Skeletal stochastic differential equations for superprocesses

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DEFINITION OF ψ-CSBP.

A CSBP $(X, \mathbb{P}_x)$ is a non-negative valued strong Markov process with probabilities $(\mathbb{P}_x, x \geq 0)$ such that for any $x, y \geq 0$, $\mathbb{P}_{x+y} = \mathbb{P}_x * \mathbb{P}_y$.

In particular

$$\mathbb{E}_x(\mathrm{e}^{-\theta X_t}) = \mathrm{e}^{-x u_t(\theta)}, \qquad x, \theta, t \geq 0,$$

where $u_t(\theta)$ uniquely solves the evolution equation

$$u_t(\theta) + \int_0^t \psi(u_s(\theta))\mathrm{d}s = \theta, \qquad t \geq 0.$$

Here, we assume that the so-called branching mechanism ψ takes the form

$$\psi(\theta) = -\alpha\theta + \beta\theta^2 + \int_{(0,\infty)} (\mathrm{e}^{-\theta x} - 1 + \theta x)\Pi(\mathrm{d}x),\ \ \theta \geq 0,$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and Π is a measure concentrated on $(0, \infty)$ which satisfies

$$\int_{(0,\infty)} (x \wedge x^2)\Pi(\mathrm{d}x) < \infty$$

.

PROPERTIES.

We assume that the process is conservative, i.e.

$$\int_{0+} \frac{1}{|\psi(\xi)|} \mathrm{d}\xi = \infty.$$

It is easily verified that

$$\mathbb{E}_x[X_t] = x\mathrm{e}^{-\psi'(0+)t}, \qquad t, x \geq 0.$$

We say that the CSBP is supercritical, critical or subcritical accordingly as $-\psi'(0+) = \alpha$ is strictly positive, equal to zero or strictly negative.

For a **supercritical** ψ-CSBP the probability of extinction is

$$\mathbb{P}_x(\lim_{t\uparrow\infty} X_t = 0) = \mathrm{e}^{-\lambda^* x},$$

where λ^* is the unique root on $(0, \infty)$ of the equation $\psi(\theta) = 0$.

Prolific skeleton (Supercritical CSBP)

The supercritical ψ-CSBP is equal in law to the total mass process obtained by the following construction.

- Initiate $\mathrm{Po}(\lambda^* x)$ independent Galton-Watson processes with branching generator

$$q\left(\sum_{k\geq 0} p_k r^k - r\right) = \frac{1}{\lambda^*}\psi(\lambda^*(1-r)), \qquad r\in[0,1],$$

  where $q=\psi'(\lambda^*)$, $p_0=p_1=0$ and for $k\geq 2$

$$p_k = \frac{1}{\lambda^*\psi'(\lambda^*)}\left\{\beta(\lambda^*)^2\mathbf{1}_{\{k=2\}} + (\lambda^*)^k\int_{(0,\infty)}\frac{r^k}{k!}\mathrm{e}^{-\lambda^* r}\Pi(\mathrm{d}r)\right\}.$$

- Along the edges immigrate CSBPs at rate

$$2\beta\mathrm{d}\mathbb{Q}^* + \int_0^\infty y\mathrm{e}^{-\lambda^* y}\Pi(\mathrm{d}y)\mathrm{d}\mathbb{P}^*_y,$$

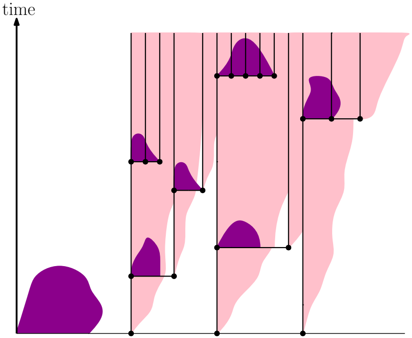
  where $\mathbb{P}^*_x,\ x\geq 0$ is the law of the CSBP with branching mechanism $\psi^*(\lambda)=\psi(\lambda+\lambda^*)$ (i.e. the process conditioned to die out) and $\mathbb{Q}^*$ is the associated excursion measure.

