Refracted Lévy processes

Andreas E. Kyprianou [1]

Department of Mathematical Sciences, University of Bath

[1] Joint work with Ronnie Loeffen

Basic data

- $X = \{X_t : t \geq 0\}$ with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ will always denote a spectrally negative Lévy process (i.e. $\Pi(0, \infty) = 0$ and $-X$ is not a subordinator).

Basic data

- $X = \{X_t : t \geq 0\}$ with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ will always denote a spectrally negative Lévy process (i.e. $\Pi(0, \infty) = 0$ and $-X$ is not a subordinator).
- For $\theta \geq 0$ we may work with the Laplace exponent

$$\psi(\theta) := \log \mathbb{E}_0(e^{\theta X_1}).$$

Basic data

- $X = \{X_t : t \geq 0\}$ with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ will always denote a spectrally negative Lévy process (i.e. $\Pi(0, \infty) = 0$ and $-X$ is not a subordinator).
- For $\theta \geq 0$ we may work with the Laplace exponent

$$\psi(\theta) := \log \mathbb{E}_0(e^{\theta X_1}).$$

- For each $q \geq 0$, the, so-called, q-scale function $W^{(q)} : \mathbb{R} \mapsto [0, \infty)$ is defined by $W^{(q)}(x) = 0$ for $x < 0$ and otherwise is continuous satisfying

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q}$$

for all β sufficiently large.

Basic data

- $X = \{X_t : t \geq 0\}$ with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ will always denote a spectrally negative Lévy process (i.e. $\Pi(0,\infty) = 0$ and $-X$ is not a subordinator).
- For $\theta \geq 0$ we may work with the Laplace exponent

$$\psi(\theta) := \log \mathbb{E}_0(e^{\theta X_1}).$$

- For each $q \geq 0$, the, so-called, q-scale function $W^{(q)} : \mathbb{R} \mapsto [0,\infty)$ is defined by $W^{(q)}(x) = 0$ for $x < 0$ and otherwise is continuous satisfying

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)dx = \frac{1}{\psi(\beta) - q}$$

for all β sufficiently large.

- For convenience we shall write W for $W^{(0)}$.

Sample fluctuation identities

For example, if $\tau_0^- = \inf\{t > 0 : X_t < 0\}$ and $\tau_a^+ = \inf\{t > 0 : X_t > a\}$ then

- The oldest one in the book (Takács 1966, Zolotarev 1964) (the 'ruin probability' - in fact the Pollaczek-Khintchine formula in disguise)

$$\mathbb{P}_x(\tau_0^- < \infty) = 1 - (\mathbb{E}_0(X_1) \vee 0)\, W(x)$$

for $x \geq 0$.

Sample fluctuation identities

For example, if $\tau_0^- = \inf\{t > 0 : X_t < 0\}$ and $\tau_a^+ = \inf\{t > 0 : X_t > a\}$ then

- The oldest one in the book (Takács 1966, Zolotarev 1964) (the 'ruin probability' - in fact the Pollaczek-Khintchine formula in disguise)

$$\mathbb{P}_x(\tau_0^- < \infty) = 1 - (\mathbb{E}_0(X_1) \vee 0)\, W(x)$$

  for $x \geq 0$.

- Resolvent in a strip: For any $a > 0$, $x, y \in [0, a]$, $q \geq 0$

$$\int_0^\infty e^{-qt}\mathbb{P}_x(X_t \in dy,\, t < \tau_a^+ \wedge \tau_0^-)\, dt = \left\{ \frac{W^{(q)}(x)\, W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y) \right\} dy.$$

Controlled Lévy risk processes

- Think of X as the wealth of an insurance company (X can be a classical Cramér-Lundberg processes if you want).

Controlled Lévy risk processes

- Think of X as the wealth of an insurance company (X can be a classical Cramér-Lundberg processes if you want).
- Suppose that $\xi = \{L_t^\xi : t \geq 0\}$ is a stream of dividend payments or 'dividend strategy': left continuous, non-negative, non-decreasing process adapted to the filtration generated by X.

Controlled Lévy risk processes

- Think of X as the wealth of an insurance company (X can be a classical Cramér-Lundberg processes if you want).
- Suppose that $\xi = \{L_t^\xi : t \geq 0\}$ is a stream of dividend payments or 'dividend strategy': left continuous, non-negative, non-decreasing process adapted to the filtration generated by X.
- Define the aggregate process $U_t^\xi = X_t - L_t^\xi$ when paying dividends with strategy ξ and let

$$\sigma^\xi = \inf\{t > 0 : U_t^\xi < 0\}$$

be the ruin time of the aggregate process.

