Self-similar Markov Processes and Stable Processes

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Declaration of any previous Submission of the Work

The material presented here for examination for the award of a higher degree by research has not been incorporated into a submission for another degree.

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Declaration of Authorship

I am the author of this thesis, and the work described therein was carried out by myself personally, with the exception of Chapters 2 to 4, which contain research articles that originated from collaboration with my supervisors Andreas E. Kyprianou and Victor M. Rivero.

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Weerapat Satitkanitkul

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Contents

Chapter 1

Introduction

This thesis concerns the development of some new results in the setting of self-similar Markov processes. In particular, we give a special attention to stable processes. For this reason, we start our exposition with a review of some standard definitions and theory.

1.1 Basic definitions and notations

We use the notion of a standard Markov process in the sense of [8] albeit with a slight simplification to suit the context of the thesis. Let S be a locally compact and separable Banach space with norm $||\cdot||$ and σ-algebra $\mathcal{S}$. We write $S_\Delta = S \cup \{\Delta\}$ where Δ is a cemetery state.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with outcomes $\omega \in \Omega$, collection of events $\mathcal{F}$ and probability law $\mathbb{P}$. Assume $\mathcal{F}$ to be complete in the sense that whenever $N \in \mathcal{F}$ with $\mathbb{P}(N) = 0$, then $S \in \mathcal{F}$ with $\mathbb{P}(S) = 0$ for all $S \subseteq N$. A complete right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ is a family of $\mathbb{P}$ σ-algebras such that, for all $0 < s < t$,

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F} \text{ and } \mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u,$$

and, for all $N \in \mathcal{F}$ with $\mathbb{P}(N) = 0$, we have that $N \in \mathcal{F}_0$. We also say that such a set N is a $\mathbb{P}$-null set. A *stochastic process* is a $\mathcal{F}$-measurable function $X : \omega \mapsto X(\omega)$, for $\omega \in \Omega$ such that

$$X(\omega) : [0, \infty) \to S_\Delta \text{ is a random path with } X(\omega) : t \mapsto X_t(\omega), \text{ for } t \geq 0.$$

We say that X is $(\mathcal{F}_t)_{t\geq 0}$-adapted if $(X_u(\omega))_{u\leq t}$ is $\mathcal{F}_t$-measurable, for all $t \geq 0$. Let $\mathbb{X}$ be the set of all paths in S_Δ defined as $\mathbb{X} = \{X : [0, \infty) \to S_\Delta\}$. We define $\mathbb{D}(S) \subseteq \mathbb{X}$ to contain paths $X : [0, \infty) \to S_\Delta$ that satisfy the following;

- paths have a lifetime $\varsigma := \inf\{s \geq 0 : X_s = \Delta\}$ where $X_s = \Delta$ for all $s \geq \varsigma$, and
- for $0 < s < \varsigma$, X_{s-} exists and $X_{s+} = X_s$.

For $\tau : \Omega \to [0,\infty) \cup \{\infty\}$ an $\mathcal{F}$-measurable function, let θ_τ, $\mathtt{k}_\tau$ be operators on paths such that

$$(\theta_\tau X)_s = X_{\tau+s}, \text{ for } s \geq 0 \text{ and } (\mathtt{k}_\tau X)_s = \begin{cases} X_s & \text{if } s < \tau \\ \Delta & \text{otherwise.} \end{cases}$$

We call $(\theta_t)_{t\geq 0}$ the shift operator and $(\mathtt{k}_t)_{t\geq 0}$ the killing operator. We say τ is a *stopping time* if

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t, \text{ for each } t \geq 0.$$

Finally, we write

$$\mathcal{F}_\tau = \{A : A \cap \{\omega : \tau(\omega) < t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Intuitively, $\mathcal{F}_\tau$ describes the information available to us up to time τ. For convenience, we write $\mathbb{P}_x(\cdot) := \mathbb{P}(\cdot|X_0 = x)$, for $x \in S_\Delta$. Define the σ-algebra generated by X as

$$\begin{aligned} \mathcal{X} &= \{\mathcal{A} \in \mathbb{X} : \{\omega \in \Omega : X(\omega) \in \mathcal{A}\} \in \mathcal{F}\}, \text{ and,} \\ \mathcal{X}_t &= \{\mathcal{A} \in \mathbb{X} : \{\omega \in \Omega : X(\omega) \in \mathcal{A}\} \in \mathcal{F}_t\}. \end{aligned}$$

There is a subtle difference between $\mathcal{F}$ and $\mathcal{X}$ in the sense that the former has collections of outcomes as events while the latter has collections of paths as members.

Definition 1. A stochastic process $(X, \mathbb{P})$ is a *standard Markov process* absorbed at Δ if it is a $(\mathcal{F}_t)_{t\leq 0}$-adapted process with the additional properties that

1. for $x \in S_\Delta$, $\mathbb{P}_x(X(\omega) \in \mathbb{D}(S)) = 1$,
2. it satisfies the *strong Markov property* i.e. for stopping time τ,

$$\mathbb{P}_x\left(\theta_\tau X(\omega) \in \mathcal{A} | \mathcal{F}_\tau \cap \{\tau < \infty\}\right) = \mathbb{P}_{X_\tau}\left(X \in \mathcal{A}\right) \mathbf{1}_{(\tau<\infty)}, \text{ for } \mathcal{A} \in \mathcal{X}, \tag{1.1}$$

Proposition 1. Any standard Markov process X is *quasi-left continuous*. This means for any stopping time $\tau < \varsigma$ and any sequence of stopping times $(\tau_n)_{n\geq 1}$ such that $\tau_n \uparrow \tau$, we have that

$$\mathbb{P}\left(X_\tau(\omega) = \lim_{n\to\infty} X_{\tau_n}(\omega)\right) = 1.$$

Proof. This can be done following the same argument as Lemma 3.2 from [26]. □

We can see that the strong Markov property is highly dependent on the filtration. The sample space Ω may be taken as just the space of paths $\mathbb{X}$ with the σ-algebras given by $\mathcal{F} = \mathcal{X}$ and $(\mathcal{F}_t)_{t\geq 0} = (\mathcal{X}_t)_{t\geq 0}$. However, we will stick with a more general class of filtrations. We have also abused our notation by referring to $(X, \mathbb{P})$ as a standard Markov process without mentioning the filtration. We will also be writing $X \in \mathcal{A}$ for the event $\{\omega \in \Omega : X(\omega) \in \mathcal{A}\}$. One can find a reference about continuous time Markov processes in general in [21].

We also use this opportunity to define the notion of a Feller semi-group. Let

$$C(S_\Delta) := \{f : S_\Delta \to \mathbb{R} \text{ such that } f \text{ is continuous and } \lim_{||x||\to\infty} f(x) = 0\}.$$

Further, we define a *transition operator* as a linear map $T : f \mapsto Tf$ such that, for each $x \in S_\Delta$, there exists a probability measure μ_x on S_Δ such that, for $f \in C(S_\Delta)$,

$$Tf(x) = \int_{S_\Delta} \mu_x(\mathrm{d}y) f(y).$$

A *semi-group* is an indexed family of transition operators $(\mathcal{P}_t)_{t\geq 0}$ that satisfies the *Chapman-Komogorov* equality:

$$\mathcal{P}_{t+s} f(x) = \mathcal{P}_t \mathcal{P}_s f(x) \text{ for } t, s > 0 \text{ and } f \in C(S_\Delta). \tag{1.2}$$

A *Feller semi-group* is a semi-group $(\mathcal{P}_t)_{t\geq 0}$, such that for $f \in C(S_\Delta)$, we have the followings:

1. for $t \geq 0$, $\mathcal{P}_t f(x)$ is continuous with respect to $x \in S_\Delta$, and
2. for $x \in S_\Delta$, $\lim_{t\to 0} \mathcal{P}_t f(x) = f(x)$.

Example 1.1.1 (Brownian motion)**.** Take $S = \mathbb{R}$. A family of transition operators $(\mathcal{P}_t)_{t\geq 0}$ specified by

$$\mathcal{P}_t f(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{(x-y)^2}{2t}} f(y) \mathrm{d}y, \tag{1.3}$$

is a Feller semi-group. The proof was left as an exercise at the end of Section 2.2 in [15].

Proposition 2. Given a Feller semi-group $(\mathcal{P}_t)_{t\geq 0}$, there exists a standard Markov process $(X, \mathbb{P})$ such that

$$\mathcal{P}_t f(x) = \mathbb{E}_x[f(X_t)], \text{ for } f \in C(S_\Delta). \tag{1.4}$$

We say that X is a *Càdlag modification* of $(\mathcal{P}_t)_{t\geq 0}$.

Proof. Using Kolmogorov's consistency, see Theorem 14.26 in [2], we can construct a stochastic process X satisfying (1.4). Then, appealing to Proposition 5 in Section 2.2 and Theorem 1 in Section 2.3 from [15] to verify the remaining conditions. □

Definition 2. We say $(X, \mathbb{P})$ is a ($\mathbb{R}^d$-valued) *self-similar Markov process* if it is a standard Markov process absorbed at $\Delta = 0$ taking values in $\mathbb{R}^d$ such that X satisfies the *scaling property* i.e.

There exists $\alpha > 0$ such that, for $x \in \mathbb{R}^d$, $c > 0$ and $\mathcal{A} \in \mathcal{X}$,

$$\mathbb{P}_{cx}\left((X_t)_{t\geq 0} \in \mathcal{A}\right) = \mathbb{P}_x\left((cX_{c^{-\alpha}t})_{t\geq 0} \in \mathcal{A}\right). \tag{1.5}$$

The use of the term "self-similar" has appeared in [24, 32] before it is studied by Lamperti in [28] under the name "semi-stable".

1.2 Self-similar processes as scaling limits

In a lot of literature, a stochastic process with a scaling property can occur as a scaling limit of another stochastic process, say $(Y_t)_{t\geq 0}$. Lamperti [28] proved semi-stable processes form a class of asymptotes of general stochastic processes. The treatment did not assume the process Y to be a Markov process. Semi-stable processes were later renamed as self-similar Markov processes due to their scaling property.

A *slowly varying function* is a function $S : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{S(ct)}{S(t)} = 1, \text{ for all } c > 0.$$

We denote $\mathcal{B}(\mathbb{R}^d)$ as the set of Borel sets in $\mathbb{R}^d$. Let (Y, P) be a $\mathbb{R}^d$-stochastic process. Assume that there exists a function $f : \mathbb{R} \to \mathbb{R}$ and $(X_t)_{t\geq 0}$ such that

$$Y_t^L := \frac{Y_{Lt}}{f(L)} \to X_t \text{ as } L \to \infty \text{ for finite dimensional distributions.} \tag{1.6}$$

This means for, $n \in \mathbb{N}$, $A_1, A_2, A_3, \ldots, A_n \in \mathcal{B}(\mathbb{R}^d)$ and $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$,

$$\lim_{L\to\infty} \mathbb{P}\left(Y_{t_i}^L \in A_i, \forall i = 0, 1, 2, \ldots, n\right) = \mathbb{P}\left(X_{t_i} \in A_i, \forall i = 0, 1, 2, \ldots, n\right).$$

Proposition 3 (Lamperti 1962)**.** Let (Y, P) be an $\mathbb{R}^d$-stochastic process. Suppose there exists $f : \mathbb{R}^+ \to \mathbb{R}^+$ and a non-degenerate $(X_t)_{t\geq 0}$ such that condition (1.6) holds. Then, necessarily, f must take the form,

$$f(t) = t^{1/\alpha} S(t), \text{ with } S \text{ slowly varying.}$$

Moreover, for $i = 0, 1, 2, \ldots, n \in \mathbb{N}$, let $A_i \in \mathcal{B}(\mathbb{R}^d)$ and $0 \leq t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$. We have that

$$\mathbb{P}\left(X_{t_i} \in A_i, \forall i = 0, 1, 2, \ldots, n\right) = \mathbb{P}\left(cX_{c^{-\alpha}t_i} \in A_i, \forall i = 0, 1, 2, \ldots, n\right). \tag{1.7}$$

We say that a process X satisfying (1.7) is self-similar in finite dimensional sense with index α. This is a weaker condition than that specified in (4.29) where the statement is on the whole path. The definition of index of similarity α in this thesis is different from those definitions appearing in [28]. Our index of similarity is what is known as Hurst index in other literature.

Lamperti did not specify where the process starts. However, from the construction

$$\mathbb{P}(X_t \in A) = \mathbb{P}(t^{1/\alpha} X_1 \in A), \text{ for } A \in \mathcal{B}(\mathbb{R}^d), \text{ and } t \geq 0.$$

Therefore, it follows that $X_{0+} = 0$ almost surely.

As a process, we want X to start at an arbitrary $x \in \mathbb{R}^d \setminus \{0\}$. Further, we want to consider the case where X is a Markov process. We need to modify the setting slightly. Let (Y, P) be a Markov process. Assume that there exists $f : \mathbb{R} \to \mathbb{R}$ and a Feller semi-group $(\mathcal{P}_t)_{t\geq 0}$ such that, for $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\lim_{L\to\infty} P_{f(L)x}\left(\frac{Y_{Lt}}{f(L)} \in A\right) = \mathcal{P}_t \mathbf{1}_A(x), \tag{1.8}$$

where

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}.$$

It then follows from Proposition 1.7 that $(\mathcal{P}_t)_{t\geq 0}$ satisfies

$$\mathcal{P}_t f(cx) = \mathcal{P}_{c^{-\alpha}t}(H_c f)(x), \text{ where } (H_c f)(x) := f(cx) \text{ for all } x \in \mathbb{R}^d.$$

Further, if we assume that $(\mathcal{P}_t)_{t\geq 0}$ is a Feller semi-group, there exists a version $(X, \mathbb{P})$ that is a self-similar Markov process defined in Definition 2.

The first example that can be constructed this way is Brownian motion. The existence of the scaling limit is just a consequence of Central Limit Theorem with $\alpha = 2$.

Example 1.2.1 (Brownian motion)**.** Let $(\chi_i)_{i\geq 1}$ be (i.i.d.) real random variables with $E[\chi_1] = 0$ and $E[\chi_1^2] = 1$. Define the sequence

$$S_n = \sum_{i=1}^{n} \chi_n, \text{ for } n \geq 1,$$

with $S_0 = 0$ and the interpolation

$$Y_t = S_{\lfloor t\rfloor} + (t - \lfloor t\rfloor)\chi_{\lfloor t\rfloor+1}, \text{ for } t \geq 0.$$

Under $\mathbb{P}_x$, $x \in \mathbb{R}$, we can construct $(B_t)_{t\geq 0}$ using the Central Limiting Theorem,

$$B_t := x + \lim_{n\to\infty} \frac{Y_{nt}}{n^{1/2}},$$

where B_t has an explicit semi-group specified in (1.3). It can be checked from the density that $(B_t)_{t\geq 0}$ has the scaling property. For $c > 0$ and $x \in \mathbb{R}$,

$$\mathbb{P}_x(cB_{c^{-2}t} \in A) = \frac{c}{\sqrt{2\pi t}} \int_{c^{-1}A} \mathrm{e}^{-\frac{c^2}{2t}(y-x)^2} \mathrm{d}y = \mathbb{P}_{cx}(B_t \in A), \ A \in \mathcal{B}(\mathbb{R}).$$

Further, there exists a standard Markov process whose distribution is consistent with B_t, $t \geq 0$. The process is known as a *Brownian Motion* is a simple example of ssMp with index 2.

Example 1.2.2 (Continuous-state branching process with interactions)**.** Consider a population model with the number of individuals at time $t \geq 0$ given by Y_t. We set the evolution of Y in discrete time steps as the following

1. start with $Y_0 = y_0 \in \mathbb{N}$,

2. at each time step n, the kth-individual living in this generation dies and produces $Z_{k,n}$ offsprings, where $(Z_{k,n})_{k\in\mathbb{N}}$ are independent with distribution given by

$$P(Z_{k,n} = a) = p_a(Y_n), \text{ where } \sum_{a\geq 0} p_a(x) = 1 \text{ for all } x \in (0,\infty).$$

If we can find a function f such that there exists a non degenerate limit of

$$\frac{Y_{Lt}}{f(L)} \text{ given } Y_0 = f(L)x, \text{ as } L \to \infty.$$

Then, the result that comes out must be a positive self-similar Markov process. The setting could also extend to multi-type population with slight complications in notations. However, it would not change the intuition that one could expect the result to be a self-similar Markov process. The question of what conditions on $(p_a(\cdot))_{a\in\mathbb{Z}}$ are required for this to exist still remains rather open ended. Lamperti [29] proved convergence for the cases when the offspring distribution does not depend on the current population i.e. Galton-Watson process.