PROLIFIC SKELETON (SUPERCRITICAL CSBP)

- Given that an individual dies and branches into $k \geq 2$ offspring, an independent ψ^*-CSBP is immigrated with initial mass r with probability

$$\eta_k(\mathrm{d}r) = \frac{1}{p_k \lambda^* \psi'(\lambda^*)} \left\{ \beta(\lambda^*)^2 \delta_0(\mathrm{d}r) \mathbf{1}_{\{k=2\}} + (\lambda^*)^k \frac{r^k}{k!} \mathrm{e}^{-\lambda^* r} \Pi(\mathrm{d}r) \right\}.$$

- Finally an independent ψ^*-CSBP is issued at time zero with initial mass x.



CSBP SDE.

The process $(X, \mathbb{P}_x)$, $x > 0$, can be represented as the unique strong solution to the stochastic differential equation (SDE)

$$X_t = x + \alpha \int_0^t X_{s-} \mathrm{d}s + \sqrt{2\beta} \int_0^t \int_0^{X_{s-}} W(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_0^\infty \int_0^{X_{s-}} r\tilde{N}(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu),$$

for $x > 0, t \geq 0$, where

- $W(\mathrm{d}s, \mathrm{d}u)$ is a white noise process on $(0, \infty)^2$ based on the Lebesgue measure $\mathrm{d}s \otimes \mathrm{d}u$,
- $N(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu)$ is a Poisson point process on $(0, \infty)^3$ with intensity $\mathrm{d}s \otimes \Pi(\mathrm{d}r) \otimes \mathrm{d}\nu$, and $\tilde{N}(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu)$ the compensated measure of $N(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu)$.

Prolific skeletal SDE decomposition

Theorem (Fekete-Fontbona-K. (2017))

Suppose that ψ corresponds to a supercritical branching mechanism (i.e. $\alpha > 0$). Consider the coupled system of SDEs

$$\begin{aligned}\begin{pmatrix}\Lambda_t\\ Z_t\end{pmatrix} = \begin{pmatrix}\Lambda_0\\ Z_0\end{pmatrix} - \psi'(\lambda^*)\int_0^t \begin{pmatrix}\Lambda_{s-}\\ 0\end{pmatrix}\mathrm{d}s + \sqrt{2\beta}\int_0^t\int_0^{\Lambda_{s-}}\begin{pmatrix}1\\ 0\end{pmatrix}W(\mathrm{d}s,\mathrm{d}u)\\
+\int_0^t\int_0^\infty\int_0^{\Lambda_{s-}}\begin{pmatrix}r\\ 0\end{pmatrix}\tilde{\mathrm{N}}^0(\mathrm{d}s,\mathrm{d}r,\mathrm{d}\nu)\\
+\int_0^t\int_0^\infty\int_1^{Z_{s-}}\begin{pmatrix}r\\ 0\end{pmatrix}\mathrm{N}^1(\mathrm{d}s,\mathrm{d}r,\mathrm{d}j) + 2\beta\int_0^t\begin{pmatrix}Z_{s-}\\ 0\end{pmatrix}\mathrm{d}s\\
+\int_0^t\int_0^\infty\int_0^\infty\int_1^{Z_{s-}}\begin{pmatrix}r\\ k-1\end{pmatrix}\mathrm{N}^2(\mathrm{d}s,\mathrm{d}r,\mathrm{d}k,\mathrm{d}j), \qquad t\geq 0,\end{aligned}$$

with $\Lambda_0 \geq 0$ given and fixed. Under the assumption that Z_0 is an independent random variable which is Poisson distributed with intensity $\lambda^*\Lambda_0$ the coupled system of SDEs above has a unique strong solution such that:

(i) *For $t \geq 0$, $Z_t|\mathcal{F}_t^\Lambda$ is Poisson distributed with intensity $\lambda^*\Lambda_t$, where $\mathcal{F}_t^\Lambda := \sigma(\Lambda_s : s \leq t)$;*

(ii) *The process $(\Lambda_t, t \geq 0)$ is a weak solution to the CSBP SDE.*

Driving sources of randomness

Let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\sharp(\mathrm{d}\ell) = \sum_{i \in \mathbb{N}_0} \delta_i(\mathrm{d}\ell)$, $\ell \geq 0$.