Controlled Lévy risk processes

- Think of X as the wealth of an insurance company (X can be a classical Cramér-Lundberg processes if you want).
- Suppose that $\xi = \{L_t^\xi : t \geq 0\}$ is a stream of dividend payments or 'dividend strategy': left continuous, non-negative, non-decreasing process adapted to the filtration generated by X.
- Define the aggregate process $U_t^\xi = X_t - L_t^\xi$ when paying dividends with strategy ξ and let

$$\sigma^\xi = \inf\{t > 0 : U_t^\xi < 0\}$$

  be the ruin time of the aggregate process.
- A strategy ξ is called admissible if $L_{t+}^\xi - L_t^\xi \leq U_t^\xi$ for $t < \sigma^\xi$ (i.e. ruin of the aggregate process does not result as a consequence of a dividend payment).

De Finetti's control problem

An 'old' actuarial problem of the 'modern' probabilistic age proposed by de Finetti 1957: find the value function and matching dividend strategy ξ^* such that

$$v(x) = \sup_{\xi} \mathbb{E}_x \left(\int_0^{\sigma^\xi} e^{-qt} \, dL_t^\xi \right) = \mathbb{E}_x \left(\int_0^{\sigma^{\xi^*}} e^{-qt} \, dL_t^{\xi^*} \right)$$

where $q > 0$ and the supremum is taken over all admissible dividend strategies.

Reflection strategies

It has been shown that the optimal strategy is of a 'barrier type with reflection':

$$L_t^a = (a \vee \sup_{s \le t} X_s) - a$$

for some optimal level a. These cases are:

1 (Gerber 1969) Cramér-Lundberg process with exponentially distributed jumps $X_t = ct - \sum_{i=1}^{N_t} \mathbf{e}_i$,

Reflection strategies

It has been shown that the optimal strategy is of a ‘barrier type with reflection’:

$$L_t^a = (a \vee \sup_{s \le t} X_s) - a$$

for some optimal level a. These cases are:

1. (Gerber 1969) Cramér-Lundberg process with exponentially distributed jumps $X_t = ct - \sum_{i=1}^{N_t} \mathbf{e}_i$,
2. (Jeanblanc & Shiryaev 1995 and many others) Linear Brownian motion: $X_t = \mu t + \sigma B_t$.

Reflection strategies

It has been shown that the optimal strategy is of a 'barrier type with reflection':

$$L_t^a = (a \vee \sup_{s \leq t} X_s) - a$$

for some optimal level a. These cases are:

1. (Gerber 1969) Cramér-Lundberg process with exponentially distributed jumps $X_t = ct - \sum_{i=1}^{N_t} \mathbf{e}_i$,
2. (Jeanblanc & Shiryaev 1995 and many others) Linear Brownian motion: $X_t = \mu t + \sigma B_t$.
3. (Loeffen 2008) Any spectrally negative Lévy process whose jump measure has a completely monotone density.

Reflection strategies

It has been shown that the optimal strategy is of a 'barrier type with reflection':

$$L_t^a = (a \vee \sup_{s \le t} X_s) - a$$

for some optimal level a. These cases are:

1. (Gerber 1969) Cramér-Lundberg process with exponentially distributed jumps $X_t = ct - \sum_{i=1}^{N_t} \mathbf{e}_i$,
2. (Jeanblanc & Shiryaev 1995 and many others) Linear Brownian motion: $X_t = \mu t + \sigma B_t$.
3. (Loeffen 2008) Any spectrally negative Lévy process whose jump measure has a completely monotone density.
4. (K. Rivero and Song 2008) Any spectrally negative Lévy process whose jump measure has a log-convex density.

Restricted class of control strategies

- Many variations on this theme have been examined for the case of diffusions (Jeanblanc & Shiryaev 1995, Elena Boguslavskaya's Ph.D. thesis 2005) as well as the Cramér-Lundberg case with exponential jumps (Gerber & Shiu 2006) including the following:

Restricted class of control strategies

- Many variations on this theme have been examined for the case of diffusions (Jeanblanc & Shiryaev 1995, Elena Boguslavskaya's Ph.D. thesis 2005) as well as the Cramér-Lundberg case with exponential jumps (Gerber & Shiu 2006) including the following:
- The class of admissible strategies is further restricted to the case that

$$U_t^\phi = X_t - L_t^\phi = X_t - \int_0^t \phi(U_s^\phi)ds \tag{1}$$

where ϕ is measurable and uniformly bounded by, say, $\delta > 0$. Should now think of ϕ as the control.