1.3 Lamperti's transformation

In this part, we aim to describe an invertible map between a self-similar Markov process and a Markov additive process (defined later) given in [1]. This allows us to deduce the properties for Markov additive processes from self-similar Markov processes. Conversely, one might also learn more about self-similar Markov processes from Markov additive processes. We give a heuristic deriviation of the Lamperti-Kiu transform where the choice of time change is explained.

Let X be an $\mathbb{R}^d$-ssMp with index $\alpha \in (0,2)$ and probability law $(\mathbb{P}_x)_{x\in\mathbb{R}^d}$. Consider the probability,

$$\mathbb{P}_x\left(\left(\frac{X_{t+|X_t|^\alpha s}}{|X_t|}\right)_{s\geq 0} \in \mathcal{A}\,\middle|\, \mathcal{F}_t\right), \text{ for } t>0,\, x\in\mathbb{R}^d\setminus\{0\} \text{ and } \mathcal{A}\in\mathcal{X}.$$

Denoting the Markov shift of X by $\widetilde{X} = \theta_t X$, by Markov property and scaling property, this is equal to

$$\mathbb{P}_{X_t}\left(\left(\frac{\widetilde{X}_{|X_t|^\alpha s}}{|X_t|}\right)_{s\geq 0} \in \mathcal{A}\,\middle|\, \mathcal{F}_t\right) = \mathbb{P}_{\arg(X_t)}\left((X_s)_{s\geq 0}\in\mathcal{A}\right), \text{ where } \arg(x) := \frac{x}{|x|}.$$

This means that we can determine the evolution of X after time $t > 0$ with just a rescaled X starting at $\arg(X_t)$ with the factor of $|X_t|^{-\alpha}$ time adjustment. Hence, using this intuition, we introduce the time change

$$\varphi(t) := \inf\left\{0 \le s < T_0 : \int_0^s |X_u|^{-\alpha} \mathrm{d}u > t\right\}.$$

We call $\{X_{\varphi(t)} : t \ge 0\}$ a *Kiu process*. According to Kiu [23], a Kiu process is multiplicatively invariant i.e.

$$\mathbb{P}_x\left((cX_{\varphi(t)})_{t\ge 0} \in \mathcal{A}\right) = \mathbb{P}_{cx}\left((X_{\varphi(t)})_{t\ge 0} \in \mathcal{A}\right).$$

Another way of saying this is that it is self-similar with index 0.

One could make an analogy of this time change with the idea that time runs with different speeds depending on the position in space. The stopping time $\varphi(t)$ can also be considered as a "particle clock" which describes how much time the particle feels has passed from its point of view, when the "system clock" has passed precisely t units of time. The "particle clock" will be accelerated faster the closer X gets to 0 and slow down as X gets far from the origin. For example, the process might get absorbed in finite time, with $T_0 < \infty$, but the "particle clock" might need infinite time to get there, i.e. $\varphi(T_0) = \infty$.

We can turn a Kiu process into a shift invariant process by taking logarithm of the radial part. The resulting process will be called a Markov additive process (MAP). Let E be a locally compact and separable subspace of $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d \setminus \{0\} : |x| = 1\}$ with $d \ge 1$.

Definition 3 (MAP)**.** Let (ξ, Θ) be a $\mathbb{R} \times E$-valued standard Markov process absorbed at $(-\infty, \Delta)$ with probability law $\mathbf{P}$ and $\mathcal{G}_\infty$ and $(\mathcal{G}_t)_{t\ge0}$ their filtrations. We say this is a *Markov additive process*, if for $r \in \mathbb{R}$, $\theta \in E$, $s, t > 0$ and $f : \mathbb{R} \times E \to \mathbb{R}^+$ measurable,

$$\mathbf{E}_{r,\theta}\left[f\left(\xi_{t+s} - \xi_t, \Theta_{t+s}\right)\mathbf{1}_{(t+s<\zeta)}\middle|\,\mathcal{G}_t\right] = \mathbf{E}_{0,\Theta_t}\left[f(\xi_s, \Theta_s)\mathbf{1}_{(s<\zeta)}\right]\mathbf{1}_{(t<\zeta)},$$

where we set the stopping time ζ as the life-time of the process with $(\xi_s, \Theta_s) = (-\infty, \Delta)$, $\forall s \ge \zeta$.

In the literature, we say Θ is the *modulator part* and ξ is the *ordinator part*. In the degenerate case that $E = \{1\}$, ξ is nothing more than a real valued Lévy process killed after an independent and exponentially distributed time with rate in $[0, \infty)$, where the rate 0 means infinite lifetime. When E is finite, there are many known results. A reference of the case where finite E can be found in several books, for example [4, 5, 34].

Proposition 4 (Lamperti-Kiu transform)**.** Let X be an $\mathbb{R}^d$-valued self-similar Markov process. Under $\mathbb{P}_x$ with $x \in \mathbb{R}^d \setminus \{0\}$, define (ξ, Θ) as

$$\xi_t = \log|X_{\varphi(t)}| \text{ and } \Theta_t = \arg(X_{\varphi(t)}), \text{ for } t < \zeta,$$

with $\zeta = \int_0^{T_0} |X_t|^{-\alpha}\mathrm{d}t$. Then, (ξ, Θ) is a Markov additive process with $\xi_0 = \log|x|$ and $\Theta_0 = \arg(x)$. Conversely, let (ξ, Θ) be a Markov additive process with $\xi_0 = \log|x|$ and $\Theta_0 = \arg(x)$.

Define, for $t \geq 0$,

$$\psi(t) = \inf\left\{s > 0 : \int_0^s \mathrm{e}^{\alpha\xi_u} > t\right\} \text{ when } t < \int_0^\zeta \mathrm{e}^{\alpha\xi_u}\mathrm{d}u.$$

Otherwise, $\psi(t) = \infty$. Then, the process X given by,

$$X_t := \begin{cases} \mathrm{e}^{\xi_{\psi(t)}}\Theta_{\psi(t)} & \text{when } t < \int_0^\zeta \mathrm{e}^{\alpha\xi_s}\mathrm{d}s \\ 0 & \text{otherwise.} \end{cases}, \text{ is a self-similar Markov process with } X_0 = x.$$

Moreover, fixing $\alpha > 0$, this transformation is bijective. We will write $X = LK(\xi, \Theta, \alpha)$.

In 1972, Lamperti introduced this transformation in his paper [30] where he proved it for the case X taking positive real values. The generalisation for this into $\mathbb{R}$ and $\mathbb{R}^d$ case comes much later in 2010s, see [14, 1] . This transformation also explains how rich the class of self-similar Markov process is. We can just take $\xi_t = t$ and Θ to be any Markov process on $\mathbb{S}^{d-1}$. Hence, there are more self-similar Markov process on $\mathbb{R}^d$ than there are Markov processes on $\mathbb{S}^{d-1}$.

1.4 Stable processes

In section 1.2, we described the Central Limit Theorem as an example of a scaling limit of (i.i.d.) sum. This requires the increments to have finite second moment. It would be natural to ask whether or not the Central Limit Theorem can be modified to the case when the second moment is infinite. We motivate this section by introducing the scaling limit result for real-valued processes.

Example 1.4.1 ($\mathbb{R}$-stable process)**.** Let $(\chi_i)_{i\geq 1}$ be (i.i.d.) such that there exists $\alpha \in (0, 2)$, $c_-, c_+ \geq 0$, $(c_- + c_+) > 0$ and L slowly varying function satisfying

$$\lim_{x\to+\infty} \frac{P(\chi_1 > x)x^\alpha}{L(x)} = c_+ \text{ and } \lim_{x\to-\infty} \frac{P(\chi_1 < x)|x|^\alpha}{L(-x)} = c_-. \tag{1.9}$$

Define

$$Y_t = S_{\lfloor t\rfloor} + (t - \lfloor t\rfloor)\chi_{\lfloor t\rfloor+1}, \text{ for } t \geq 0.$$

Then, for each $t \geq 0$, there exists X_t such that for $A \in \mathcal{B}(\mathbb{R})$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{Y_{nt}}{n^{1/\alpha}} \in A\right) = \mathbb{P}_0(X_t \in A).$$

We say that X_t has a *stable distribution.* Indeed, from the construction, we have that

$$X_{t+s} - X_t \text{ is independent of } X_t \text{ and distributed as } X_s, \text{ for } s, t > 0. \tag{1.10}$$

Hence, we can define a standard Markov process X with transition probability given by

$$\mathbb{P}_x(X_t \in A) = \mathbb{P}_0(x + X_t \in A),\ x \in \mathbb{R},\ \text{ and }\ t \geq 0.$$

The process X satisfies the scaling property with index α given in Definition 2.

We want to push this idea of the stable distribution into $\mathbb{R}^d$-valued processes. However, it would be difficult to make sense of stable processes without first understanding Lévy processes and the Lévy-Khintchine formula. We translate the feature that appears in equation (1.10) into the context of standard Markov processes.

Definition 4 ($\mathbb{R}^d$-Lévy process)**.** Let $(Y, \mathbb{P})$ be a $\mathbb{R}^d$-valued standard Markov process. We say $(Y, \mathbb{P})$ is a *Lévy process* if

(i) it has infinite lifetime, and

(ii) it has *stationary and independent increments*, that is

$$\mathbb{P}_x\left(Y(t+s) - Y(t) \in A \middle| \mathcal{G}_t\right) = \mathbb{P}_0\left(Y(s) \in A\right). \tag{1.11}$$

for $x \in \mathbb{R}^d$, $s, t > 0$ and $A \in \mathcal{B}(\mathbb{R}^d)$.

The use of the letter $\mathbb{P}$ is intentional as we will soon consider Y as a self-similar Markov process of some form. Let $n \in \mathbb{N}$, and consider

$$Y(1) - Y(0) = \sum_{i=0}^{n} \left[Y\left(\frac{i+1}{n}\right) - Y\left(\frac{i}{n}\right) \right].$$

Using the definition, the terms in the sum are independent and distributed as $Y(1/n) - Y(0)$. This means the distribution of $Y(1) - Y(0)$ has to be *infinitely divisible* and we have that

$$\mathbb{E}_0\left[\mathrm{e}^{i<\lambda, Y(1)>}\right] = \mathbb{E}_0\left[\mathrm{e}^{i<\lambda, Y(1/n)>}\right]^n,\ \text{ for }\ \lambda \in \mathbb{R}^d.$$

We could also obtain, for $m, n \in \mathbb{N}$,

$$Y\left(\frac{m}{n}\right) - Y(0) = \sum_{i=0}^{m} \left[Y\left(\frac{i+1}{n}\right) - Y\left(\frac{i}{n}\right) \right],$$

and

$$\mathbb{E}_0\left[\mathrm{e}^{i<\lambda, Y(m/n)>}\right] = \mathbb{E}_0\left[\mathrm{e}^{i<\lambda, Y(1/n)>}\right]^m = \mathbb{E}_0\left[\mathrm{e}^{i<\lambda, Y(1)>}\right]^{m/n}.$$

For $t \geq 0$, we can find sequences of integers $(m_k)_{k \in \mathbb{N}}$ and $(n_k)_{k \in \mathbb{N}}$ such that $\frac{m_k}{n_k} \downarrow t$. Then, we can use the continuity of Y at time t, to have that

$$\mathbb{E}_0\left[\mathrm{e}^{i<\lambda, Y(t)>}\right] = \mathbb{E}_0\left[\mathrm{e}^{i<\lambda, Y(1)>}\right]^t,\ \text{ for }\ \lambda \in \mathbb{R}^d,$$

where $< \cdot, \cdot >$ is the inner scalar product in $\mathbb{R}^d$. Hence, the probability law of a Lévy process can be determined by the characteristic exponent of $Y(1)$ which is given by

$$\exp\{-\Psi(\lambda)\} = \mathbf{E}_0\left[\mathrm{e}^{i<\lambda, Y(1)>}\right], \text{ for } \lambda \in \mathbb{R}^d,$$

It is necessary that it must follow Lévy-Khintchine formula.

Theorem 1 (Lévy-Khintchine)**.** The followings are equivalent

(a) There exists a Lévy process $(Y, \mathbb{P})$ with characteristic function Ψ.

(b) There exists a *characteristic triplet* $(\mathbf{a}, Q, \Pi)$ where $\mathbf{a} \in \mathbb{R}^d$, Q a positive semi-definite $d \times d$ real matrix and Π a measure on $\mathbb{R}^d \setminus \{0\}$ with $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\Pi(\mathrm{d}x) < \infty$ such that

$$\Psi(\lambda) = i < \mathbf{a}, \lambda > + \frac{1}{2}\lambda^T Q \lambda + \int_{\mathbb{R}^d}\left(1 - \mathrm{e}^{i<\lambda,x>} + i < \lambda, x > \mathbf{1}_{(|x|<1)}\right)\Pi(\mathrm{d}x), \text{ for } \lambda \in \mathbb{R}^d.$$

We say $(\mathbf{a}, Q, \Pi)$ is the *Lévy-Khintchine representation* of Y. The representation is unique.

The first component $\mathbf{a} \in \mathbb{R}^d$ represents a linear trend of the process, the second component Q represents the Gaussian component and the third component Π represents the jump rate of the Lévy process. Using Lévy-Itô Decomposition, as presented in Chapter 2 of [26], it can be shown that

$$\mathbf{E}_x\left[\sum_{0\le s\le t}\mathbf{1}_{(Y(s)-Y(s-)\in B)}\right] = \Pi(B)t,$$

for $x \in \mathbb{R}^d$, $t \ge 0$ and $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ such that the closure $\overline{B}$ does not contain 0. More precisely, the number of jumps with increments in a set B is Poisson distributed with mean $\Pi(B)t$.

Lévy processes are very well studied due to their homogeneity in space and time. We refer to [7, 26] for some general theory.

Definition 5 ($\mathbb{R}^d$-stable process)**.** We say that Y is a *stable process* if it is a Lévy process with characteristic triplets given by $(\mathbf{a}, 0, \Pi)$ with

(i)
$$\Pi(B) = \int_{\mathbb{S}^{d-1}}\Lambda(\mathrm{d}\theta)\int_{(0,\infty)}\mathbf{1}_{(r\theta\in B)}\frac{\mathrm{d}r}{r^{\alpha+1}}, \text{ for } B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),$$
where $\alpha \in (0, 2)$ and Λ a finite measure on $\mathbb{S}^{d-1}$,

(ii) if $\alpha \in (0, 1) \cup (1, 2)$,
$$\mathbf{a} = \int_{\mathbb{R}^d} x\mathbf{1}_{(|x|<1)}\Pi(\mathrm{d}x),$$
otherwise, $\mathbf{a} \in \mathbb{R}^d$ is arbitrary.

Example 1.4.2. In the one dimensional case, we have $\mathbb{S}^{d-1} = \{-1, 1\}$. Then, the measure Λ must take the form

$$\Lambda(\mathrm{d}\theta) = c_- \delta_{\{-1\}}(\mathrm{d}\theta) + c_+ \delta_{\{1\}}(\mathrm{d}\theta), \text{ for some } c_-, c_+ > 0.$$

Referring back to Example 1.4.1, the constants c_-, c_+ are consistent with those in the equation (1.9). We say that a real stable process is symmetric when $c_- = c_+$ and $\mathbf{a} = 0$. In this case, it also follows that X has the same distribution as $-X$.