Then in the previous theorem

- $\mathtt{N}^0$ is a Poisson random measure on $(0,\infty)^3$ with intensity measure $\mathrm{d}s \otimes \mathrm{e}^{-\lambda^* r}\Pi(\mathrm{d}r) \otimes \mathrm{d}\nu$, $\tilde{\mathtt{N}}^0$ is the associated compensated version of $\mathtt{N}^0$,
- $\mathtt{N}^1(\mathrm{d}s, \mathrm{d}r, \mathrm{d}j)$ is a Poisson point process on $(0,\infty)^2 \times \mathbb{N}$ with intensity $\mathrm{d}s \otimes r\mathrm{e}^{-\lambda^* r}\Pi(\mathrm{d}r) \otimes \sharp(\mathrm{d}j)$,
- $\mathtt{N}^2(\mathrm{d}s, \mathrm{d}r, \mathrm{d}k, \mathrm{d}j)$ is a Poisson point process on $(0,\infty)^2 \times \mathbb{N}_0 \times \mathbb{N}$ with intensity $\psi'(\lambda^*)\mathrm{d}s \otimes \eta_k(\mathrm{d}r) \otimes p_k \sharp(\mathrm{d}k) \otimes \sharp(\mathrm{d}j)$, and
- $W(\mathrm{d}s, \mathrm{d}u)$ is the white noise process on $(0,\infty)^2$ based on the Lebesgue measure $\mathrm{d}s \otimes \mathrm{d}u$.

What's the point?

- An SDE inherently shows us where (the) martingales (that we already know exist) lie.
- Remember that $\alpha = -\psi'(0+)$ and (immigration = CSBP)

$$\mathbb{E}_x[\Lambda_t] = x\mathrm{e}^{-\psi'(0+)t} = x\mathrm{e}^{\alpha t}$$

and that Z (skeleton) has branching mechanism $\psi(\lambda^*(1-r))/\lambda^*$, so that the rate of growth is given by

$$q\left(\sum_{k=0}^{\infty} kp_k - 1\right) = \frac{1}{\lambda^*}\,\frac{\mathrm{d}}{\mathrm{d}r}\psi(\lambda^*(1-r))\bigg|_{r=1} = -\psi'(0+) = \alpha.$$

- As a consequence, we know that for both the CSBP and its skeleton:

$$(\mathrm{e}^{-\alpha t}\Lambda_t, t \geq 0) \qquad \text{and} \qquad (\mathrm{e}^{-\alpha t} Z_t, t \geq 0)$$

are martingales.
- Hence by rearrangement in our SDE, we should see that

$$\begin{pmatrix} \mathrm{e}^{-\alpha t}\Lambda_t \\ \mathrm{e}^{-\alpha t} Z_t \end{pmatrix} = \text{SDE} = \text{martingale}$$

- **(EXERCISE!!!!)** Should be able to show that (under the appropriate $x \log x$ conditions) the skeleton and immigration martingales converge together on the same space to the same random variable.

What's the point?