Restricted class of control strategies

- Many variations on this theme have been examined for the case of diffusions (Jeanblanc & Shiryaev 1995, Elena Boguslavskaya's Ph.D. thesis 2005) as well as the Cramér-Lundberg case with exponential jumps (Gerber & Shiu 2006) including the following:
- The class of admissible strategies is further restricted to the case that

$$U_t^\phi = X_t - L_t^\phi = X_t - \int_0^t \phi(U_s^\phi)ds \tag{1}$$

  where ϕ is measurable and uniformly bounded by, say, $\delta > 0$. Should now think of ϕ as the control.
- Immediate problem: (1) can be a stochastic differential equation of the *degenerate* type. Does it even have a unique weak solution? (possible bad cases: X has no Gaussian component).

Restricted class of control strategies

- Many variations on this theme have been examined for the case of diffusions (Jeanblanc & Shiryaev 1995, Elena Boguslavskaya's Ph.D. thesis 2005) as well as the Cramér-Lundberg case with exponential jumps (Gerber & Shiu 2006) including the following:
- The class of admissible strategies is further restricted to the case that

$$U_t^{\phi} = X_t - L_t^{\phi} = X_t - \int_0^t \phi(U_s^{\phi})ds \tag{1}$$

  where ϕ is measurable and uniformly bounded by, say, $\delta > 0$. Should now think of ϕ as the control.
- Immediate problem: (1) can be a stochastic differential equation of the *degenerate* type. Does it even have a unique weak solution? (possible bad cases: X has no Gaussian component).
- Could one at least investigate (1) for the optimal strategies that have appeared in the aforementioned articles?

Refraction strategies (Loeffen and K. 2008)

- A refraction strategy refers to the control $\phi(x) = \delta\mathbf{1}_{(x>b)}$ for some threshold level $b \geq 0$. Thus the controlled process would need to solve the stochastic differential equation

$$U_t = X_t - \delta \int_0^t \mathbf{1}_{(U_s>b)}\,ds.$$

Refraction strategies (Loeffen and K. 2008)

- A refraction strategy refers to the control $\phi(x) = \delta \mathbf{1}_{(x>b)}$ for some threshold level $b \geq 0$. Thus the controlled process would need to solve the stochastic differential equation

$$U_t = X_t - \delta \int_0^t \mathbf{1}_{(U_s>b)} ds.$$

- When X has a Gaussian part then classical theory gives us a unique strong solution.

Refraction strategies (Loeffen and K. 2008)

- A refraction strategy refers to the control $\phi(x) = \delta \mathbf{1}_{(x>b)}$ for some threshold level $b \geq 0$. Thus the controlled process would need to solve the stochastic differential equation

$$U_t = X_t - \delta \int_0^t \mathbf{1}_{(U_s > b)} ds.$$

- When X has a Gaussian part then classical theory gives us a unique strong solution.
- When X has paths of bounded variation, then solution can be constructed pathwise.

Refraction strategies (Loeffen and K. 2008)

- A refraction strategy refers to the control $\phi(x) = \delta \mathbf{1}_{(x>b)}$ for some threshold level $b \geq 0$. Thus the controlled process would need to solve the stochastic differential equation

$$U_t = X_t - \delta \int_0^t \mathbf{1}_{(U_s > b)} ds.$$

- When X has a Gaussian part then classical theory gives us a unique strong solution.
- When X has paths of bounded variation, then solution can be constructed pathwise.
- When X has unbounded variation, no Gaussian part, solution can be strongly approximated by solutions from the bounded variation case:

$$\sup_{s\in[0,1]} |X_s - X_s^{(n)}| \to 0 \Rightarrow \sup_{s\in[0,1]} |U_s^* - U_s^{(n)}| \to 0$$

as $n \uparrow \infty$ for some stochastic process U^* (which is a limit point in the $(D[0,1], ||\cdot||_\infty)$ Banach space.

When is U^* a refracted process?

- Since

$$U_t^* = X_t - \delta \lim_{n\uparrow\infty} \int_0^t \mathbf{1}_{(U_s^{(n)} > b)} ds$$

we have that U^* is a refracted process as soon as one can prove that $\mathbb{P}_x(U_s^* = b) = 0$ for Lebesgue almost every $s > 0$.

When is U^* a refracted process?

- Since
$$U_t^* = X_t - \delta \lim_{n\uparrow\infty} \int_0^t \mathbf{1}_{(U_s^{(n)} > b)} ds$$
we have that U^* is a refracted process as soon as one can prove that $\mathbb{P}_x(U_s^* = b) = 0$ for Lebesgue almost every $s > 0$.
- Do this by noting that for $\eta, q > 0$:

When is U^* a refracted process?