The scope of this thesis restricts to a class where a stable process also satisfies a scaling property. This is usually called "strictly stable processes" in the literature but we will henceforth refer to it as just stable (Lévy) processes. This requires

$$\Psi(c\lambda) = c^\alpha \Psi(\lambda), \text{ for } c > 0.$$

In such cases, we can refer to Theorem 14.10 from [36] Chapter 3 to have that

$$\Psi(\lambda) = \int_{\mathbb{S}^{d-1}} | < \lambda, \theta > |^\alpha \left(1 - i \tan\left(\frac{\pi\alpha}{2}\right) \operatorname{sgn}(< \lambda, \theta >)\right) \Lambda(\mathrm{d}\theta), \text{ for } \alpha \in (0,1) \cup (1,2),$$

and

$$\Psi(\lambda) = i < \lambda, \mathbf{a} > + \int_{\mathbb{S}^{d-1}} \left(| < \lambda, \theta > | + i\frac{2}{\pi} \operatorname{sgn}(< \lambda, \theta >) \log | < \lambda, \theta > | \right) \Lambda(\mathrm{d}\theta), \text{ for } \alpha = 1,$$

where

$$\mathbf{a} \in \mathbb{R}^d \text{ and } \int_{\mathbb{S}^{d-1}} \theta \Lambda(\mathrm{d}\theta) = 0.$$

Stable processes satisfy the scaling property. However, they still do not fit our definition of self-similar Markov processes given in Definition 2. This is because Y could hit 0 and then continue thereafter. There are many modifications we can make to get different self-similar Markov processes out of a stable process. One of the way possible is to kill it upon hitting 0.

Example 1.4.3 (Stable processes killed upon hitting 0)**.** Consider the hitting time of 0 for a stable process

$$T_0 = \inf\{s > 0 : Y(s) = 0\}.$$

If $d \geq 2$ or $\alpha \in (0, 1]$, this is infinite almost surely. If $d = 1$ and $\alpha \in (1, 2)$, $T_0 < \infty$ almost surely. The explicit probability density of T_0 in this case has been worked out as power series by Kuznetsov et. al. [25]. In any case, the process $X := \mathtt{k}_{T_0} Y$ is a self-similar Markov process.

We can also consider a different way of killing the process Y to get a self-similar Markov process that lives in a cone.

Example 1.4.4 (Stable processes killed upon exiting a cone)**.** We say $\Gamma \subseteq \mathbb{R}^d$ is a *Lipchitz cone* if

$$\Gamma = \left\{ x \in \mathbb{R}^d \setminus \{0\} : \frac{x}{|x|} \in E \right\} \text{ for } E \subseteq \mathbb{S}^{d-1} \text{ open.}$$

Then, we define the exit time from the cone and the stable process killed upon exiting a cone as

$$\kappa_\Gamma = \inf\{s > 0 : Y(s) \notin \Gamma\} \text{ and } X^\Gamma = \mathtt{k}_{\kappa_\Gamma} Y.$$

Then, the killed process X^Γ is a self-similar Markov process with representation $LK(\xi, \Theta, \alpha)$. In this thesis, we will study such killed processes as examples of when theory developed for self-similar Markov processes can be applied.

We can gain a better understanding of the stopping time κ_Γ using the Lamperti-Kiu representation. For $t > 0$ and $x \in \Gamma$,

$$\mathbb{P}_x(\kappa_\Gamma > t) = \mathbf{P}_{\log|x|, \arg(x)}\left(\int_0^\zeta \mathrm{e}^{\alpha\xi_u} \mathrm{d}u > t\right) = \mathbf{P}_{0, \arg(x)}\left(|x|^\alpha \int_0^\zeta \mathrm{e}^{\alpha\xi_u} \mathrm{d}u > t\right).$$

It can be easily verified from the equation above that κ_Γ under $\mathbb{P}_{cx}$ is the same in distribution as $c^\alpha \kappa_\Gamma$ under $\mathbb{P}_x$. The integral $\int_0^\zeta \mathrm{e}^{\alpha\xi_u} \mathrm{d}u$ is known as an *exponential functional* of a Markov additive process.

1.5 Isotropic stable processes in $\mathbb{R}^d$

The natural next step from real stable processes would be to replicate similar theory for $\mathbb{R}^d$-stable processes, $d \geq 2$. We keep our focus on *isotropic stable processes*. A stable process is isotropic if its increments, namely $X_t - X_s$ for $t > s > 0$, is distributionally invariant after orthogonal transformation. This would mean that

$$\mathbf{a} = 0 \text{ and } \Lambda \text{ is uniform on } \mathbb{S}^{d-1}.$$

The characteristic exponents of isotropic stable processes are given in Theorem 14.14 from Chapter 3 of [36].

Without loss of generality, one can choose the total mass of $\Lambda(\cdot)$ so that

$$\Psi(\lambda) = |\lambda|^\alpha, \ \lambda \in \mathbb{R}^d.$$

Equivalently, the Lévy measure is given by

$$\Pi(B) = \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(\alpha/2)|} \int_B \frac{\mathrm{d}y}{|y|^{\alpha+d}}, \ B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),$$

where $\mathrm{d}y$ denotes the d-dimensional Lebesgue measure.

Let Y be an isotropic $\mathbb{R}^d$-stable process with index $\alpha \in (0,2)$, $d \geq 2$. The process Y does not hit 0, hence $T_0 = \infty$ almost surely, according to Chapter I of [7]. So, the killed process $X = \mathtt{k}_{T_0} Y$ is basically Y i.e. no killing. We will refer to it as $(X, \mathbb{P})$ to stress that we will study it as a self-similar Markov process.

One attractive feature of an isotropic $\mathbb{R}^d$-stable process is the simplicity of its Riesz-Bogdan-Żak transform, introduced in [12]. Write $Kx = x/|x|^2$, for $x \in \mathbb{R}^d \setminus \{0\}$. Introduce a time change

$$\eta(t) = \inf\left\{ s > 0 : \int_0^s |X_u|^{-2\alpha} > t \right\}, \text{ for } t \geq 0.$$

Then, for $x \neq 0$, it follows that $(X, \mathbb{P}^\circ)$ given by

$$\mathbb{P}^\circ_x (X \in \mathcal{A}) = \mathbb{P}_{Kx}\left((KX_{\eta(s)})_{s\geq 0} \in \mathcal{A} \right), \mathcal{A} \in \mathcal{X}, \tag{1.12}$$

is also a self-similar Markov process with index α. We say that $(X, \mathbb{P}^\circ)$ is the *Riesz-Bogdan-Żak transform* of $(X, \mathbb{P})$. The idea behind this comes from the fact that if the process $(X_t)_{t\geq 0}$ has Lamperti-Kiu representation $LK(\xi, \Theta, \alpha)$, then $(X_{\eta(t)})_{t\geq 0}$ has Lamperti-Kiu representation $LK(-\xi, \Theta, \alpha)$. For an isotropic stable process, $(X, \mathbb{P}^\circ)$ can be described as a density transform of $(X, \mathbb{P})$. Specifically, according to Theorem 1 of [12], for $x \in \mathbb{R}^d \setminus \{0\}$ and $t > 0$,

$$\mathbb{P}^\circ_x (X \in \mathcal{A}, t < T_0) = \mathbb{E}_x \left[\frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}} \mathbf{1}_{(X\in\mathcal{A}, t<T_0)} \right], \mathcal{A} \in \mathcal{X}_t.$$

Theorem III.3.4 of [19] implies that a martingale change of measure remains valid at a given stopping time. This means, for $x \in \mathbb{R}^d \setminus \{0\}$ and τ a stopping time,

$$\mathbb{P}^\circ_x (X \in \mathcal{A}, \tau < T_0) = \mathbb{E}_x \left[\frac{|X_\tau|^{\alpha-d}}{|x|^{\alpha-d}} \mathbf{1}_{(X\in\mathcal{A}, \tau<T_0)} \right], \mathcal{A} \in \mathcal{X}_\tau.$$

We say that the process $(X, \mathbb{P}^\circ)$, described above, is the *Doob-h transform* of $(X, \mathbb{P})$ with respect to the *h-function* $x \mapsto |x|^{\alpha-d}$. This Doob-h transform increases the likelihood of X to be closer to the origin and decreases the likelihood of X to be further from the origin. This is clear when we consider

$$\mathbb{P}^\circ_x (X_\tau \in \mathrm{d}y, \tau < T_0) = \frac{|y|^{\alpha-d}}{|x|^{\alpha-d}} \mathbb{P}_x (X_\tau \in \mathrm{d}y, \tau < T_0), \text{ for } x, y \in \mathbb{R}^d \setminus \{0\}, \tag{1.13}$$

and the fact that $\alpha - d < 0$ when $d \geq 2$.

Example 1.5.1 (Entrance/Exit into/from a ball)**.** Let $B(r) = \{x : |x| < r\}$ a unit ball. Define the following stopping times, for $r > 0$,

$$\tau_r^\ominus = \inf\{s > 0 : X_s \notin B(r)\} \text{ and } \tau_r^\oplus = \inf\{s > 0 : X_s \in B(r)\}.$$

Define the transformed process and the stopping time

$$(\widetilde{X})_{t\geq 0} = (KX_{\eta(t)})_{t\geq 0} \text{ and } \widetilde{\tau}_1^\oplus = \inf\{s > 0 : \widetilde{X}_s \in B(1)\}.$$

From the definitions, we also have $\widetilde{\tau}_1^\oplus = \inf\{s > 0 : X_{\eta(s)} \notin B(1)\}$. Recalling that $\eta(\cdot)$ is continuous and non-decreasing, it must follow that $\eta(\widetilde{\tau}_1^{\oplus}) = \tau_1^{\ominus}$. Hence, on the event $\{\tau_1^{\ominus} < \infty\}$, we have that

$$\widetilde{X}_{\widetilde{\tau}_1^{\oplus}} = KX_{\eta(\widetilde{\tau}_1^{\oplus})} = KX_{\tau_1^{\ominus}}.$$

We appeal to (1.12) and (1.13) to have that, for $x \in B(1)^c$ and $y \in B(1)$,

$$\begin{aligned}\mathbb{P}_{Kx}(KX_{\tau_1^{\ominus}} \in \mathrm{d}y, \tau_1^{\ominus} < \infty) &= \mathbb{P}_{Kx}(\widetilde{X}_{\widetilde{\tau}_1^{\oplus}} \in \mathrm{d}y, \tau_1^{\oplus} < \infty) \\ &= \mathbb{P}_x^{\circ}(X_{\tau_1^{\oplus}} \in \mathrm{d}y, \tau_1^{\oplus} < \infty) \\ &= \frac{|y|^{\alpha-d}}{|x|^{\alpha-d}}\mathbb{P}_x(X_{\tau_1^{\oplus}} \in \mathrm{d}y, \tau_1^{\oplus} < \infty).\end{aligned}$$

Hence,

$$\mathbb{P}_x(X_{\tau_1^{\oplus}} \in \mathrm{d}y, \tau_1^{\oplus} < \infty) = \frac{|x|^{\alpha-d}}{|y|^{\alpha-d}}\mathbb{P}_{Kx}(KX_{\tau_1^{\ominus}} \in \mathrm{d}y, \tau_1^{\ominus} < \infty), \text{ for } x \in B(1) \text{ and } y \in B(1)^c.$$

This means that we can infer the distribution of $X_{\tau_1^{\oplus}}$ from the distribution of $X_{\tau_1^{\ominus}}$. There are many other computations that could be done in the same spirit. The method can be verified by using an explicit formula for the entrance/exit distribution into/from a ball, as computed in [9].

For $x \in B(1)$ and $y \in B(1)^c$,

$$\mathbb{P}_x\left(X_{\tau_1^{\ominus}} \in \mathrm{d}y\right) = C_{\alpha,d}\frac{|1-|x|^2|^{\alpha/2}}{|1-|y|^2|^{\alpha/2}|x-y|^d}\mathrm{d}y, \tag{1.14}$$

where

$$C_{\alpha,d} = \pi^{-d/2+1}\Gamma\left(\frac{d}{2}\right)\sin\frac{\pi\alpha}{2}.$$

Then, for $x \in B(1)^c$ and A a Borel subset of $B(1)$,

$$\begin{aligned}\mathbb{P}_x(X_{\tau_1^{\oplus}} \in A, \tau_1^{\oplus} < \infty) &= \int_A \mathbb{P}_x(X_{\tau_1^{\oplus}} \in \mathrm{d}y, \tau_1^{\oplus} < \infty) \\ &= \int_A \frac{|x|^{\alpha-d}}{|y|^{\alpha-d}}\mathbb{P}_{Kx}(KX_{\tau_1^{\ominus}} \in \mathrm{d}y, \tau_1^{\ominus} < \infty) \\ &= \int_{KA} \frac{|x|^{\alpha-d}}{|Kz|^{\alpha-d}}\mathbb{P}_{Kx}(X_{\tau_1^{\ominus}} \in \mathrm{d}z, \tau_1^{\ominus} < \infty) \\ &= C_{\alpha,d}\int_{KA}\frac{|x|^{\alpha-d}}{|Kz|^{\alpha-d}}\frac{|1-|Kx|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}|Kx-z|^d}\mathrm{d}z,\end{aligned}$$

where $KA := \{z \in \mathbb{R}^d : Kz \in A\}$. Apply a change of variables $y = Kz$, we have $\mathrm{d}z = |y|^{-2d}\mathrm{d}y$ and

$$\mathbb{P}_x(X_{\tau_1^{\oplus}} \in A, \tau_1^{\oplus} < \infty) = \int_A \frac{|x|^{\alpha-d}}{|y|^{\alpha-d}}\frac{|1-|Kx|^2|^{\alpha/2}}{|1-|Ky|^2|^{\alpha/2}|Kx-Ky|^d}|y|^{-2d}\mathrm{d}y,$$

which simplifies into

$$\mathbb{P}_x\left(X_{\tau_1^{\oplus}} \in \mathrm{d}y, \tau_1^{\oplus} < \infty\right) = C_{\alpha,d} \int_A \frac{|1-|x|^2|^{\alpha/2}}{|1-|y|^2|^{\alpha/2}|x-y|^d} \mathrm{d}y. \tag{1.15}$$

The density functions look exactly the same with a different range of variables.

Martin kernels for a cone

Isotropic stable processes have strong connections to α-harmonic functions, from which many properties can be deduced using potential analysis. Let X be an isotropic stable process in $\mathbb{R}^d$ with $\alpha \in (0,2)$ and $d \geq 2$. For a Lipchitz cone Γ, define the exit time

$$\tau_{\Gamma_1} = \inf\{s > 0 : X_s \notin \Gamma_1\}, \text{ where } \Gamma_1 = \Gamma \cap B(1).$$

A function $u : \mathbb{R}^d \to \mathbb{R}^+$ is regular α-harmonic in Γ_1 if it satisfies

$$u(x) = \mathbb{E}_x[f(X_{\tau_{\Gamma_1}}), \tau_{\Gamma_1} < \infty], \text{ for } x \in \Gamma.$$

The following is the main result from [10].

Proposition 5 (Boundary Harnack Principle)**.** There is a constant $C_1 = C_1(\Gamma, \alpha)$ such that for all u, v regular α-harmonic functions in Γ_1 with

(i) $u(x_0) = v(x_0)$, for some $x_0 \in \Gamma \cap B(1/2)$, and

(ii) $u(x) = v(x)$ when $x \in \Gamma^c \cap B(1)$, then we have,

$$C_1^{-1} v(x) \leq u(x) \leq C_1 v(x), \text{ for } |x| < 1/2.$$

Moreover, $\lim_{\Gamma \ni x \to 0} u(x)/v(x)$ exists.

The Boundary Harnack Principle gives us an idea of how fast an α-harmonic function decays as we take $\Gamma \ni x \to 0$. We note specifically that the constant C_1 does not depend on the choice of u, v. A more recent result from [6] shows the existence of a regular α-harmonic function M of the form

$$M(x) = |x|^\beta M\left(\frac{x}{|x|}\right), \text{ for some } \beta > 0 \text{ for all } x \in \Gamma.$$

We say M is a *Martin Kernel (with pole at infinity)* for Γ. The choice of M is unique up to multiplicative constant. The function M is bounded on any bounded set and vanishes outside Γ. There is no explicit formula for the function M. However, we know there are special cases where M is explicit, according to [6].

