \mathtt{a}/\lambda$. Monotone travelling waves exist uniquely up to linear shift in the argument if and only if $|c_\lambda| \geq \sqrt{2\mathtt{a}}$ in which case, when $|\lambda| < \sqrt{2\mathtt{a}}$ ($\Rightarrow |c_\lambda| > \sqrt{2\mathtt{a}}$),

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- (Durrett/Liggett/Biggins/K./Liu) For BRW, if positions at generation n are given by $\{\zeta_i^n : i \geq 1\}$ then a "travelling wave" $\phi : \mathbb{R} \to [0,1]$ is a solution to the functional equation

$$\phi(x) = \mathbb{E}\prod_i \phi(x + \zeta_i^n + cn)$$

  and can be similarly analysed by comparing against the behaviour of Biggins' martingale $W_n(\lambda) := \sum_i e^{-\lambda\zeta_i^n}/m(\lambda)^n$.

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- **Notation:** Its probabilities are denoted by $\{\mathbb{P}_{\mathbf{s}} : \mathbf{s} \in \nabla\}$ and, for $s \in (0, 1]$, we shall reserve the special notation $\mathbb{P}_s$ as short hand for $\mathbb{P}_{(s,0,\cdots)}$ and in particular write $\mathbb{P}$ for $\mathbb{P}_1$.

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- **Markov (fragmentation) property:** Given that $\mathbf{X}(t) = (s_1, s_2, \cdots)$, where $t \geq 0$, then for $u > 0$, $\mathbf{X}(t + u)$ has the same law as the variable obtained by ranking in decreasing order the sequences $\mathbf{X}^{(1)}(u), \mathbf{X}^{(2)}(u), \cdots$ where the latter are independent, random mass partitions with values in ∇ having the same distribution as $\mathbf{X}(u)$ under $\mathbb{P}_{s_1}, \mathbb{P}_{s_2}, \cdots$ respectively.

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- **Rate of fragmentation:** Fragmentation is governed by a measure ν on ∇ such that an individual block of mass $s \leq 1$ in the process $\mathbf{X}$ at time t will dislocate into an array of fragments $s \times \mathbf{s}$ with rate $\nu(\mathrm{d}\mathbf{s}) \times \mathrm{d}t + o(\mathrm{d}t)$.

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- As $h \downarrow 0$

$$\begin{aligned} & u(x,t+h) - u(x,t) \\ &= \mathbb{E}\left(\prod_i u(x - \log X_i(h), t)\right) - u(x,t) \\ &= \int_\nabla \left\{\prod_i u(x - \log s_i, t) - u(x,t)\right\} \nu(\mathrm{ds}) h + o(h). \end{aligned}$$

Travelling wave equation for fragmentation ctd...

- This suggestively leads us to the integro-differential equation, the KPP equation for fragmentation processes:

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- Hence a travelling wave $\psi : \mathbb{R} \to [0,1]$ with wave speed $c \in \mathbb{R}$ solves the equation

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- With some further restriction on the class in which ψ sits, one can show through stochastic calculus for semi-martingales (Poisson random fields) that ψ is a travelling wave with speed c iff

$$M_t(c) := \prod_i \psi(x - \log X_i(t) - ct),\ t \geq 0$$

is a martingale.

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- For each $t \geq 0$, $\mathbf{X}(t)$ is a (random) probability distribution,

$$\mathbb{E}\left(\sum_i X_i(t) g(-\log X_i(t))\right) = E(g(\xi_t))$$

where $\{\xi_t : t \geq 0\}$ under P is a pure jump subordinator with Laplace exponent

$$-\frac{1}{t}\log E(e^{-q\xi_t}) = \Phi(q) = \int_{\nabla_1}\left(1 - \sum_{i=1}^{\infty} s_i^{q+1}\right)\nu(ds),\ \ q > \underline{p},$$

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- Without major restriction, we assume $\underline{p} < 0$ and that $\Phi(\underline{p}) = -\infty$.

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- **Range of speeds:** Let $c_p = \Phi(p)/(p+1)$. There exists a unique solution to the equation $(p+1)\Phi'(p) = \Phi(p)$, denoted by $\overline{p}$. Then wave speeds exist for $c \in (c_{\underline{p}}, c_{\overline{p}}]$.
  Note
  $$\lim_{t\uparrow\infty} \frac{-\log X_1(t)}{t} = c_{\overline{p}}, \text{ a.s.}$$

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- **Supercritical speeds:** Note that if ψ for speeds $c > c_{\underline{p}}$,

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- **Subcritical speeds:** Biggins' martingale convergence theorem (Bertoin-Rouault) for additive martingales, $p \in (\underline{p}, \overline{p})$,
  $$W(t,p) := \sum_i X_i(t)^{p+1} e^{\Phi(p)t} \stackrel{t\uparrow\infty}{\to} W(\infty, p), \text{ a.s.}, L^1.$$
  $$\psi(x) = \mathbb{E}(\exp\{-e^{-(p+1)x} W(\infty,p)\}) \text{ is a travelling wave.}$$
- **Critical speeds:** Replace $W(\infty, p)$ by $-\partial W(\infty, \underline{p})/\partial p$.