- Since

$$U_t^* = X_t - \delta \lim_{n\uparrow\infty} \int_0^t \mathbf{1}_{(U_s^{(n)} > b)} ds$$

  we have that U^* is a refracted process as soon as one can prove that $\mathbb{P}_x(U_s^* = b) = 0$ for Lebesgue almost every $s > 0$.
- Do this by noting that for $\eta, q > 0$:

$\{U_t^* = b\} \subseteq \{U_t^{(n)} \in (b-\eta, b+\eta) \text{ e.v.}\}$

When is U^* a refracted process?

- Since

$$U_t^* = X_t - \delta \lim_{n\uparrow\infty} \int_0^t \mathbf{1}_{(U_s^{(n)} > b)} ds$$

we have that U^* is a refracted process as soon as one can prove that $\mathbb{P}_x(U_s^* = b) = 0$ for Lebesgue almost every $s > 0$.

- Do this by noting that for $\eta, q > 0$:

$\{U_t^* = b\} \subseteq \{U_t^{(n)} \in (b-\eta, b+\eta) \text{ e.v.}\}$

When is U^* a refracted process?

- Since

$$U^*_t = X_t - \delta \lim_{n\uparrow\infty} \int_0^t \mathbf{1}_{(U^{(n)}_s > b)} ds$$

  we have that U^* is a refracted process as soon as one can prove that $\mathbb{P}_x(U^*_s = b) = 0$ for Lebesgue almost every $s > 0$.
- Do this by noting that for $\eta, q > 0$:

$\{U^*_t = b\} \subseteq \{U^{(n)}_t \in (b-\eta, b+\eta) \text{ e.v.}\}$

$\int_0^\infty e^{-qt}\mathbb{P}_x(U^*_t = b)dt \leq \int_0^\infty e^{-qt} \liminf_n \mathbb{P}_x(U^{(n)}_t \in (b-\eta, b+\eta))dt$

When is U^* a refracted process?

- Since

$$U_t^* = X_t - \delta \lim_{n\uparrow\infty} \int_0^t \mathbf{1}_{(U_s^{(n)} > b)} ds$$

we have that U^* is a refracted process as soon as one can prove that $\mathbb{P}_x(U_s^* = b) = 0$ for Lebesgue almost every $s > 0$.

- Do this by noting that for $\eta, q > 0$:

$\{U_t^* = b\} \subseteq \{U_t^{(n)} \in (b - \eta, b + \eta) \text{ e.v.}\}$

$\int_0^\infty e^{-qt}\mathbb{P}_x(U_t^* = b)dt \leq \int_0^\infty e^{-qt} \liminf_n \mathbb{P}_x(U_t^{(n)} \in (b - \eta, b + \eta))dt$

When is U^* a refracted process?

- Since
$$U_t^* = X_t - \delta \lim_{n\uparrow\infty} \int_0^t \mathbf{1}_{(U_s^{(n)} > b)} ds$$
we have that U^* is a refracted process as soon as one can prove that $\mathbb{P}_x(U_s^* = b) = 0$ for Lebesgue almost every $s > 0$.
- Do this by noting that for $\eta, q > 0$:

$$\{U_t^* = b\} \subseteq \{U_t^{(n)} \in (b-\eta, b+\eta) \text{ e.v.}\}$$

$$\int_0^\infty e^{-qt}\mathbb{P}_x(U_t^* = b)dt \le \int_0^\infty e^{-qt} \liminf_n \mathbb{P}_x(U_t^{(n)} \in (b-\eta, b+\eta))dt$$

$$\int_0^\infty e^{-qt}\mathbb{P}_x(U_t^* = b)dt$$
$$\le \limsup_{\eta\downarrow 0} \liminf_n \int_0^\infty e^{-qt}\mathbb{P}_x(U_t^{(n)} \in (b-\eta, b+\eta))dt$$

When is U^* a refracted process?

- Since

$$U_t^* = X_t - \delta \lim_{n\uparrow\infty} \int_0^t \mathbf{1}_{(U_s^{(n)} > b)} ds$$

we have that U^* is a refracted process as soon as one can prove that $\mathbb{P}_x(U_s^* = b) = 0$ for Lebesgue almost every $s > 0$.