Example 1.5.2 (Half-space)**.** Define the half-space by

$$H = \{(x_1, x_2, x_3, \dots, x_d) \in \mathbb{R}^d : x_d > 0\}.$$

Then, we have a Martin kernel of H given by

$$M^H(x) = |x_d|^{\alpha/2}\mathbf{1}_{(x_d>0)}, \text{ for } x = (x_1, x_2, x_3, \dots, x_d) \in \mathbb{R}^d.$$

Example 1.5.3 (Sliced domain)**.** When $\alpha \in (1, 2)$, let

$$K = \{(x_1, x_2, x_3, \dots, x_d) \in \mathbb{R}^d : x_d \neq 0\}.$$

There is a Martin kernel of K is given by

$$M^K(x) = |x_d|^{\alpha-1}\mathbf{1}_{(x_d\neq0)}, \text{ for } x = (x_1, x_2, x_3, \dots, x_d) \in \mathbb{R}^d.$$

The exponent $\beta = \beta(\alpha, \Gamma) \in (0, \alpha)$ depends on just the index of similarity and the cone. It is decreasing in Γ in the sense that if $\gamma \subseteq \Gamma$ then $\beta(\gamma, \alpha) \geq \beta(\Gamma, \alpha)$, shown in [6]. We can see from the examples straight away that $\beta(H, \alpha) > \beta(K, \alpha)$ and $H \subseteq K$.

The analysis extends futher to [11] where Bogdan, Palmowski and Wang proved that, for $y \in \Gamma$,

$$\lim_{\Gamma \ni x \to 0} \frac{\mathbb{P}_x(X_t \in \mathrm{d}y, t < \kappa_\Gamma)}{M(x)} = n_t(\mathrm{d}y) \text{ exists.}$$

It also follows from the scaling property that

$$\lim_{\Gamma \ni x \to 0} \frac{\mathbb{P}_x(t < \kappa_\Gamma)}{t^{-\beta/\alpha}M(x)} \text{ exists and is not dependent on } t \geq 0.$$

This means, as $\Gamma \ni x \to 0$, the probability of X^Γ surviving for at least a time t behaves like $t^{-\beta/\alpha}M(x)$ which tends to 0 as $\Gamma \ni |x| \to 0$. Hence, a process that starts near the origin will exit the cone very early. In chapter 4, we will develop the idea of a self-similar Markov process starting at 0. This requires us to construct another process that could escape 0. The function M will play a major role in the computation we will do in the future.

1.6 Lévy system of a Markov additive process

This section will serve as a review of results proved in [18, 16] where the general theory of Markov additive process was developed. Let (ξ, Θ) be a Markov additive process with probabilities $(\mathbf{P}_{r,\theta})_{\theta\in E, r\in\mathbb{R}}$ and filtration $(\mathcal{G}_t)_{t\geq0}$ and $\mathcal{G}$. It follows from the definition that Θ is a Markov process in its own right. This means

$$\mathbf{P}_{0,\theta}\left(\theta_t\Theta \in \mathcal{A} \middle| \mathcal{G}_t \cap \{t < \zeta\}\right) = \mathbf{P}_{0,\Theta_t}(\Theta \in \mathcal{A})\mathbf{1}_{(t<\zeta)},$$

for $t \geq 0$, $\theta \in E$ and $\mathcal{A}$ a measurable collection of paths.

Let $\mathcal{K} := \sigma(\{\Theta_u : u \geq 0\})$ and $\mathcal{K}_t := \sigma(\{\Theta_u : 0 \leq u \leq t\})$, for $t \geq 0$. In this section, the starting point of the process may not be relevant as we will be considering the probability law given $\mathcal{K}$. Hence, we shall omit the index on $\mathbf{P}$ when it is not necessary.

Referring to [16], we say that ξ given $\mathcal{K}$ has independent increments meaning we have

$$\mathbf{E}\left[\prod_{i=1}^{n} h_i(\xi_{t_i} - \xi_{t_{i-1}}) \middle| \mathcal{K}\right] = \prod_{i=0}^{n} \mathbf{E}\left[h_i(\xi_{t_i} - \xi_{t_{i-1}}) \middle| \mathcal{K}\right],$$

for $n \in \mathbb{N}$, $0 \leq t_1 < t_2 < t_3 < \cdots < t_n$ and $h_1, h_2, h_3, \ldots, h_n$ positive measurable functions.

Ezhov and Skorohod [18] used the notation for a Markov Additive Process as (Θ, ξ) where the first component is the Markov part and the second component is additive. Hence, the name "Markov Additive". The ordinator part can be characterised using the multiplicative functional

$$M_t^{\lambda}(\xi) = \mathbf{E}\left[\mathrm{e}^{-i\lambda\xi_t} \middle| \mathcal{K}\right].$$

The characterisation of $M_t^{\lambda}(\xi)$ is done with the help of the *additive functionals*. Let A be a $(\mathcal{K}_t)_{t\geq 0}$-adapted process, we write $A_t := A_t(\Theta)$. We say that $(A_t)_{t\geq 0}$ with $A_0 = 0$ is an additive functional of Θ if

(a) the path of A is almost surely right continuous with left limits with $A_t = A_\zeta$ for all $t \geq \zeta$, and

(b) almost surely $A_{t+s}(\Theta) = A_t(\Theta) + A_s(\theta_t\Theta)$.

One may consider an additive functional as a predictable process given Θ.

Example 1.6.1 (Integrated functional)**.** Suppose $f : E \to \mathbb{R}^+$ is measurable. Consider the integrated functional

$$A_t := \int_0^{t\wedge\zeta} f(\Theta_s)\mathrm{d}s. \tag{1.16}$$

Then, $(A_t)_{t\geq 0}$ is an additive functional of Θ.

Example 1.6.2 (Conditional drift)**.** Let (ξ, Θ) be a Markov additive process. The expectation $\mathbf{E}\left[\xi_{t\wedge\zeta} - \xi_0 | \mathcal{K}\right]$ is also an additive functional.

It is tempting to conclude that every additive functional is an integrated functional. However, this is not true in general.

The jump structure of a Markov additive process can be analysed using the Lévy system, see [17]. This is applicable for every MAP with quasi-left continuity. Indeed, we have already assumed a standard Markov process to be quasi-left continuous and a MAP to be a standard Markov process in our definition. The definition in Çinlar did not assume a MAP to be quasi-left continuous. We begin by reviewing the results from [17] and then using the result to simplify those given in [16].

For a non-decreasing additive functional $(H_s)_{s\geq 0}$ we may write its derivative as $\mathrm{d}H_s$. For arbitrary sets A and B, a *transition kernel* from A to B is a family of measures $(\mu_a)_{a\in A}$ on the Borel sets of B.

Proposition 6 (Çinlar (1975))**.** There exists a kernel Π from E to $\mathbb{R}\times E$ and additive functional H of Θ such that

$$\mathbf{E}_{0,\theta}\left[\sum_{0\leq s\leq t}\mathbf{1}_{((\xi_{s-},\Theta_{s-})\neq(\xi_s,\Theta_s))}f(\xi_s-\xi_{s-},\Theta_{s-},\Theta_s)\right]$$
$$=\mathbf{E}_{0,\theta}\left[\int_0^t \mathrm{d}H_s\int_{\mathbb{R}}\int_E \Pi(\Theta_s,\mathrm{d}r,\mathrm{d}\phi)f(r,\Theta_s,\phi)\right], \tag{1.17}$$

for $t\geq 0$ and $f:\mathbb{R}\times E\times E\to\mathbb{R}^+$ measurable. Moreover, we have the following

1. for $\theta\in E$,
$$\Pi(\theta,\{0\},\{\theta\})=0,$$
2. for $\theta\in E$,
$$\int_{-1}^{1}|r|^2\Pi(\theta,\mathrm{d}r,\{\theta\})<\infty,$$
and
3. let K be as defined as
$$K(\theta,A)=\Pi(\theta,A\setminus\{\theta\},\mathbb{R})\,\theta\in E.$$
Then, we have that
$$\mathbf{E}_{0,\theta}\left[\sum_{0\leq s\leq t}\mathbf{1}_{(\Theta_{s-}\neq\Theta_s)}f(\Theta_{s-},\Theta_s)\right]=\mathbf{E}_{0,\theta}\left[\int_0^t\mathrm{d}H_s\int_E K(\Theta_s,\mathrm{d}\phi)f(\Theta_s,\phi)\right],$$
for $t\geq 0$, $\theta\in E$ and $f:\mathbb{R}\times E\times E\to\mathbb{R}^+$ measurable.

We say that (H,Π) is the *Lévy system* of (ξ,Θ).

Intuitively, $\Pi(\theta,A,B)$ represents the rate of jumps with $(\xi_t-\xi_{t-},\Theta_t)\in A\times B$ while $\Theta_{t-}=\theta$. We can write Π as the sum of two Kernels

$$\Pi(\theta,\mathrm{d}\phi,\mathrm{d}r)=\mathbf{1}_{(\theta=\phi)}\Pi^d(\theta,\mathrm{d}r)+\mathbf{1}_{(\theta\neq\phi)}K(\theta,\mathrm{d}\phi)F_{\theta,\phi}(\mathrm{d}r),$$

for $\theta,\phi\in E$, $r\in\mathbb{R}$ and a family of measures $(F_{\theta,\phi})_{\theta,\phi\in E}$. This categorises the jumps of ξ into two categories, those simultaneous with jumps of Θ and those that are not. We can now go back to the results from [16] to make sense of MAP decomposition.

Proposition 7 (Çinlar (1972))**.** Let (ξ, Θ) be a MAP with a Lévy system (H, Π). The ordinator part ξ can be decomposed into

$$\xi_t = A_t + \xi_t^f + \xi_t^c + \xi_t^d, \text{ for } 0 \le t < \zeta,$$

where $\sigma(\{\xi_s^f : s \ge 0\})$, $\sigma(\{\xi_s^c : s \ge 0\})$ and $\sigma(\{\xi_s^d : s \ge 0\})$ are conditionally independent given $\mathcal{K}$. Each component satisfies the following statements

(a) The process $(A_t)_{t \ge 0}$ is an additive functional of Θ.

(b) Let $\mathbb{T} := \{s \ge 0 : \Theta_s \ne \Theta_{s-}\}$. We have

$$\xi_t^f = \sum_{u \in \mathbb{T} \cap [0,t]} \Delta^{(u)}, \; t < \zeta,$$

where $(\Delta^{(u)})_{u \in \mathbb{T}}$ are independent given $\mathcal{K}$ with $\Delta^{(u)}$ sampled from the measure $F_{\Theta_{u-}, \Theta_u}$.

(c) For $s, t \ge 0$, $\xi_{t+s}^c - \xi_t^c$ is independent of $\{\xi_u^c : 0 \le u \le t\}$, and it has normal distribution with variance $Q_{t+s} - Q_t$ where Q_t is an increasing continuous additive functional of Θ.

(d) For $t \ge 0$ and $A \in \mathcal{B}(\mathbb{R})$, define random measures

$$N_t(A) = \sum_{s \le t} \mathbb{1}_{(\xi_s^d - \xi_{s-}^d \in A \setminus \{0\})} \text{ and } B_t(A) = \int_0^s \Pi^d(\Theta_s, A) \mathrm{d}H_s. \tag{1.18}$$

For $A \in \mathcal{B}(\mathbb{R})$, $B_t(A)$ is an additive functional of Θ with $\int_{\mathbb{R}} (1 \wedge |r|^2) B_t(\mathrm{d}r) < \infty$. Moreover,

$$\xi_t^d = \lim_{\epsilon_n \downarrow 0} \left(\int_{\mathbb{R} \setminus (-\epsilon_n, \epsilon_n)} r N_t(\mathrm{d}r) - \int_{(-1,1) \setminus (-\epsilon_n, \epsilon_n)} r B_t(\mathrm{d}r) \right), \text{ for } t < \zeta, \tag{1.19}$$

for some random $\epsilon_n \downarrow 0$ which depends on t and is $\mathcal{K}_t$-measurable.

Further, we have that $M_t^\lambda(\xi)$ follows an analogue of the Lévy-Khintchine formula, that is

$$M_t^\lambda(\xi) = \left[\prod_{u \in \mathbb{T} \cap [0,t]} F_{\Theta_{u-}, \Theta_u}(\lambda) \right] \exp \left\{ i\lambda A_t - \frac{\lambda^2}{2} Q_t + \int_{\mathbb{R}} \left(\mathrm{e}^{i\lambda r} - 1 - i\lambda r \mathbf{1}_{(|r|<1)} \right) B_t(\mathrm{d}r) \right\},$$

for $t < \zeta$.

Remark. Intuitively, we can say that $N_t(A)$ specified in (1.18), given $\mathcal{K}$, has Poisson distribution with parameter $B_t(A)$, for $A \in \mathcal{B}(\mathbb{R})$ and $t \ge 0$. The convergence in (1.19) can be proved in a similar fashion to Theorem 2.10 from [26].

This characterisation of a MAP gives us a better understanding. However, this characterisation relies heavily on how much we know about the modulator part Θ. When E is finite, MAP is rather well studied, see [4, 5, 20, 34] for references.

However, if we take E to be infinite, especially uncountable, the development of theory becomes much harder in general. We will look to further develop the theory of MAP alongside the theory of ssMp through some concrete examples, isotropic stable processes.

Example 1.6.3 (Isotropic stable MAP, Theorem 3.13 from [27])**.** Let X be an isotropic stable process written as $LT(\xi, \Theta, \alpha)$. Then, (ξ, Θ) has a Lévy system (H, Π) given by

$$H_t = t \wedge \zeta \text{ and } \Pi(\theta, \mathrm{d}r, \mathrm{d}\phi) = \frac{\mathrm{e}^{dr}}{|\mathrm{e}^r \phi - \theta|^{(\alpha+d)}} \mathrm{d}r\mathrm{d}\phi, \tag{1.20}$$

for $t > 0$, $r \in \mathbb{R}$ and $\theta, \phi \in \mathbb{S}^d$.

1.7 Fluctuation identities for self-similar Markov processes

We are interested in the manner by which new radial maxima of a ssMp are attained. Our understanding in this respect benefits from classical excursion theory presented e.g. in a paper by Maisonneuve (1975) [31]. Let X be a $\mathbb{R}^d$-ssMp with representation

$$X = LK(\xi, \Theta, \alpha).$$

It follows from the Lamperti-Kiu transformation that X has the same range as $\mathrm{e}^{\xi}\Theta$. Hence, a new radial maximum for X will also correspond to a new maximum for ξ. For example, under $\mathbb{P}_x$ with $x \in B(1)$, we can write

$$X_{\tau_1^{\ominus}} = \mathrm{e}^{\xi_{T_{(0,\infty)}}} \Theta_{T_{(0,\infty)}}, \text{ where } T_{(0,\infty)} := \inf\{s \geq 0 : \xi_s > 0\}.$$

Hence, it is possible to study the behaviour at radial maximum of X from the behaviour of ξ at its maximum. Let $\overline{\xi}_t = \sup_{u \leq t} \xi_u$, for $t \geq 0$. According to Section 3 of [22], it is true that $(\overline{\xi} - \xi, \Theta)$ is a standard Markov process.