- **Still not a motivation!**
- BUT! What if we could develop a similar SDE for a skeletal decomposition of a superprocess?
- **Roughly speaking (more careful description coming next):** We want $(X_t(\cdot), t \geq 0)$ is an appropriately defined superprocess (measure-valued Markov process), then there exists a branching particle diffusion (written as an atomic measure-valued process) $(Z_t(\cdot), t \geq 0)$ and a measure-valued process of immigration $(\Lambda_t(\cdot), t \geq 0)$, such that for all "suitably smooth" functions f, g:

$$\begin{pmatrix} \langle f, \Lambda_t \rangle \\ \langle g, Z_t \rangle \end{pmatrix} = \mathrm{SDE}$$

- **Still not a motivation!**
- BUT! If we are in a scenario where we can define a "generalised principal eigenvalue for the linear operator of the system" (say λ_c) for which

$$\begin{pmatrix} \mathrm{e}^{-\lambda_c t} \langle f, \Lambda_t \rangle \\ \mathrm{e}^{-\lambda_c t} \langle g, Z_t \rangle \end{pmatrix} = \mathrm{SDE}$$

- And hence we get a new tool for producing (**OPEN PROBLEM!!!!!!**) a Strong Law of Large Numbers for the coupled system.
- Variant of the previous open problem: Previous studies have shown that when $(\mathrm{e}^{-\lambda_c t} \langle g, Z_t \rangle, t \geq 0)$ has a SLLN, then $(\mathrm{e}^{-\lambda_c t} \langle f, \Lambda_t \rangle, t \geq 0)$ has a SLLN.

Superprocess

- Let E be a domain of $\mathbb{R}^d$, and denote by $\mathcal{M}(E)$ the space of finite Borel measures on E.
- Want a strong Markov process X on E taking values in $\mathcal{M}(E)$.
- The process is characterised by two quantities $\mathcal{P}$ and ψ.
- $\mathcal{P} = (\mathcal{P}_t)_{t\geq 0}$ is the semigroup of an $\mathbb{R}^d$-valued diffusion killed on exiting E.
- For technical reasons we assume that $\mathcal{P}$ is a Feller semigroup whose generator takes the form
$$\mathcal{L} = \frac{1}{2}\nabla \cdot a(x)\nabla + b(x)\cdot\nabla,$$
where $a : E \to \mathbb{R}^{d\times d}$ is the diffusion matrix that takes values in the set of symmetric, positive definite matrices, and $b : E \to \mathbb{R}^d$ is the drift term.
- ψ is the so-called branching mechanism. The latter takes the form
$$\psi(x,z) = -\alpha(x)z + \beta(x)z^2 + \int_{(0,\infty)} \left(\mathrm{e}^{-zu} - 1 + zu\right) m(x,\mathrm{d}u), \quad x\in E,\ z\geq 0,$$
where α and $\beta \geq 0$ are bounded continuous mappings from E to $\mathbb{R}$ and $[0,\infty)$ respectively, and $(u\wedge u^2)m(x,\mathrm{d}u)$ is a bounded kernel from E to $(0,\infty)$.

Semigroup representation of superprocess

- For all $\mu \in \mathcal{M}(E)$ and $f \in B^+(E)$, where $B^+(E)$ denotes the non-negative measurable functions on E, we have

$$\mathbb{E}_\mu\left[\mathrm{e}^{-\langle f, X_t\rangle}\right] = \exp\left\{-\int_E u_f(x,t)\mu(\mathrm{d}x)\right\}, \quad t \geq 0,$$

- here, $u_f(x,t)$ is the unique non-negative solution to the integral equation

$$u_f(x,t) = \mathcal{P}_t[f](x) - \int_0^t \mathrm{d}s \cdot \mathcal{P}_s[\psi(\cdot, u_f(\cdot, t-s))](x), \quad x \in E,\ t \geq 0. \tag{1}$$

  Here we use the notation

$$\langle f, \mu\rangle = \int_E f(x)\mu(\mathrm{d}x), \quad \mu \in \mathcal{M}(E),\ f \in B^+(E).$$

- For each $\mu \in \mathcal{M}(E)$ we denote by $\mathbb{P}_\mu$ the law of the process X issued from $X_0 = \mu$. The process $(X, \mathbb{P}_\mu)$ is called a $(\mathcal{P}, \psi)$-superprocess.