- Do this by noting that for $\eta, q > 0$:

$$\{U_t^* = b\} \subseteq \{U_t^{(n)} \in (b-\eta, b+\eta) \text{ e.v.}\}$$

$$\int_0^\infty e^{-qt}\mathbb{P}_x(U_t^* = b)dt \leq \int_0^\infty e^{-qt} \liminf_n \mathbb{P}_x(U_t^{(n)} \in (b-\eta, b+\eta))dt$$

$$\int_0^\infty e^{-qt}\mathbb{P}_x(U_t^* = b)dt$$
$$\leq \limsup_{\eta\downarrow 0} \liminf_n \int_0^\infty e^{-qt}\mathbb{P}_x(U_t^{(n)} \in (b-\eta, b+\eta))dt$$

- Amazingly this can be done because a expression for the resolvent can be found semi-explicitly in terms of scale functions.

Resolvent

- Suppose that X has paths of bounded variation and $R^{(q)}(x, \cdot)$ is the resolvent measure of U under $\mathbb{P}_x$.

Resolvent

- Suppose that X has paths of bounded variation and $R^{(q)}(x,\cdot)$ is the resolvent measure of U under $\mathbb{P}_x$.
- For $x, b \in \mathbb{R}$, Borel B and $q > 0$,

$$
\begin{aligned}
&\mathbb{E}_x\left(\int_0^\infty e^{-qt}\mathbf{1}_{\{U_t\in B\}}ds\right) \\
&= \int_{B\cap[b,\infty)}\Bigg\{\left(\mathrm{e}^{\Phi(q)(x-b)}+\delta\Phi(q)\mathrm{e}^{-\Phi(q)b}\mathbf{1}_{\{x\geq b\}}\int_b^x \mathrm{e}^{\Phi(q)z}\mathbb{W}^{(q)}(x-z)\mathrm{d}z\right) \\
&\qquad\qquad \cdot\frac{\varphi(q)-\Phi(q)}{\delta\Phi(q)}\mathrm{e}^{-\varphi(q)(y-b)}-\mathbb{W}^{(q)}(x-y)\Bigg\}\mathrm{d}y \\
&+ \int_{B\cap(-\infty,b)}\Bigg\{\left(\mathrm{e}^{\Phi(q)(x-b)}+\delta\Phi(q)\mathrm{e}^{-\Phi(q)b}\mathbf{1}_{\{x\geq b\}}\int_b^x \mathrm{e}^{\Phi(q)z}\mathbb{W}^{(q)}(x-z)\mathrm{d}z\right) \\
&\qquad\qquad \cdot\frac{\varphi(q)-\Phi(q)}{\Phi(q)}\mathrm{e}^{\varphi(q)b}\int_b^\infty \mathrm{e}^{-\varphi(q)z}W^{(q)\prime}(z-y)\mathrm{d}z \\
&\qquad\qquad -\left(W^{(q)}(x-y)+\delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z-y)\mathrm{d}z\right)\Bigg\}\mathrm{d}y.
\end{aligned}
$$

Uniqueness

- We have established existence of a strong solution for all driving spectrally negative Lévy processes X.

Uniqueness

- We have established existence of a strong solution for all driving spectrally negative Lévy processes X.
- Uniqueness: Suppose that $U^{(1)}$ and $U^{(2)}$ are two strong solutions. Then writing

$$\Delta_t = U_t^{(1)} - U_t^{(2)} = -\delta \int_0^t (\mathbf{1}_{\{U_s^{(1)} > b\}} - \mathbf{1}_{\{U_s^{(2)} > b\}}) ds,$$

it follows from classical calculus that

$$\Delta_t^2 = -2\delta \int_0^t \Delta_s (\mathbf{1}_{\{U_s^{(1)} > b\}} - \mathbf{1}_{\{U_s^{(2)} > b\}}) ds.$$

Now note that thanks to the fact that $\mathbf{1}_{\{x>b\}}$ is an increasing function, it follows from the above representation that, for all $t \geq 0$, $\Delta_t^2 \leq 0$ and hence $\Delta_t = 0$ almost surely.

Sample identities for U

Some nice identities fall out of this analysis. Suppose that

$$\kappa_0^- := \inf\{t > 0 : U_t < 0\}.$$

For $q \geq 0$ and $x \geq 0$

$$\begin{aligned}
&\mathbb{E}_x\left(\int_0^{\kappa_0^-} e^{-qt}\delta \mathbf{1}_{\{U_t > b\}}\, ds\right) \\
&= -\delta \int_0^{(x-b)\vee 0} \mathbb{W}^{(q)}(z)\, dz \\
&\quad + \frac{W^{(q)}(x) + \delta \mathbf{1}_{\{x \geq b\}} \int_b^x \mathbb{W}^{(q)}(x-y)\, W^{(q)\prime}(y)\mathrm{d}y}{\varphi(q)\int_0^\infty \mathrm{e}^{-\varphi(q) y}\, W^{(q)\prime}(y+b)\mathrm{d}y}.
\end{aligned}$$