We will work with a further assumption that ξ is regular for $(0, \infty)$ i.e.

$$\mathbb{P}_{0,\theta} \left(\inf\{u \geq 0 : \xi_u \in (0, \infty)\} = 0\right) = 1, \text{ for all } \theta \in \mathbb{S}^{d-1}. \tag{1.21}$$

For $t > 0$, define the left and the right supremum points of t as

$$\mathtt{g}_t = \sup\left\{s < t : \xi_s = \overline{\xi}_s\right\} \text{ and } \mathtt{d}_t = \inf\left\{s > t : \xi_s = \overline{\xi}_s\right\}. \tag{1.22}$$

The quantity $\mathtt{g}_t$ is well-defined as the time when the supremum of ξ up to time t is attained in a sense that either $\overline{\xi}_t = \xi_{\mathtt{g}_t}$ or $\overline{\xi}_t = \xi_{\mathtt{g}_t -}$ almost surely. In general, the quantities $\xi_{\mathtt{g}_t -}$ and $\xi_{\mathtt{g}_t}$ could be different. For $t > 0$, let $R_t = \mathtt{d}_t - t$ and the set of random times

$$G := \{t > 0 : R_{t-} = 0, \, R_t > 0\} = \{\mathtt{g}_s : s \geq 0 \text{ and } \mathtt{d}_s > \mathtt{g}_s\}.$$

Moreover, we will now show - using the strong Markov property and (1.21) - that for each $\mathtt{g} \in G$, we have that $\xi_{\mathtt{g}} \leq \xi_{\mathtt{g}-}$ almost surely. For $\delta > 0$, define a stopping time $T^{(\delta)} = \inf\{\mathtt{g} \geq 0 : R_{\mathtt{g}-} = 0, \xi_{\mathtt{g}} - \xi_{\mathtt{g}-} > \delta\}$. Then, by the strong Markov property, we have

$$\begin{aligned}\mathbf{P}_{0,\theta}\left(\exists \mathtt{g} \in G, \xi_{\mathtt{g}} - \xi_{\mathtt{g}-} > \delta\right) &= \mathbf{P}_{0,\theta}\left(R_{T^{(\delta)}-} = 0, R_{T^{(\delta)}} > 0, T^{(\delta)} < \infty\right)\\ &= \mathbf{E}_{0,\theta}\left[\mathbf{1}_{(T^{(\delta)}<\infty)}\mathbf{P}_{0,\Theta_{T^{(\delta)}}}(R_0 > 0)\right] = 0 \text{ for } \theta \in \mathbb{S}^{d-1}.\end{aligned} \tag{1.23}$$

This is true for all $\delta > 0$, so by the dominated convergence theorem, we have that

$$\mathbf{P}_{0,\theta}\left(\exists \mathtt{g} \in G, \xi_{\mathtt{g}} > \xi_{\mathtt{g}-} > 0\right) = \lim_{\delta \to 0} \mathbb{P}_{0,\theta}\left(\exists \mathtt{g} \in G, \xi_{\mathtt{g}} - \xi_{\mathtt{g}-} > \delta\right) = 0.$$

In any case, it follows that, for all $\mathtt{g} \in G$, $\overline{\xi}_{\mathtt{g}} = \xi_{\mathtt{g}-}$ almost surely.

For $t \geq 0$ with $\mathtt{g}_t < \mathtt{d}_t \leq \infty$, we define an *excursion process*

$$(\epsilon_{\mathtt{g}_t}(s), \Theta^{\epsilon}_{\mathtt{g}_t}(s)) := (\xi_{\mathtt{g}_t+s} - \xi_{\mathtt{g}_t}, \Theta_{\mathtt{g}_t+s}),\ s \leq \zeta_{\mathtt{g}_t} := \mathtt{d}_t - \mathtt{g}_t. \tag{1.24}$$

This codes the excursion of $(\overline{\xi} - \xi, \Theta)$ from the set $(0, \mathbb{S}^{d-1})$. The excursion lives in the space $\mathbb{U}(\mathbb{R} \times \mathbb{S}^{d-1})$, the space of Càdlàg paths with life time $\zeta = \inf\{s \geq 0 : \epsilon(s) > 0\}$. This is different to $\mathbb{D}(\mathbb{R} \times \mathbb{S}^{d-1})$ as the cemetery state is not fixed.

Proposition 8 (Maisonneuve (1975) [31])**.** There exists an additive functional $(\ell_t)_{t\geq 0}$ whose values only increase on the set G, with $\sup_{\theta \in \mathbb{S}^{d-1}} \mathbf{E}_{0,\theta}\left[\int_0^\infty \mathrm{e}^{-t}\mathrm{d}\ell_t\right] < \infty$ and a family of *excursion measures*, $(\mathbb{N}_\theta)_{\theta \in \mathbb{S}^{d-1}}$ such that:

1. The kernels $(\mathbb{N}_\theta)_{\theta \in \mathbb{S}^{d-1}}$ are measures on $\mathbb{D}(\mathbb{S}^{d-1} \times \mathbb{R})$ with support on the set

$$\{\zeta > 0\}, \text{ and } \mathbb{N}_\theta(1 - \mathrm{e}^{-\zeta}) < \infty, \text{ for all } \theta \in \mathbb{S}^{d-1}.$$

2. We have the exit formula

$$\begin{aligned}&\mathbf{E}_{r,\theta}\left[\sum_{\mathtt{g} \in G} F((\xi_s, \Theta_s) : s < \mathtt{g}) H((\epsilon_{\mathtt{g}}, \Theta^{\epsilon}_{\mathtt{g}}))\right]\\ &= \mathbf{E}_{r,\theta}\left[\int_0^\infty F((\xi_s, \Theta_s) : s < t)\mathbb{N}_{\Theta_{t-}}(H(\epsilon, \Theta^{\epsilon}))\mathrm{d}\ell_t\right],\end{aligned} \tag{1.25}$$

for $r \in \mathbb{R}$, $\theta \in \mathbb{S}^{d-1}$, F positive continuous on $\mathbb{D}(\mathbb{R} \times \mathbb{S}^{d-1})$ and H is measurable on $\mathbb{U}(\mathbb{R} \times \mathbb{S}^{d-1})$.

3. For all $t > 0$ and $\theta \in \mathbb{S}^{d-1}$, the process $(\epsilon, \Theta^\epsilon)$ is Markovian in the sense that

$$\mathbb{N}_\theta\left(H((\epsilon_{t+s}, \Theta^{\epsilon}_{t+s})_{s\geq 0}), t < \zeta\right) = \mathbb{N}_\theta\left(\mathbf{E}_{\epsilon(t),\Theta^\epsilon(t)}[H(\xi, \Theta)], t < \zeta\right),$$

where H is measurable on $\mathbb{U}(\mathbb{R} \times \mathbb{S}^{d-1})$.

We say $(\ell, (\mathbb{N}_\theta)_{\theta\in\mathbb{S}^{d-1}})$ is an exit system for $(\overline{\xi}-\xi,\Theta)$ from $(0,\mathbb{S}^{d-1})$.

Remark. We note that ℓ and $(\mathbb{N}_\theta)_{\theta\in\mathbb{S}^{d-1}}$ are not necessarily unique. Indeed, by writing

$$\mathrm{d}\widetilde{\ell}_t = f(\Theta_t)\ell_t \text{ and } \widetilde{\mathbb{N}}_\theta(\cdot) = \frac{1}{f(\theta)}\mathbb{N}_\theta(\cdot), \text{ for } \theta\in\mathbb{S}^{d-1} \text{ and } f \text{ bounded and measurable.}$$

It is obvious that $(\widetilde{\ell}, (\widetilde{N}_\theta)_{\theta\in\mathbb{S}^{d-1}})$ is another exit system.

Remark. It is possible to have $\mathbb{N}_\theta((\epsilon_0,\Theta_0^\epsilon)\neq(0,\theta))>0$, possibly infinite, for some $\theta\in\mathbb{S}^{d-1}$. This corresponds to the case where it is possible that $(\xi_{\mathtt{g}-},\Theta_{\mathtt{g}-})\neq(\xi_{\mathtt{g}},\Theta_{\mathtt{g}})$, for some $\mathtt{g}\in G$.

Ascending ladder MAP

We denote ℓ^{-1} as the right inverse of ℓ given by

$$\ell_t^{-1} = \inf\{s\geq 0 : \ell_s > t\}.$$

An advantage of defining ℓ^{-1} as the right inverse is that $(\xi_{\ell_t^{-1}})_{t\geq 0}$ has the same range of motion as $(\overline{\xi}_t)_{t\geq 0}$ in a sense that

$$\left\{\xi_{\ell_t^{-1}} : t\geq 0\right\} = \left\{\overline{\xi}_t : t\geq 0\right\}.$$

Define a process (H^+,Θ^+) as

$$H_t^+ = \xi_{\ell_t^{-1}} \text{ and } \Theta_t^+ = \Theta_{\ell_t^{-1}}, \text{ for } t<\ell_\infty.$$

Then, (H^+,Θ^+) is a Markov additive process with lifetime $\zeta^+=\ell_\infty$, see Section 3 of [22]. In particular, we call (H^+,Θ^+) the *ascending ladder MAP* which is a MAP with respect to filtration $(\mathcal{F}_{\ell_t^{-1}})_{t\geq 0}$. Define the potential measure,

$$U_\theta^+(\mathrm{d}r,\mathrm{d}\phi) = \mathbf{E}_{0,\theta}\left[\int_0^\infty \mathbf{1}_{(\xi_{t-}\in\mathrm{d}r,\Theta_{t-}\in\mathrm{d}\phi)}\mathrm{d}\ell_t\right], \text{ for } \theta,\phi\in\mathbb{S}^{d-1} \text{ and } r>0. \quad (1.26)$$

This measure can be written in a more intuitive form. We note that ℓ is continuous and (ξ,Θ) is right-continuous, so we can replace $t-$ in (1.26) by just t. Moreover, we can perform a change of variable with $s=\ell_t$ so we have that

$$U_\theta^+(\mathrm{d}r,\mathrm{d}\theta) = \mathbf{E}_{0,\theta}\left[\int_0^{\zeta^+} \mathbf{1}_{(H_s^+\in\mathrm{d}r,\Theta_s^+\in\mathrm{d}\theta)}\mathrm{d}s\right].$$

Intuitively, $U_\theta^+(A,B)$ describes the expected occupation time of (H^+,Θ^+) in the set $A\times B$, where $A\in\mathcal{B}(\mathbb{R}^+)$ and $B\in\mathcal{B}(\mathbb{S}^{d-1})$.

The measures $(U_\theta^+)_{\theta\in\mathbb{S}^{d-1}}$ can simplify the exit formula when the only quantity we are interested in is the position of $(\xi_{\mathtt{g}-},\Theta_{\mathtt{g}-})$.

Example 1.7.1 (Fluctuation identities for isotropic stable processes)**.** Let X be an isotropic stable process with $X = LK(\xi, \Theta, \alpha)$. By rotational invariance, we have that ξ is a Lévy process. By Proposition 2.3 of [33], ξ is regular for both $(-\infty, 0)$ and $(0, \infty)$ in the sense that,

$$0 = \inf\{s > 0 : \xi_s \in (-\infty, 0)\} = \inf\{s > 0 : \xi_s \in (0, \infty)\}\ \mathbf{P}_{0,\theta}\text{-almost surely, for all } \theta \in \mathbb{S}^{d-1}. \tag{1.27}$$

We will show that, under $\mathbf{P}_{\log|x|,\arg(x)}$ with $|x| < 1$, the first passage time $T_{(0,\infty)}$ is a right supremum point ξ. Hence, $X_{\tau_1^\ominus}$ can be written in terms of the excursion process defined in (1.24).

Proposition 9. For $x \in B(1)$, $\mathbf{P}_{\log|x|,\arg(x)}(\mathtt{g}_{T_{(0,\infty)}} \in G) = 1$.

Proof. By the definition of G, it suffices to show that $\mathtt{g}_{T_{(0,\infty)}} < T_{(0,\infty)}$. On the event that $\mathtt{g}_{T_{(0,\infty)}} = T_{(0,\infty)}$, this implies that $\xi_{T_{(0,\infty)}-} = \overline{\xi}_{T_{(0,\infty)}-}$. By Theorem 3 from [13], $\xi_{T_{(0,\infty)}} \neq \xi_{T_{(0,\infty)}-}$ almost surely. Hence, we are only left with the possibility that $\xi_{T_{(0,\infty)}-} = \overline{\xi}_{T_{(0,\infty)}-} < \xi_{T_{(0,\infty)}}$. We use the fact that $(\xi, \overline{\xi})$ is a standard Markov process, see [22], and (2.29) from [17] to compute that

$$\begin{aligned}\mathbf{P}_{\log|x|,\arg(x)}(\mathtt{g}_{T_{(0,\infty)}} = T_{(0,\infty)}) &= \mathbf{P}_{\log|x|,\arg(x)}\left(\xi_{T_{(0,\infty)}-} = \overline{\xi}_{T_{(0,\infty)}-} < \xi_{T_{(0,\infty)}}\right) \\ &= \mathbf{E}_{\log|x|,\arg(x)}\left(\int_0^{T_{(0,\infty)}} \mathbb{1}_{(\xi_{s-} = \overline{\xi}_{s-} < 0)} \Pi((-\xi_{s-}, \infty))\mathrm{d}s\right),\end{aligned}$$

where Π is the Lévy measure of ξ. This is 0 as a result of Theorem 6.7 from [26] that

$$\mathbf{P}_{\log|x|,\arg(x)}\left(\int_0^{T_{(0,\infty)}} \mathbb{1}_{(\xi_s = \overline{\xi}_s)}\mathrm{d}s = 0\right) = 1.$$

Hence, $\mathbf{P}_{\log|x|,\arg(x)}(\mathtt{g}_{T_{(0,\infty)}} = T_{(0,\infty)}) = 0$. So, $\mathbf{P}_{\log|x|,\arg(x)}(\mathtt{g}_{T_{(0,\infty)}} < T_{(0,\infty)}) = 1$. □

Proposition 10. For $x \in B(1)$ and $f : B(1)^c \to [0, \infty)$ be a measurable function. It follows that

$$\mathbb{E}_x\left[f(X_{\tau_1^\ominus})\right] = \int_0^{-\log|x|} \int_{\mathbb{S}^{d-1}} U_\theta^+(\mathrm{d}r, \mathrm{d}\phi)\mathbb{N}_\phi\left(f(|x|\mathrm{e}^{r+\epsilon(\zeta)}\Theta^\epsilon(\zeta))\mathbf{1}_{(\epsilon(\zeta) > -\log|x| - r)}\right).$$

Proof. For $x \in B(1)$, using Lamperti-Kiu representation, we can write

$$\begin{aligned}\mathbb{E}_x\left[f(X_{\tau_1^\ominus})\right] &= \mathbf{E}_{\log|x|,\arg(x)}\left[f(\mathrm{e}^{T_{(0,\infty)}}\Theta_{T_{(0,\infty)}})\right] \\ &= \mathbf{E}_{\log|x|,\arg(x)}\left[\sum_{\mathtt{g}\in G} \mathbf{1}_{(\xi_{\mathtt{g}-} < 0, \epsilon_{\mathtt{g}}(\zeta_{\mathtt{g}}) > -\xi_{\mathtt{g}-})} f(\mathrm{e}^{\xi_{\mathtt{g}-} + \epsilon_{\mathtt{g}}(\zeta_{\mathtt{g}})}\Theta^\epsilon_{\mathtt{g}}(\zeta_{\mathtt{g}}))\right] \\ &= \mathbf{E}_{0,\arg(x)}\left[\sum_{\mathtt{g}\in G} \mathbf{1}_{(\xi_{\mathtt{g}-} < -\log|x|)}\mathbf{1}_{(\epsilon_{\mathtt{g}}(\zeta_{\mathtt{g}}) > -\log|x| - \xi_{\mathtt{g}-})} f(|x|\mathrm{e}^{\xi_{\mathtt{g}-} + \epsilon_{\mathtt{g}}(\zeta_{\mathtt{g}})}\Theta^\epsilon_{\mathtt{g}}(\zeta_{\mathtt{g}}))\right].\end{aligned}$$

Then, we can appeal to (1.25), (1.26) and Fubini's theorem so this is equal to

$$\begin{aligned}
&\mathbf{E}_{0,\arg(x)}\left[\int_0^\infty \mathbf{1}_{(\xi_{t-}<-\log|x|)}\mathbb{N}_{\Theta_{t-}}\left(f(|x|e^{\xi_{t-}+\epsilon(\zeta)}\Theta^\epsilon(\zeta))\mathbf{1}_{(\epsilon(\zeta)>-\log|x|-\xi_{t-})}\right)\mathrm{d}\ell_t\right]\\
&\quad=\mathbf{E}_{0,\arg(x)}\left[\int_0^\infty \mathbf{1}_{(H_s^+<-\log|x|,s<\zeta^+)}\mathbb{N}_{\Theta_s^+}\left(f(|x|e^{H_s^++\epsilon(\zeta)}\Theta^\epsilon(\zeta))\mathbf{1}_{(\epsilon(\zeta)>-\log|x|-H_s^+)}\right)\mathrm{d}s\right]\\
&\quad=\int_0^\infty\left(\mathbf{E}_{0,\arg(x)}\left[\mathbf{1}_{(H_s^+<-\log|x|,s<\zeta^+)}\mathbb{N}_{\Theta_s^+}\left(f(|x|e^{H_s^++\epsilon(\zeta)}\Theta^\epsilon(\zeta))\mathbf{1}_{(\epsilon(\zeta)>-\log|x|-H_s^+)}\right)\right]\right)\mathrm{d}s\\
&\quad=\int_0^\infty\int_0^{-\log|x|}\int_{\mathbb{S}^{d-1}}\mathbf{P}_{0,\arg(x)}\left(H_s^+\in\mathrm{d}r,\Theta^+\in\mathrm{d}\phi,s<\zeta^+\right)\\
&\qquad\qquad\times\left[\mathbb{N}_\phi\left(f(|x|e^{r+\epsilon(\zeta)}\Theta^\epsilon(\zeta))\mathbf{1}_{(\epsilon(\zeta)>-\log|x|-r)}\right)\right]\mathrm{d}s\\
&\quad=\int_0^{-\log|x|}\int_{\mathbb{S}^{d-1}}U_\theta^+(\mathrm{d}r,\mathrm{d}\phi)\mathbb{N}_\phi\left(f(|x|e^{r+\epsilon(\zeta)}\Theta^\epsilon(\zeta))\mathbf{1}_{(\epsilon(\zeta)>-\log|x|-r)}\right).
\end{aligned}$$

□

On the one hand, this means the distribution of $X_{\tau_1^\ominus}$ can be computed from $(U_\theta^+)_{\theta\in\mathbb{S}^{d-1}}$ and $(\mathbb{N}_\theta)_{\theta\in\mathbb{S}^{d-1}}$. On the other hand, this also suggests that we may be able to obtain some information about $(U_\theta^+)_{\theta\in\mathbb{S}^{d-1}}$ and $(\mathbb{N}_\theta)_{\theta\in\mathbb{S}^{d-1}}$ from a given distribution of $X_{\tau_1^\ominus}$, which has been computed by Blumenthal–Getoor–Ray [9] in this case.