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$$\sum_i X_i(t)^{p+1} e^{\Phi(p)t} L_p(x - \log X_i(t) - ct) \sim L_p(\alpha t) \sum_i X_i(t)^{p+1} e^{\Phi(p)t}$$

for some α, then $-\log M_t(c_p)/W(t,p) \sim L(\alpha t) \Rightarrow L_p \sim k_p \in (0, \infty)$ and uniqueness follows.

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- Problem: "$-\log X_i(t) - ct$" behaves like a Lévy process with no positive jumps drifting to $+\infty$. Too difficult to control all of them uniformly.

Stopping lines

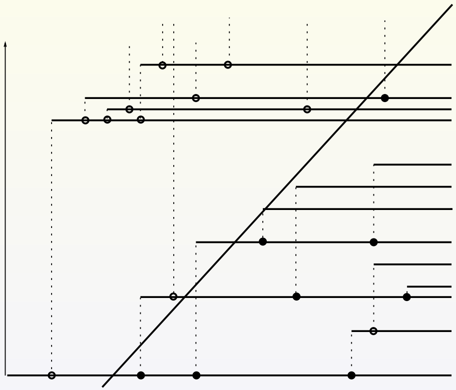

Figure: Freeze fragments as soon as $-\log X(t) - c_p t \geq z$ with $p \in (0, \overline{p})$. Collection of block sizes and their "freezing time" denoted $\{(B_i(z), \ell_i(z)) : i \geq 1\}$.

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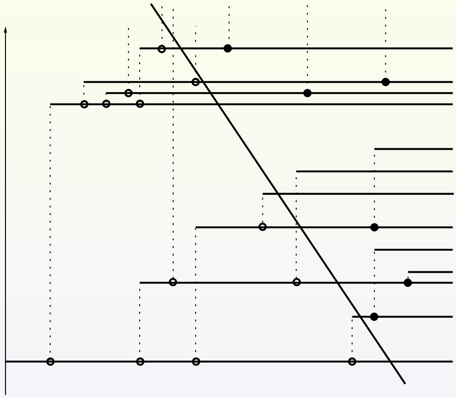

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- All martingales concerned are uniformly integrable and their limits can be "projected back" on to the stopping lines to give "stopped" versions of martingales. For $z \geq 0$

$$M_{\ell_z}(c_p) := \prod_i \psi(x - \log B_i(z) - c_p \ell_i(z)) \text{ and } W(\ell_z, p) := \sum_i B_i(z)^{(p+1)} e^{\Phi(p)\ell_i(z)}.$$

Working with stopping lines

- All martingales concerned are uniformly integrable and their limits can be "projected back" on to the stopping lines to give "stopped" versions of martingales. For $z \geq 0$

$$M_{\ell_z}(c_p) := \prod_i \psi(x - \log B_i(z) - c_p \ell_i(z)) \text{ and } W(\ell_z, p) := \sum_i B_i(z)^{(p+1)} e^{\Phi(p)\ell_i(z)}.$$

- Now much easier to compare $-\log M_{\ell_z}$ against $W(\ell_z, p)$ $(x - \log B_i(z) - c_p \ell_i(z) \geq x + z$ uniformly in $i)$ and deduce that, as $z \uparrow \infty$,

$$\begin{aligned} \frac{-\log M_{\ell_z}(c_p)}{W(\ell_z, p)} &\sim e^{-(p+1)x} \sum_i \frac{B_i(z)^{(p+1)} e^{\Phi(p)\ell_i(z)}}{W(\ell_z, p)} L_p(x - \log B_i(z) - c_p \ell_i(z)) \\ &\sim e^{-(p+1)x} L_p(x + z) \end{aligned}$$

and our naive argument can be made rigorous.

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- For all exponentially bounded positive functions f and $p \in (\underline{p}, \overline{p}]$,

$$\sum_i B_i(z)^{(p+1)} e^{\Phi(p)\ell_i(z)} f(x - \log B_i(z) - c_p \ell_i(z)) \sim Q_p(f) W(\infty, p)$$

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- When $p \in (0, \overline{p})$ this result can in fact be deduced from Nerman's classical strong law.