w-SKELETON

- Suppose that $w(x) > 0$, for all $x \in E$, $\sup_{x\in E} w(x) < \infty$ and
$$\mathbb{E}_\mu\left(\mathrm{e}^{-\langle w, X_t\rangle}\right) = \mathrm{e}^{-\langle w,\mu\rangle}, \qquad \text{for all} \qquad \mu \in \mathcal{M}(E), t \geq 0.$$
- w-skeleton $Z = (Z_t, t \geq 0)$ is a Markov branching process with diffusion semigroup $\mathcal{P}^w$ is "formally" associated to the generator
$$\mathcal{L}^w := w^{-1}\mathcal{L}(wu) - w^{-1}\mathcal{L}w$$
and branching generator
$$F(x,s) = q(x)\sum_{n\geq 0} p_n(x)(s^n - s), \quad x \in E,\ s \in [0,1],$$
where
$$q(x) = \psi'(x, w(x)) - \frac{\psi(x, w(x))}{w(x)},$$
and $p_0(x) = p_1(x) = 0$, and for $n \geq 2$
$$p_n(x) = \frac{1}{w(x)q(x)}\left\{\beta(x)w^2(x)\mathbf{1}_{\{n=2\}} + w^n(x)\int_{(0,\infty)} \frac{y^n}{n!}\mathrm{e}^{-w(x)y}m(x,\mathrm{d}y)\right\}.$$
Here we used the notation
$$\psi'(x, w(x)) := \frac{\partial}{\partial z}\psi(x,z)\Big|_{z=w(x)}, \quad x \in E.$$
We refer to the process Z as the $(\mathcal{P}^w, F)$ skeleton.

Immigration

- The function
$$\psi^*(x,z) = \psi(x, z + w(x)) - \psi(x, w(x)), \quad x \in E,$$
is a branching mechanism which can be written as
$$\psi^*(x,z) = -\alpha^*(x)z + \beta(x)z^2 + \int_{(0,\infty)} (\mathrm{e}^{-zu} - 1 + zu)m^*(x, \mathrm{d}u), \quad x \in E,$$
where
$$\alpha^*(x) = -\psi'(x, w(x)) \text{ and } m^*(x, \mathrm{d}u) = \mathrm{e}^{-w(x)u} m(x, \mathrm{d}u).$$
- Dress the branches of the spatial tree that describes the trajectory of Z in such a way that a particle at the space-time position $(x,t) \in E \times [0,\infty)$ has an independent $\mathbb{D}([0,\infty) \times \mathcal{M}(E))$-valued trajectory grafted on to it with rate
$$2\beta(x)\mathrm{d}\mathbb{N}^*_x + \int_{(0,\infty)} y\mathrm{e}^{-w(x)y} m(x, \mathrm{d}y) \times \mathrm{d}\mathbb{P}^*_{y\delta_x}.$$
Here $\mathbb{N}^*_x$ is the excursion measure on the space $\mathbb{D}([0,\infty) \times \mathcal{M}(E))$ associated to $\mathbb{P}^*$.
- When a particle in Z dies and gives birth to $n \geq 2$ offspring at spatial position $x \in E$, with probability $\eta_n(x, \mathrm{d}y)\mathbb{P}^*_{y\delta_x}$ an additional independent $\mathbb{D}([0,\infty) \times \mathcal{M}(E))$-valued trajectory is grafted on to the space-time branching point, where
$$\eta_n(x, \mathrm{d}y) = \frac{1}{w(x)q(x)p_n(x)} \left\{ \beta(x) w^2(x) \delta_0(\mathrm{d}y) \mathbf{1}_{\{n=2\}} + w^n(x) \frac{y^n}{n!} \mathrm{e}^{-w(x)y} m(x, \mathrm{d}y) \right\}.$$

w-SKELETAL PATH DECOMPOSITION

Theorem (Engländer-Pinsky (1999), K.-Pérez-Ren (2015))

Suppose that $\mu \in \mathcal{M}(E)$, and let Z be a $(\mathcal{P}^w, F)$-Markov branching process with initial configuration consisting of a Poisson random field of particles in E with intensity $w(x)\mu(\mathrm{d}x)$.