1.8 Markov additive renewal theory

The aim of this section is to discuss some asymptotic properties of the potential measures $(U_\theta^+)_{\theta\in\mathbb{S}^{d-1}}$. This can be described using Markov additive renewal theorem. Suppose that (H^+,Θ^+) has infinite lifetime and π^+ is the stationary distribution of Θ^+ with

$$\mu^+ := \int_{\mathbb{S}^{d-1}}\pi^+(\mathrm{d}\theta)\mathbf{E}_{0,\theta}[H_1^+]\in(0,\infty).$$

The main idea of this section is to establish that

$$U_\theta^+(\mathrm{d}r,\mathrm{d}\phi)\ \text{ behaves like }\ \frac{1}{\mu^+}\mathrm{d}r\otimes\pi^+(\mathrm{d}\phi)\ \text{ as }\ r\to\infty.$$

This can be done with the help of the main results in [3]. The results there are for a discrete time analogue of Markov additive processes. Let $(S_n,\Xi_n)_{n\geq 0}$ be a discrete time Markov chain with values in $\mathbb{R}\times\mathbb{S}^{d-1}$ under measure P. Write $P_\theta(\cdot) := P(\cdot|S_0=0,\Xi_0=\theta)$ and $P_\mu(\cdot)=\int_{\mathbb{S}^{d-1}}\mu(\mathrm{d}\theta)P_\theta(\cdot)$, for measure μ. We say (S,Ξ) is a *Markov additive renewal process* if

$$P\left(S_{n+1}-S_n\in A,\Xi_{n+1}\in B|\,(S_i,\Xi_i)_{i\leq n}\right)=P_\theta\left(S_1\in A,\Xi_1\in B\right)\big|_{\theta=\Xi_n},$$

for all $A \in \mathcal{B}(\mathbb{R})$, $B \in \mathcal{B}(\mathbb{S}^{d-1})$ and $n \geq 0$. Assume π is the stationary distribution of Ξ, we say the Markov additive renewal process is *non-arithemetic* if the support of S_1 under P_π is non-lattice. Define the Markov renewal function as

$$U_\theta(A, B) = E_\theta\left[\sum_{n\geq 0} \mathbf{1}_{(S_n \in A, \Xi_n \in B)}\right],$$

for $\theta \in \mathbb{S}^{d-1}$, $A \in \mathcal{B}(\mathbb{R})$ and $B \in \mathcal{B}(\mathbb{S}^{d-1})$.

In the framework of Alsmeyer [3], the Markov chain $(\Xi_n)_{n\geq 0}$ is assumed to be *strongly aperiodic Harris recurrent*, in the sense that there exists a probability measure, $\rho(\cdot)$ on $\mathcal{B}(\mathbb{S}^{d-1})$ such that, for some $\lambda > 0$,

$$P_\theta(\Xi_1 \in E) \geq \lambda\rho(E), \text{ for all } E \in \mathcal{B}(\mathbb{S}^{d-1}) \text{ and } \rho\text{-almost surely } \theta \in \mathbb{S}^{d-1}.$$

It also follows from [4] that Ξ has a stationary distribution π satisfying

$$\pi(\mathrm{d}\phi) = \lim_{n\to\infty} P_\theta(\Xi_1 \in \mathrm{d}\phi) \text{ for } \phi \in \mathbb{S}^{d-1} \text{ and } \rho\text{-almost surely } \theta \in \mathbb{S}^{d-1}.$$

Proposition 11 (Markov additive renewal theorem)**.** Let $(S_n, \Xi_n)_{n\geq 0}$ be a non-arithemetic Markov additive renewal process where Ξ_n is aperiodic Harris recurrent with stationary distribution π. If $\mu^+ = E_\pi[S_1] \in (0, \infty) \cup \{\infty\}$ and $g : \mathbb{R} \times \mathbb{S}^{d-1} \to \mathbb{R}$ measurable satisfying

1. $g(\cdot, \theta)$ is continuous almost everywhere for π-almost surely $\theta \in \mathbb{S}^{d-1}$,

2. there exists $\rho > 0$ such that

$$\int_{\mathbb{S}^{d-1}} \sum_{n\in\mathbb{Z}} \sup_{n\rho \leq r < (n+1)\rho} |g(r, \theta)|\pi(\mathrm{d}\theta) < \infty,$$

then

$$\lim_{t\to\infty} E_\theta\left[\sum_{n\geq 0} g(t - S_n, \Xi_n)\right] = \frac{1}{\mu^+}\int_{\mathbb{S}^{d-1}} \pi(\mathrm{d}\phi)\int_{\mathbb{R}} g(r, \phi)\mathrm{d}r,$$

for π-almost all $\theta \in \mathbb{S}^{d-1}$.

Example 1.8.1 (Ascending ladder MAP)**.** Let (H^+, Θ^+) be a continuous time ascending ladder MAP. We could make a discrete time MAP by sampling at every exponential time with mean 1. More precisely, let $(\mathbf{e}_i)_{i\geq 0}$ be (i.i.d.) exponential random variable with mean 1 and $T_n = \sum_{i\geq 1}^n \mathbf{e}_i$. Setting $S_0 = 0$, $\Xi_0 = \Theta_0$ and

$$S_n = H^+_{T_n} \text{ and } \Xi_n = \Theta_n \text{ for } n \geq 1.$$

Then, (S, Ξ) is a discrete time MAP with potential that corresponds to the potential of (H^+, Θ^+)

$$U_\theta^+(A, B) = \sum_{n \geq 0} E_\theta \left[\sum_{n \geq 0} \mathbf{1}_{(S_n \in A, \Xi_n \in B)} \right],$$

for $\theta \in \mathbb{S}^{d-1}$, $A \in \mathcal{B}(\mathbb{R})$ and $B \in \mathcal{B}(\mathbb{S}^{d-1})$.

The ascending ladder MAP (H^+, Θ^+) could be complicated. It is already a difficult task to show that Θ^+ has a stationary distribution even if Θ has a stationary distribution.

The Markov renewal theorem turns out to be a very important ingredient to compute many asymptotic results. However, this comes at the cost of conditions being difficult to check. In Chapter 4, we use a different way of sampling (ξ, Θ) which allows us to bypass many difficult conditions for applying Markov renewal theorem.

1.9 Research Chapters outline

This thesis aims to develop further theory of self-similar Markov processes and Markov additive processes from the themes listed in the introduction. The research chapters of this thesis each contain a research article. The articles included in the thesis are results of collaboration with my supervisors Andreas E. Kyprianou and Victor M. Rivero.

1. **Conditioned real self-similar Markov processes (Chapter 2).** The journey starts with an exploration of real-valued self-similar Markov processes. The aim is to get better understanding of absorption behaviour. This proves to be very useful in later chapters. More precisely, let X be a $\mathbb{R}$-ssMp with Lamperti-Kiu representation $LK(\xi, \Theta, \alpha)$. In this case, we have that the modulator Θ only takes values in $\{-1, 1\}$. We can characterise (ξ, Θ) using the matrix exponent $\mathbf{F}(\cdot)$ which satisfies

$$\mathbf{E}_{0,\theta}\left[\mathrm{e}^{z\xi_t}, \Theta_t = \phi\right] = (\mathrm{e}^{\mathbf{F}(z)t})_{\theta,\phi},$$

for $\theta, \phi \in \{-1, 1\}$ and z where the left-hand side is defined. Suppose that $\mathbf{F}(z)$ has the leading eigenvalue $\chi(z)$ with the corresponding eigenvector $\boldsymbol{\nu}(z) = (\nu_1(z), \nu_{-1}(z))$. Then, with respect to its filtration, there exists a martingale of the form

$$\frac{\nu_{\Theta_t}(z)}{\nu_{\Theta_0}(z)} \mathrm{e}^{z(\xi_t - \xi_0) - \chi(z)t}, \ t \geq 0 \ \text{and } z \ \text{when this is defined.}$$

We say that $\beta \in \mathbb{R}$ is a Cramér number if $\chi(\beta) = 0$. It is well-known that

$$\lim_{t \to \infty} \xi_t = \begin{cases} -\infty & \text{when } \beta > 0 \\ +\infty & \text{when } \beta < 0 \end{cases}.$$

So, for $\beta \neq 0$, X will have two different absorption behaviours

(A) if $\beta > 0$, X has finite lifetime with $X_{T_0-} = 0$,

(B) if $\beta < 0$, X has infinite lifetime with $\lim_{t\to\infty} |X_t| = \infty$.

Under suitable assumptions, we condition X in category (A) to be in category (B) and vice versa. The conditioning was done with two different limiting approaches. Our first approach is to condition X to exit an asymptotically large ball around the origin, when $\beta > 0$, and to enter an asymptotically small ball around the origin, when $\beta < 0$. In each of the cases, one requires an estimate of

$$\mathbb{P}_x(\tau_r^{\ominus} < \infty) \text{ as } r \to \infty \text{ and } \mathbb{P}_x(\tau_r^{\oplus} < \infty) \text{ as } r \to 0.$$

The article also presents another conditioning for the case (A). Intuitively, this is done by conditioning on the event $\{T_0 > t\}$ and then take $t \to \infty$. Using the Lamperti-Kiu representation, we can write

$$T_0 = |x|^\alpha \int_0^\infty \mathrm{e}^{\alpha \xi_t} \mathrm{d}t.$$

We call the integral on the right exponential functionals. We adapted some of the existing techniques for exponential functionals of Lévy processes to compute the tail distribution of an exponential functional of Markov additive process.
Preprint arXiv:1510.01781, *to appear in Stochastic Processes and their Applications.*

2. **Deep factorisation of the stable process III (Chapter 3).** This chapter aims to develop the excursion theory of $(\overline{\xi} - \xi, \Theta)$ from the set $\{0\} \times \mathbb{S}^{d-1}$. We consider an isotropic $\mathbb{R}^d$-stable process $(X, \mathbb{P})$. In high dimensions, $\lim_{t\to\infty} |X_t| = \infty$ and so there exists a point of closest reach to the origin which is almost surely non-zero. Let X have Lamperti-Kiu representation $LT(\xi, \Theta, \alpha)$ and $(\ell, (\mathbb{N}_\theta)_{\theta \in \mathbb{S}^{d-1}})$ be an exit system of $(\xi - \underline{\xi}, \Theta)$ from $(0, \mathbb{S}^{d-1})$. Then, we can define H^- in similar way to H^+ with H^- tracking all the minimum points of ξ.

   Let $\mathtt{G}(\infty) = \sup\{s \geq 0 : |X_s| = \inf_{u\geq 0} |X_u|\}$ and the point of closest reach to the origin denoted as $X_{\mathtt{G}(\infty)}$. In this case, it is known that $X_{\mathtt{G}(\infty)} = X_{\mathtt{G}(\infty)-}$ and $|X_{\mathtt{G}(\infty)}| = \inf_{u\geq 0} |X_u|$ almost surely. Using exit formula from [31], it follows that the distribution of $X_{\mathtt{G}(\infty)}$ can be computed using the potential

$$\mathbb{P}_\theta(X_{\mathtt{G}(\infty)} \in \mathrm{d}y) = U_\theta^-(\mathrm{d}y) \mathbb{N}_{\arg(y)}(\zeta = \infty), \text{ for } \theta \in \mathbb{S}^{d-1} \text{ and } y \in B(1),$$

   where

$$U_\theta^-(\mathrm{d}y) := \mathbf{E}_{0,\theta}\left[\int_0^\infty \mathbf{1}(\mathrm{e}^{H_t^-}\Theta_t^- \in \mathrm{d}y)\mathrm{d}t\right].$$

   Without loss of generality, we can choose ℓ so that $\mathbb{N}_\theta(\zeta = \infty) = 1$ for all $\theta \in \mathbb{S}^{d-1}$. This means that the potential measure U^- can be directly implied from the distribution of the point of closest reach to the origin. The distribution of $|X_{\mathtt{G}(\infty)}|$ has been previously computed in [13] where

$$\mathbb{P}_x(|X_{\mathtt{G}(\infty)}| \in \mathrm{d}r) = -\frac{\mathrm{d}}{\mathrm{d}r}\mathbb{P}_x(\tau_r^{\oplus} < \infty), \text{ for } x \in \mathbb{R}^d,\, r > 0.$$

We need a different method to workout the distribution of $X_{\mathtt{G}(\infty)} = |X_{\mathtt{G}(\infty)}| \times \arg(X_{\mathtt{G}(\infty)})$. Heuristically, we compute the distribution of the point of closest reach using the estimation

$$\frac{1}{\delta}\mathbb{E}_\theta\left[f(\arg(X_{\tau_r^{\oplus}})), |X_{\tau_r^{\oplus}}| \in [r-\delta, \delta], \tau_{r-\delta}^{\oplus} = \infty\right], \text{ for } \theta \in \mathbb{S}^{d-1},\, r < 1, \text{ and } f \text{ measurable},$$

as $\delta \to 0$. We argue that, on the event $\{\tau_{r-\delta}^{\oplus} = \infty\}$, $X_{\tau_r^{\oplus}}$ would be a good approximation to the point of closest reach. The main ingredient in verifying this was the classic formula given by Blumenthal-Getoor-Ray [9]. We have also give some explicit formula for $\mathbb{N}$ in this case. The main idea is to show that $U_\theta^-(\mathrm{d}y)$ behaves like a point mass on small annuli. We can then have that

$$\mathbb{N}_\theta(\mathrm{e}^{\epsilon(\zeta)}\Theta(\zeta) \in \mathrm{d}y) = \lim_{\delta\to 0}\frac{\mathbb{P}_\theta(X_{\tau_{1-\delta}^{\oplus}} \in \mathrm{d}y)}{\mathbb{P}_\theta(\tau_{1-\delta}^{\oplus} = \infty)} \text{ for } \theta \in \mathbb{S}^{d-1},\, y \in B(1).$$

We can also derive some further explicit expression for isotropic $\mathbb{R}^d$-stable processes in similar way. Finally, we take anvantage of the fact that the process $(KX_{\eta(t)})_{t\geq 0}$ has Lamperti-Kiu representation $LK(-\xi, \Theta, \alpha)$ and is the Doob-h transform of X with respect to the h-function $x \mapsto |x|^{\alpha-d}$. In a similar fashion to Example 1.5.1, we can work out the counterpart identities for excursion of $(\bar{\xi} - \xi, \Theta)$ from $\{0\} \times \mathbb{S}^{d-1}$.
Preprint arXiv:1706.09924

3. **Stable processes in a cone (Chapter 4).** This chapter aims at developing ideas used in Chapter 2 to apply to a higher dimensional setting. Let X be an isotropic $\mathbb{R}^d$-stable process killed upon exiting a Lipchitz cone, namely $\mathtt{k}_{\kappa_\Gamma}Y$. However, in this case, we do not have the martingale assumptions. We start this work based on proven results from [6, 11]. Bogdan et. al. proved that, for $t > 0$ and $y \in \Gamma$, there exists a density $n_t(y)$ such that

$$\lim_{\Gamma \ni x \to 0}\frac{\mathbb{P}_x(X_t \in \mathrm{d}y)}{M(x)} = n_t(y)\mathrm{d}y.$$

We use this result as the base assumption to condition X to stay in the cone, where we denote the resulting process as $(X, \mathbb{P}^{\triangleleft})$. We show that there exists a Skorohod limit

$$\lim_{\Gamma \ni x \to 0}\mathbb{P}_x^{\triangleleft}(\cdot) =: \mathbb{P}_0^{\triangleleft}(\cdot).$$

On the other hand, X can be conditioned to get absorbed at 0 continuously whilst remaining in the cone which denote by $(X, \mathbb{P}^{\triangleright})$. The two measures $\mathbb{P}^{\triangleleft}$ and $\mathbb{P}^{\triangleright}$ can be considered counterparts in many different ways. First of all, if the process $(X, \mathbb{P}^{\triangleleft})$ has Lamperti-Kiu representation

$LK(\xi, \Theta, \alpha)$, then the process $(X, \mathbb{P}^{\triangleright})$ has Lamperti-Kiu representation $LK(-\xi, \Theta, \alpha)$. We also establish that $(X, \mathbb{P}^{\triangleright})$ is the time reversal of $(X, \mathbb{P}^{\triangleleft})$ starting from 0. Finally, in the sense of [35], we construct recurrence extentions of $(X, \mathbb{P})$ after κ_Γ and also of $(X, \mathbb{P}^{\triangleright})$ after T_0 given that the cone is "wide enough".
Preprint arXiv:1804.08393

This thesis is presented in the alternative format which includes publications. This means the research chapters are developed independent of the introduction and supposed to be self-contained. Hence, it will be inevitable to have some inconsistency in notations and redundant contents to the introduction chapter.

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Chapter 2

Conditioned real self-similar Markov processes

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Abstract

In recent work, Chaumont et al. [10] showed that it is possible to condition a stable process with index $\alpha \in (1, 2)$ to avoid the origin. Specifically, they describe a new Markov process which is the Doob h-transform of a stable process and which arises from a limiting procedure in which the stable process is conditioned to have avoided the origin at later and later times. A stable process is a particular example of a real self-similar Markov process (rssMp) and we develop the idea of such conditionings further to the class of rssMp. Under appropriate conditions, we show that the specific case of conditioning to avoid the origin corresponds to a classical Cramér-Esscher-type transform to the Markov Additive Process (MAP) that underlies the Lamperti–Kiu representation of a rssMp. In the same spirit, we show that the notion of conditioning a rssMp to continuously absorb at the origin also fits the same mathematical framework. In particular, we characterise the stable process conditioned to continuously absorb at the origin when $\alpha \in (0, 1)$. Our results also complement related work for positive self-similar Markov processes in [11].

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2.1 Introduction

This work concerns conditionings of real self-similar Markov processes (rssMp) and so we start by characterising this class of stochastic processes.

A rssMp with *index of self-similarity* $\alpha > 0$ is a standard Markov process $X = (X_t)_{t\geq 0}$ (in the sense of [6]) with probability laws $(\mathbb{P}_x)_{x\in\mathbb{R}}$ and filtration $(\mathcal{F}_t)_{t\geq 0}$, which satisfies the *scaling property* that for all $x \in \mathbb{R}\setminus\{0\}$ and $c > 0$,

$$\text{the law of } (cX_{tc^{-\alpha}})_{t\geq 0} \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{cx}.$$

In the language of the classical paper by Lamperti [27], where self-similar Markov processes were first analysed at depth, this corresponds to the class of semi-stable Markov process with order (or Hurst index) $1/\alpha$. The structure of real self-similar Markov processes has been investigated by [13] in the symmetric case, and [10] in general. Here, we give an interpretation of these authors' results in terms of Markov additive process (MAP) with a two-state modulating Markov chain and therefore we make an immediate digression to introduce such processes.

2.1.1 Markov Additive Processes

Let E be a finite state space and $(\mathcal{G}_t)_{t\geq 0}$ a standard filtration. A càdlàg process (ξ, J) in $\mathbb{R}\times E$ with law $\mathbf{P}$ is called a *Markov additive process (MAP)* with respect to $(\mathcal{G}_t)_{t\geq 0}$ if $(J(t))_{t\geq 0}$ is a continuous-time Markov chain in E, and the following property is satisfied, for any $i \in E$, $s, t \geq 0$:

$$\begin{gathered}\text{given } \{J(t) = i\}, \text{ the pair } (\xi(t+s) - \xi(t), J(t+s)) \text{ is independent of } \mathcal{G}_t,\\ \text{and has the same distribution as } (\xi(s) - \xi(0), J(s)) \text{ given } \{J(0) = i\}.\end{gathered} \tag{2.1}$$

Aspects of the theory of Markov additive processes are covered in a number of texts, among them [5] and [4]. More classical work includes [15, 14, 3] amongst others. We will mainly use the notation of [17], where it was principally assumed that ξ is spectrally negative; the results which we quote are valid without this hypothesis, however.

Let us introduce some notation. For $x \in \mathbb{R}$, write $\mathbf{P}_{x,i} = \mathbf{P}(\cdot \,|\, \xi(0) = x, J(0) = i)$. If μ is a probability distribution on E, we write $\mathbf{P}_{x,\mu} = \sum_{i\in E} \mu_i \mathbf{P}_{x,i}$. We adopt a similar convention for expectations.

It is well-known that a Markov additive process (ξ, J) also satisfies (2.1) with t replaced by a stopping time, albeit on the event that the stopping time is finite. The following proposition gives a characterisation of MAPs in terms of a mixture of Lévy processes, a Markov chain and a family of additional jump distributions; see [4, §XI.2a] and [17, Proposition 2.5].

Proposition 12. The pair (ξ, J) is a MAP (as described above) if and only if, J is a continuous-time Markov chain in E, for each $i, j \in E$, there exist a sequence of iid Lévy processes $(\xi_i^n)_{n\geq 0}$ independent of the chain J and a sequence of iid random variables $(\Delta_{i,j}^n)_{n\geq 0}$, independent of the chain J, such that, if $\sigma_0 = 0$ and $(\sigma_n)_{n\geq 1}$ are the jump times of J, then the process ξ has the representation

$$\xi(t) = \mathbf{1}_{(n>0)}(\xi(\sigma_n-) + \Delta^n_{J(\sigma_n-),J(\sigma_n)}) + \xi^n_{J(\sigma_n)}(t-\sigma_n), \qquad t \in [\sigma_n, \sigma_{n+1}),\, n \geq 0.$$

For each $i \in E$, it will be convenient to define ξ_i as a Lévy process whose distribution is the common law of the ξ^n_i processes in the above representation; and similarly, for each $i, j \in E$, define $\Delta_{i,j}$ to be a random variable having the common law of the $\Delta^n_{i,j}$ variables.

Henceforth, we confine ourselves to irreducible (and hence ergodic) Markov chains J. Let the state space E be the finite set $\{1, \ldots, N\}$, for some $N \in \mathbb{N}$. Denote the transition rate matrix of the chain J by $\boldsymbol{Q} = (q_{i,j})_{i,j\in E}$. For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i. To be more precise, for all $z \in \mathbb{C}$ for which it exists,

$$\psi(z) := \log \int_{\mathbb{R}} \mathrm{e}^{zx} \mathrm{P}(\xi(1) \in \mathrm{d}x).$$

For each pair of $i, j \in E$, define the Laplace transform $G_{i,j}(z) = \mathrm{E}[\mathrm{e}^{z\Delta_{i,j}}]$ of the jump distribution $\Delta_{i,j}$, whenever this exists. Write $\boldsymbol{G}(z)$ for the $N \times N$ matrix whose (i, j)-th element is $G_{i,j}(z)$. We will adopt the convention that $\Delta_{i,j} = 0$ if $q_{i,j} = 0$, $i \neq j$, and also set $\Delta_{ii} = 0$ for each $i \in E$.

The multidimensional analogue of the Laplace exponent of a Lévy process is provided by the matrix-valued function

$$\boldsymbol{F}(z) = \mathrm{diag}(\psi_1(z), \ldots, \psi_N(z)) + \boldsymbol{Q} \circ \boldsymbol{G}(z), \tag{2.2}$$

for all $z \in \mathbb{C}$ such that the elements on the right are defined, where $\circ$ indicates elementwise multiplication, also called Hadamard multiplication. It is then known that

$$\mathbf{E}_{0,i}(\mathrm{e}^{z\xi(t)}; J(t) = j) = \left(\mathrm{e}^{\boldsymbol{F}(z)t}\right)_{i,j}, \qquad i,\, j \in E, t \geq 0,$$

such that the right-hand side of the equality is defined. For this reason, $\boldsymbol{F}$ is called the *matrix exponent* of the MAP (ξ, J). Note, using standard convexity properties of regular Laplace transforms, if we can garantee, for $a, b \in \mathbb{R}$ with $a < b$, that $\boldsymbol{F}(a), \boldsymbol{F}(b)$ are defined and finite (element wise), then, $\boldsymbol{F}(z)$ is well defined and finite (element wise) for $\mathrm{Re}(z) \in (a, b)$.

The role of $\boldsymbol{F}$ is analogous to the role of the Laplace exponent of a Lévy process. Similarly in this respect, one might also regard the *leading eigenvalue* associated to $\boldsymbol{F}$ (sometimes referred to as the *Perron–Frobenius eigenvalue*, see [4, §XI.2c] and [17, Proposition 2.12]) as also playing this role.

Proposition 13. Suppose that $z \in \mathbb{R}$ is such that $\boldsymbol{F}(z)$ is defined. Then, the matrix $\boldsymbol{F}(z)$ has a real simple eigenvalue $\chi(z)$, which is larger than the real part of all its other eigenvalues. Furthermore, the corresponding right-eigenvector $\boldsymbol{v} = (v_1(z), \cdots, v_N(z))$ may be chosen so that $v_i(z) > 0$ for every $i = 1, \cdots N$, and normalised such that

$$\boldsymbol{\pi} \cdot \boldsymbol{v}(z) \quad = \quad 1 \tag{2.3}$$

where $\boldsymbol{\pi} = (\pi_1, \cdots, \pi_N)$ is the equilibrium distribution of the chain J.

One sees the leading eigenvalue appearing in a number of key results. We give two such below that will be of pertinence later on. The first one is the strong law of large numbers for (ξ, J), in which the leading eigenvalue plays the same role as the Laplace exponent of a Lévy process does in analogous result for that setting. The following result is taken from [4, Proposition 2.10].

Proposition 14. If $\chi'(0)$ is well defined (either as a left or right derivative), then we have

$$\lim_{t\to\infty} \frac{\xi(t)}{t} = \chi'(0) = \mathbf{E}_{0,\boldsymbol{\pi}}[\xi(1)] := \sum_{i\in E} \boldsymbol{\pi}_i \mathbf{E}_{0,i}[\xi(1)] \tag{2.4}$$

almost surely. In that case, there is a trichotomy which dictates whether $\lim_{t\to\infty} \xi(t) = \infty$ almost surely, $\lim_{t\to\infty} \xi(t) = -\infty$ almost surely or $\limsup_{t\to\infty} \xi(t) = -\liminf_{t\to\infty} \xi(t) = \infty$ accordingly as $\chi'(0) > 0$, < 0 or $= 0$, respectively.

The leading eigenvalue also features in the following probabilistic result, which identifies a martingale (also known as the generalised Wald martingale) and associated exponential change of measure corresponding to an Esscher-type transformation of a Lévy process; cf. [4, Proposition XI.2.4, Theorem XIII.8.1].

Proposition 15. Let $\mathcal{G}_t = \sigma\{(\xi(s), J(s)) : s \leq t\}$, $t \geq 0$, and

$$M(t,\gamma) = \mathrm{e}^{\gamma(\xi(t)-\xi(0))-\chi(\gamma)t} \frac{v_{J(t)}(\gamma)}{v_{J(0)}(\gamma)}, \qquad t \geq 0, \tag{2.5}$$

for some γ such that $\chi(\gamma)$ is defined. Then, $M(\cdot, \gamma)$ is a unit-mean martingale with respect to $(\mathcal{G}_t)_{t\geq 0}$. Moreover, under the change of measure

$$\left.\frac{\mathrm{d}\mathbf{P}^{\gamma}_{x,i}}{\mathrm{d}\mathbf{P}_{x,i}}\right|_{\mathcal{G}_t} = M(t,\gamma), \qquad t \geq 0,$$

the process (ξ, J) remains in the class of MAPs and, where defined, its characteristic exponent is given by

$$\boldsymbol{F}_\gamma(z) = \boldsymbol{\Delta}_{\boldsymbol{v}}(\gamma)^{-1} \boldsymbol{F}(z+\gamma) \boldsymbol{\Delta}_{\boldsymbol{v}}(\gamma) - \chi(\gamma)\mathbf{I}, \tag{2.6}$$

where $\mathbf{I}$ is the identity matrix and $\boldsymbol{\Delta}_{\boldsymbol{v}}(\gamma) = \mathrm{diag}(\boldsymbol{v}(\gamma))$. It is straightforward to deduce that, when it exists, the associated leading eigenvalue associated to $\boldsymbol{F}_\gamma(z)$ is given by $\chi_\gamma(z) = \chi(z+\gamma) - \chi(\gamma)$.

The following properties of χ, lifted from [22, Proposition 3.4], will also prove useful in relating the last two results together.

Proposition 16. Suppose that $\boldsymbol{F}$ is defined in some open interval D of $\mathbb{R}$. Then, the function $\chi : z \mapsto \chi(z)$ the leading eigenvalue of $\boldsymbol{F}(z)$ is smooth and convex on D.

On account of the fact that $\boldsymbol{F}(0) = \boldsymbol{Q}$, it is easy to see that we always have $\chi(0) = 0$. If we assume that there exists $\theta \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{F}$ is defined on $D = \{t\theta : t \in (0, 1)\}$ with

$$\chi(\theta) = 0, \tag{2.7}$$

then by the proposition above, we can conclude that χ is defined and convex on the interval D. Henceforth the value satisfying (2.7) will be denoted by θ and referred to as the *Cramér number*.

If $\theta > 0$, then $\chi'(0+)$ is well defined and convexity dictates that it must be negative. In that case $\lim_{t\to\infty} \xi(t) = -\infty$ almost surely. Moreover, if we take $\gamma = \theta$ in Proposition 15, then, as $\chi'_\theta(0-) = \chi'(\theta-) > 0$, and under the associated change of measure, $\lim_{t\to\infty} \xi(t) = \infty$ almost surely.

Conversely, if $\theta < 0$, then $\chi'(0-)$ is well defined and convexity dictates that it must be positive. In that case $\lim_{t\to\infty} \xi(t) = \infty$ almost surely. Again, if we take $\gamma = \theta$ in Proposition 15, then $\chi'_\theta(0+) = \chi'(\theta+) < 0$. Hence, under the associated change of measure, $\lim_{t\to\infty} \xi(t) = -\infty$ almost surely. In both cases, the change of measure (2.5) using $\gamma = \theta$ exchanges the long-term drift of the underlying MAP from $\pm\infty$ to $\mp\infty$.

2.1.2 Real self-similar Markov processes

In [10] the authors confine their attention to rssMp in 'class **C.4**'. A rssMp X is in **C.4** if, for all $x \neq 0$, $\mathbb{P}_x(\exists t > 0 : X_t X_{t-} < 0) = 1$; that is, with probability one, the process X changes sign infinitely often. The reason behind this is to ensure that the chain J in the Lamperti–Kiu representation is recurrent. Otherwise, we have that $+1$ or -1, possibly both, is an absorbing state.