Define Λ_t as the total mass from the dressing present at time t together with the mass present at time t from an independent copy of $(X, \mathbb{P}^*_\mu)$ issued at time 0.

Denote the law of (Λ, Z) by $\mathbf{P}_\mu$. Then $(\Lambda, \mathbf{P}_\mu)$ is is equal in law to $(X, \mathbb{P}_\mu)$. Furthermore, under $\mathbf{P}_\mu$, conditionally on Λ_t, the measure Z_t is a Poisson random measure with intensity $w(x)\Lambda_t(\mathrm{d}x)$.

(Less exotic versions were proved before and more exotic versions with non-local branching mechanisms have since been proved)

SDE REPRESENTATION OF SUPERPROCESS

- Define $H(x, \mathrm{d}\nu)$ as the natural extension of m from $(0,\infty)$ to $\mathcal{M}(E)\setminus\{0\}$. More precisely H is concentrated on measures of the form $\{u\delta_x\}$ and assume it satisfies the integrability condition

$$\sup_{x\in E}\int_{\mathcal{M}(E)}(\langle 1,\nu\rangle \wedge \langle 1,\nu\rangle^2)H(x,\mathrm{d}\nu) = \sup_{x\in E}\int_{(0,\infty)}(u\wedge u^2)m(x,\mathrm{d}u) < \infty.$$

- Let $C_0(E)^+$ denote the space of non-negative continuous functions on E vanishing at infinity. We assume $x \mapsto (\langle 1,\nu\rangle \wedge \langle 1,\nu\rangle^2)H(x,\mathrm{d}\nu)$ is continuous in the sense of weak convergence on $\mathcal{M}(E)\setminus\{0\}$, and

$$f \mapsto \int_{\mathcal{M}(E)}(\langle f,\nu\rangle \wedge \langle f,\nu\rangle^2)H(x,\mathrm{d}\nu) = \int_{(0,\infty)}(uf(x)\wedge u^2 f(x)^2)m(x,\mathrm{d}u)$$

  maps $C_0(E)^+$ into itself.
- Next let $N(\mathrm{d}s,\mathrm{d}\nu)$ be the optional random measure on $[0,\infty)\times\mathcal{M}(E)$ defined by

$$N(\mathrm{d}s,\mathrm{d}\nu) = \sum_{s>0}\mathbf{1}_{\{\Delta X_s\neq 0\}}\delta_{(s,\Delta X_s)}(\mathrm{d}s,\mathrm{d}\nu),$$

  where $\Delta X_s = X_s - X_{s-}$, and let $\hat{N}(\mathrm{d}s,\mathrm{d}\nu)$ denote the predictable compensator of $N(\mathrm{d}s,\mathrm{d}\nu)$. It can be shown that $\hat{N}(\mathrm{d}s,\mathrm{d}\nu) = K(X_{s-},\mathrm{d}\nu)\mathrm{d}s$ with

$$K(\mu,\mathrm{d}\nu) = \int_E \mu(\mathrm{d}x)H(x,\mathrm{d}\nu).$$

Superprocess SDE

- If we denote the compensated measure by $\tilde{N}(\mathrm{d}s, \mathrm{d}\nu)$, then for any $f \in D_0(\mathcal{L})$ (the set of functions in $C_0(E)$ that are also in the domain of $\mathcal{L}$) we have

  $$\langle f, X_t\rangle = \langle f, X_0\rangle + M_t^c(f) + M_t^d(f) + \int_0^t \langle \mathcal{L}f + \alpha f, X_s\rangle \mathrm{d}s, \quad t \geq 0,$$

  where $t \mapsto M_t^c(f)$ is a continuous local martingale with quadratic variation $\int_0^t \langle 2\beta f^2, X_{s-}\rangle \mathrm{d}s$ and

  $$t \mapsto M_t^d(f) = \int_0^t \int_{\mathcal{M}(E)} \langle f, \nu\rangle \tilde{N}(\mathrm{d}s, \mathrm{d}\nu), \quad t \geq 0,$$

  is a purely discontinuous local martingale.

w-skeletal SDE decomposition

Theorem (Fekete-Fontbona-K. (2019))

Suppose that $w(x) > 0$, for all $x \in E$, $w \in D_0(\mathcal{L})$, $\sup_{x\in E} w(x) < \infty$ and

$$\mathbb{E}_\mu\left(\mathrm{e}^{-\langle w, X_t\rangle}\right) = \mathrm{e}^{-\langle w,\mu\rangle}, \qquad \textit{for all} \qquad \mu \in \mathcal{M}(E), t \geq 0.$$

Consider the following system of SDEs for $f, h \in D_0(\mathcal{L})$,

$$\begin{aligned}
\begin{pmatrix} \langle f, \Lambda_t\rangle \\ \langle h, Z_t\rangle \end{pmatrix} &= \begin{pmatrix} \langle f, \Lambda_0\rangle \\ \langle h, Z_0\rangle \end{pmatrix} - \int_0^t \begin{pmatrix} \langle \psi'(\cdot, w(\cdot))f(\cdot), \Lambda_{s-}\rangle \\ 0 \end{pmatrix} \mathrm{d}s + \begin{pmatrix} U_t^c(f) \\ V_t^c(h) \end{pmatrix} \\
&\quad + \int_0^t \int_{\mathcal{M}(E)} \begin{pmatrix} \langle f, \nu\rangle \\ 0 \end{pmatrix} \tilde{\mathrm{N}}^0(\mathrm{d}s, \mathrm{d}\nu) + \int_0^t \begin{pmatrix} \langle \mathcal{L}f, \Lambda_{s-}\rangle \\ \langle \mathcal{L}^w h, Z_{s-}\rangle \end{pmatrix} \mathrm{d}s \\
&\quad + \int_0^t \int_{\mathcal{M}(E)} \begin{pmatrix} \langle f, \nu\rangle \\ 0 \end{pmatrix} \mathrm{N}^1(\mathrm{d}s, \mathrm{d}\nu) + \int_0^t \begin{pmatrix} \langle 2\beta(\cdot)f(\cdot), Z_{s-}\rangle \\ 0 \end{pmatrix} \mathrm{d}s \\
&\quad + \int_0^t \int_{\mathcal{M}_a(E)} \int_{\mathcal{M}(E)} \begin{pmatrix} \langle f, \nu\rangle \\ \langle h, \rho\rangle \end{pmatrix} \mathrm{N}^2(\mathrm{d}s, \mathrm{d}\rho, \mathrm{d}\nu), \qquad t \geq 0,
\end{aligned}$$

where $\Lambda_0 \in \mathcal{M}(E)$ is given and fixed. Then under the assumption that Z_0 is a Poisson random measure with intensity $w(x)\Lambda_0(\mathrm{d}x)$ we have the following:

(i) *$Z_t|\mathcal{F}_t^\Lambda$ (where $\mathcal{F}_t^\Lambda = \sigma(\Lambda_s : s \leq t)$) is a Poisson random measure with intensity $w(x)\Lambda_t(\mathrm{d}x)$;*

(ii) *The process $(\Lambda_t, t \geq 0)$ is Markovian and a weak solution to the superprocess SDE.*

Driving sources of randomness

- Let $\mathtt{N}^0(\mathrm{d}s, \mathrm{d}\nu)$ be an optional random measure on $[0,\infty) \times \mathcal{M}(E)\backslash\{0\}$ with predictable compensator

$$\hat{\mathtt{N}}^0(\mathrm{d}s, \mathrm{d}\nu)|_{\nu=u\delta_x} = \mathrm{d}s \int_E \Lambda_{s-}(\mathrm{d}x) \mathrm{e}^{-w(x)u} m(x, \mathrm{d}u),$$

and $\tilde{\mathtt{N}}^0(\mathrm{d}s, \mathrm{d}\nu)$ is its compensated version,
- $\mathtt{N}^1(\mathrm{d}s, \mathrm{d}\nu)$ be an optional random measure on $[0,\infty) \times \mathcal{M}(E)\backslash\{0\}$ with predictable compensator

$$\hat{\mathtt{N}}^1(\mathrm{d}s, \mathrm{d}\nu)|_{\nu=u\delta_x} = \mathrm{d}s \int_E Z_{s-}(\mathrm{d}x) u \mathrm{e}^{-w(x)u} m(x, \mathrm{d}u),$$

- and $\mathtt{N}^2(\mathrm{d}s, \mathrm{d}\rho, \mathrm{d}\nu)$ an optional random measure on $[0,\infty) \times \mathcal{M}_a(E) \times \mathcal{M}(E)\backslash\{0\}$ with predictable compensator

$$\hat{\mathtt{N}}^2(\mathrm{d}s, \mathrm{d}\rho, \mathrm{d}\nu)|_{\nu=u\delta_x, \rho=(k-1)\delta_x} = \mathrm{d}s \int_E Z_{s-}(\mathrm{d}x) q(x) p_k(x) \eta_k(x, \mathrm{d}u) \pi(x, \mathrm{d}k),$$

$\pi(x, \mathrm{d}k) = \#(\mathrm{d}(k-1))\delta_x$ so that $\int_{\{2,3,\cdots\}} \langle h, \pi(x,\cdot)\rangle = \sum_{k\geq 2} h(x, k-1)$.

- Finally let $(U^c_t(f), t \geq 0)$ be a continuous local martingale with quadratic variation $2\langle \beta f^2, \Lambda_{t-}\rangle \mathrm{d}t$, and $(V^c_t(h), t \geq 0)$ be a continuous local martingale with quadratic variation $\langle (\nabla h)^{\mathrm{t}} a \nabla h, Z_{t-}\rangle \mathrm{d}t$.

SLLN

- Typically want to assume that there exists $\lambda_c > 0$ such that

$$\lambda_c = \inf\{\lambda \in \mathbb{R} : \exists \text{ smooth } h > 0 \text{ and } (\mathcal{L} + \alpha - \lambda)h = 0\}$$

  and associated to this eigenvalue are the left- and right-eigenfunctions $\tilde{\varphi}$ and φ, normalised so that $\langle \tilde{\varphi}, \varphi \rangle = 1$.
- **(Roughly speaking):** We have that

$$W_t^{\varphi}(\Lambda) := \mathrm{e}^{-\lambda_c t}\langle \varphi, \Lambda_t \rangle \text{ and } W_t^{\varphi/w}(Z) := \mathrm{e}^{-\lambda_c t}\langle \varphi/w, Z_t \rangle, \qquad t \geq 0,$$

  are martingales.
- One should be able to prove that

$$W_\infty^{\varphi}(\Lambda) = W_\infty^{\varphi/w}(Z) =: \Delta$$

- The SLLN would say that for $f \leq \varphi$ and $g \leq \varphi/w$,

$$\lim_{t\to\infty} \begin{pmatrix} \mathrm{e}^{-\lambda_c t}\langle f, \Lambda_t \rangle \\ \mathrm{e}^{-\lambda_c t}\langle g, Z_t \rangle \end{pmatrix} = \begin{pmatrix} \langle f, \tilde{\varphi} \rangle \\ \langle g, w\tilde{\varphi} \rangle \end{pmatrix} \Delta$$

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