Assume that both ± 1 are absorbing states, we can consider $(X, \mathbb{P}_x)$ as a positive self-similar Markov process specified by the sign of the starting point i.e. whether $x > 0$ or $x < 0$. If $x < 0$, then $-X$ is a positive self-similar Markov process. If $x < 0$, then X is a positive self-similar Markov process.

Otherwise, assume that $+1$ is the only absorbing state, then $(X, \mathbb{P}_x)$ can be considered as a positive self-similar Markov process once it crosses to a positive value and if it never crosses $-X$ is a positive self-similar Markov process. In particular, if it starts with a negative value X will cross the origin once and remain positive. If it starts with a positive value, it will remain positive.

Henceforth we will rename the class **C.4** as the class of *infinite crossing rssMp*. Such a process may be identified with a MAP via a deformation of space and time which we call the *Lamperti–Kiu representation* of X. The following result is a simple consequence of [10, Theorem 6]. In it, we will use the notation

$$\tau^{\{0\}} = \inf\{t \geq 0 : X_t = 0\},$$

for the time to absorption at the origin.

Proposition 17. Let X be an infinite crossing rssMp and fix $x \neq 0$. Then there exists a time-change σ, adapted to the filtration of X, such that, under the law $\mathbb{P}_x$, the process

$$(\xi(t), J(t)) = (\log|X_{\sigma(t)}|, \operatorname{sign}(X_{\sigma(t)})), \qquad t \geq 0,$$

is a MAP with state space $E = \{-1, 1\}$ under the law $\mathbf{P}_{\log|x|,\operatorname{sign}(x)}$. Furthermore, the process X under $\mathbb{P}_x$ has the representation

$$X_t = J(\varphi(t))\mathrm{e}^{\xi(\varphi(t))}, \qquad 0 \leq t < \tau^{\{0\}},$$

where φ is the inverse of the time-change σ, and may be given by

$$\varphi(t) = \inf\left\{s > 0 : \int_0^s \exp(\alpha\xi(u))\,\mathrm{d}u > t\right\}, \qquad t < \tau^{\{0\}}. \tag{2.8}$$

In short, up to an endogenous time change, a rssMp has a polar decomposition in which $\exp\{\xi\}$ describes the radial distance from the origin and J describes its orientation (positive or negative).

To make the connection with the previous subsection, let us understand how the existence of a Cramér number for the underlying MAP to a rssMp affects path behaviour of the latter. Revisiting the discussion at the end of the previous subsection, we see that if $\theta > 0$ then $\lim_{t\to\infty}\xi(t) = -\infty$. In that case, we deduce from the strong law of large numbers for ξ and the Lamperti–Kiu transform, that

$$\tau^{\{0\}} = \int_0^\infty \mathrm{e}^{\alpha\xi(t)}\mathrm{d}t < \infty \qquad \text{and} \qquad X_{\tau^{\{0\}}-} = 0$$

almost surely (irrespective of the point of issue of X). Said another way, the rssMp will be continuously absorbed in the origin after an almost surely finite time. Moreover, this implies that $\varphi(t) < \infty$ if and only if $t < \int_0^\infty \mathrm{e}^{\alpha\xi(s)}\mathrm{d}s$.

In the case that there is a Cramér number which satisfies $\theta < 0$, then, again referring to the limiting behaviour of ξ and the Lamperti–Kiu transform, we have

$$\tau^{\{0\}} = \int_0^\infty \mathrm{e}^{\alpha\xi(t)}\mathrm{d}t = \infty \tag{2.9}$$

almost surely (irrespective of the point of issue of X). Hence, the associated rssMp never touches the origin. Moreover, $\varphi(t) < \infty$ for all $t \geq 0$.

We can also reinterpret Proposition 15 in light of the Lamperti–Kiu representation and the fact that the quantity $\varphi(t)$ in (2.8) is also a stopping time, as well as the fact that $(\mathcal{F}_t)_{t\geq 0} = (\mathcal{G}_{\varphi(t)})_{t\geq 0}$. Theorem III.3.4 of [18] states that a martingale change of measure remains valid at a given stopping time, providing one restricts measurement to the set that the stopping time is finite. Accordingly we have that when $\theta > 0$, respectively $\theta < 0$,

$$M(\varphi(t), \theta) = \frac{v_{J(\varphi(t))}(\theta)}{v_{\text{sign}(x)}(\theta)} \mathrm{e}^{\theta(\xi(\varphi(t)) - \log|x|)} \mathbf{1}_{(\varphi(t)<\infty)} = \frac{v_{\text{sign}(X_t)}(\theta)}{v_{\text{sign}(x)}(\theta)} \frac{|X_t|^\theta}{|x|^\theta} \mathbf{1}_{(t<\tau^{\{0\}})}, \qquad t \geq 0. \quad (2.10)$$

is a $\mathbb{P}_x$-martingale, respectively, a $\mathbb{P}_x$-supermartingale.

2.2 Main results

Throughout the remainder of the paper we make following assumption.

(A): The process X is a rssMp whose underlying MAP does not have lattice support and has a leading eigenvalue χ with Cramér number $\theta \neq 0$ such that $\chi'(\theta)$ exists in $\mathbb{R}$.

Under this assumption, our objective is to construct conditioned versions of X. When $\theta > 0$, through a limiting procedure, we will build the process X conditioned to avoid the origin. Similarly when $\theta < 0$, we will build the process X conditioned to be continuously absorbed at the origin. Accordingly, in both cases, we shall show the existence of a harmonic function for the process X which is used to make a Doob h-transform in the representation of the conditioned processes.

In this respect, our work is reminiscent of density transforms which have been considered in the setting of positive self-similar Markov processes (pssMp); see [30]. In that case, the density transform plays a crucial role in the construction of an entrance law or recurrent extension from 0. Similar ideas appear in [28] when constructing a Bessel-3 process from a Brownian motion killed upon hitting 0.

Theorem 2. Suppose that X is a rssMp under assumption (A), $\mathcal{F} := \sigma(X_s : s \geq 0)$ and $\mathcal{F}_t := \sigma(X_s : s \leq t)$, $t \geq 0$ is its natural filtration. Define

$$h_\theta(x) := v_{\text{sign}(x)}(\theta)|x|^\theta, \qquad x \in \mathbb{R},$$

and, for Borel set D, let $\tau^D := \inf\{s \geq 0 : X_s \in D\}$.

(a) If $\theta > 0$, then, we define

$$\mathbb{P}_x^\circ(A) := \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)} \mathbf{1}_{(A,\, t<\tau^{\{0\}})}\right], \quad (2.11)$$

for $t > 0$, $x \neq 0$ and $A \in \mathcal{F}_t$. This can be extended to define $\mathbb{P}_x^\circ(A)$, for $A \in \mathcal{F}$, such that $(X, \mathbb{P}_x^\circ)$, $x \in \mathbb{R}\backslash\{0\}$, is a rssMp. Moreover, for all $A \in \mathcal{F}_t$,

$$\mathbb{P}_x^\circ(A) = \lim_{a\to\infty} \mathbb{P}_x(A \cap \{t < \tau^{(-a,a)^c}\} \,|\, \tau^{(-a,a)^c} < \tau^{\{0\}}). \quad (2.12)$$

(b) If $\theta < 0$, then, we define

$$\mathbb{P}^{\circ}_x(A,\, t < \tau^{\{0\}}) := \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)}\mathbf{1}_A\right],$$

for all $t > 0$, $x \neq 0$ and $A \in \mathcal{F}_t$. This can be extended to define $\mathbb{P}^{\circ}_x(A)$, for $A \in \mathcal{F}$, such that $(X, \mathbb{P}^{\circ}_x)$, $x \in \mathbb{R}\backslash\{0\}$, is a rssMp. Moreover, for all $t > 0$ and $A \in \mathcal{F}_t$

$$\mathbb{P}^{\circ}_x(A,\, t < \tau^{\{0\}}) = \lim_{a\to 0} \mathbb{P}_x(A \cap \{t < \tau^{(-a,a)}\} \,|\, \tau^{(-a,a)} < \infty). \tag{2.13}$$

In case (a) of the above theorem, as $\theta > 0$, the Doob h-transform rewards paths that drift far from the origin. Indeed the limiting procedure (2.12) conditions the paths of the rssMp to explore further and further distances from the origin before being absorbed at the origin. In this sense, we refer to the process described in part (a) as the rssMp *conditioned to avoid the origin.* In case (b) of the theorem above, the Doob h-transform rewards paths that stay close to the origin. Moreover, the limiting procedure (2.13) conditions the paths of the rssMp to ultimately visit smaller and smaller balls centred around the origin. We therefore refer to the process described in part (b) as the rssMp *conditioned to absorb continuously at the origin.*

The above theorem constructs the conditioned processes via limiting spatial requirements. For the case of conditioning to avoid the origin, we can give a second sense in which the Doob h-transform emerges as the result of a conditioning procedure. The latter is done by conditioning the first visit to the origin to occur later and later in time.

Theorem 3. Suppose that X is a rssMp under assumption (A) and $\theta > 0$. Then for $x \in \mathbb{R}\backslash\{0\}$ $t > 0$, and $A \in \mathcal{F}_t$, we have

$$\mathbb{P}^{\circ}_x(A) = \lim_{s\to\infty} \mathbb{P}(A \,|\tau^{\{0\}} > t + s), \tag{2.14}$$

where $\mathbb{P}^{\circ}_x$, $x \in \mathbb{R}\backslash\{0\}$, is given by (2.11).

In order to approach the asymptotic conditioning in Theorem 3, we need to understand the tail behaviour of the probabilities $\mathbb{P}_x(\tau^{\{0\}} > t)$, as $t \to \infty$, for all $x \neq 0$. Indeed, the Markov property tells us that, for any $t \geq 0$, $A \in \mathcal{F}_t$, and $x \in \mathbb{R}\backslash\{0\}$, we have

$$\lim_{s\to\infty} \mathbb{P}_x(A \,|\tau^{\{0\}} > t + s) = \lim_{s\to\infty} \mathbb{E}_x\left[\mathbf{1}(A,\, t < \tau^{\{0\}})\frac{\mathbb{P}_{X_t}(\tau^{\{0\}} > s)}{\mathbb{P}_x(\tau^{\{0\}} > t + s)}\right]. \tag{2.15}$$

We are thus compelled to consider the asymptotic behaviour of $\mathbb{P}_x(\tau^{\{0\}} > t)$ as $t \to \infty$. To that end, we recall that by the Lamperti–Kiu representation of X, under $\mathbb{P}_x$, $x \neq 0$, we can identify $\tau^{\{0\}}$ in terms of the MAP (ξ, J), under $\mathbf{P}_{0,\mathrm{sgn}(x)}$, via the relation

$$\tau^{\{0\}} = |x|^\alpha \int_0^\infty \mathrm{e}^{\alpha\xi(s)}\mathrm{d}s. \tag{2.16}$$

More precisely, the dependency of the law of $\tau^{\{0\}}$ on $|x|$, where x is the point of issue, can be seen directly in (2.16) through a simple multiplicative factor of $|x|^\alpha$. Hence, we should determine the asymptotic behaviour of the right tail distribution of the exponential functional of ξ, $I := \int_0^\infty e^{\alpha\xi(s)}ds$. In particular, we will prove the following result, which is more general than needed and of intrinsic interest. In order to state the result, we introduce the notation $f \sim g$ as $t \to \infty$ to mean $\lim_{t\to\infty} f/g = 1$.

Theorem 4. Let E be a finite state space and (ξ, J) a MAP with values in $\mathbb{R} \times E$. Assume that (ξ, J) does not have lattice support and has a leading eigenvalue χ with Cramér number $\theta > 0$ such that $\chi'(\theta)$ exists in $\mathbb{R}$. Define

$$I = \int_0^\infty e^{\alpha\xi(s)}ds.$$

We have that $\mathbf{E}_{0,k}[I^{\theta/\alpha-1}] < \infty$, $k \in E$, and

$$\mathbf{P}_{0,k}(I > t) \sim v_k(\theta)t^{-\theta/\alpha}\sum_{j\in E}\pi_j^\theta \frac{\mathbf{E}_{0,j}[I^{\theta/\alpha-1}]}{\mu_\theta|\alpha-\theta|v_j(\theta)}, \text{ as } t \to \infty,$$

where $\mu_\theta = \sum_{j\in E}\pi_j^\theta \mathbf{E}_{0,j}^\theta[\xi(1)]$ and $\boldsymbol{\pi}^\theta = (\pi_j^\theta, j \in E)$ is the stationary distribution of J under $\mathbf{P}_{x,i}^\theta$, $x \in \mathbb{R}$, $i \in E$.

The above result specialized to the setting in Theorem 3 gives that

$$\mathbb{P}_x(\tau^{\{0\}} > t) \sim v_{\mathrm{sign}(x)}(\theta)|x|^\theta t^{-\theta/\alpha}\sum_{j=\pm 1}\pi_j^\theta \frac{\mathbf{E}_{0,j}[I^{\theta/\alpha-1}]}{\mu_\theta|\alpha-\theta|v_j(\theta)}, \text{ as } t \to \infty.$$

This fact, together with the argument in (2.15) easily leads to the proof of Theorem 3. Indeed, to end the proof one should justify that it is possible to pass the limit through the expectation on the right-hand side of (2.15). This is done reasoning as in the proof of Theorem 2, which will be shown later in Section 2.5.

2.3 Remarks

We have a number of remarks pertaining to the suite of results in the previous section.

Unconditioning: It is natural to ask what happens if one takes a self-similar Markov process conditioned e.g. to continuously absorb at the origin, and condition it to avoid the origin. Does this reverse the effect of the original conditioning?

Suppose that X under $\mathbb{P}_x$, $x \neq 0$, is a self-similar process satisfying (A), with underlying MAP (ξ, J) with probabilities $\mathbf{P}_{x,i}$, $x \in \mathbb{R}$, $i \in \{-1, 1\}$. Now consider the matrix exponent of (ξ, J) under $\mathbf{P}_{x,i}^\theta$, $x \in \mathbb{R}$, $i \in \{-1, 1\}$, is given by

$$\boldsymbol{F}_\theta(z) = \boldsymbol{\Delta}_{\boldsymbol{v}}(\theta)^{-1}\boldsymbol{F}(z+\theta)\boldsymbol{\Delta}_{\boldsymbol{v}}(\theta).$$

This has leading eigenvector $\boldsymbol{\Delta}_{\boldsymbol{v}}(\theta)^{-1}\boldsymbol{v}(z+\theta)$ with eigenvalue $\chi_\theta(z) = \chi(z+\theta)$. Because we have assumed (A), it also follows that $\chi_\theta(-\theta) = 0$. To show that the MAP (ξ, J) under $\mathbf{P}^\theta_{x,i}$, $x \in \mathbb{R}$, $i \in \{-1, 1\}$, satisfies assumption (A), we need a further assumption that $\chi'_\theta(-\theta) = \chi'(0)$ exists and takes a finite value. In that case, if e.g. $\theta > 0$, then necessarily $\chi'(0) < 0$ and if we condition $(X, \mathbb{P}^\circ_x)$, $x \neq 0$, to be continuously absorbed at 0, then from Theorem 2 (b), the resulting MAP representing X can be identified via the Doob h-transform. For $t \geq 0$ and $A \in \mathcal{F}_t$,

$$\mathbb{P}^{\circ\circ}_x(A,\, t < \tau^{\{0\}}) := \mathbb{E}^\circ_x\left[\frac{h^\circ_\theta(X_t)}{h^\circ_\theta(x)}\mathbf{1}_A\right],$$

with

$$h^\circ_\theta(x) := \frac{v_{\mathrm{sign(x)}}(0)}{v_{\mathrm{sign}(x)}(\theta)}|x|^{-\theta} = \frac{1}{h_\theta(x)}, \qquad x \in \mathbb{R},$$

where the second equality holds because $\boldsymbol{F}(0) = \boldsymbol{Q}$ and hence $v_1(0) = v_{1-}(0) = 1$. As a consequence, we see that, changing measure in the stye of part (b) of Theorem 2 after a change of measure in the style of part (a), we get

$$\begin{aligned}\mathbb{P}^{\circ\circ}_x(A,\, t < \tau^{\{0\}}) &:= \mathbb{E}^\circ_x\left[\frac{h^\circ_\theta(X_t)}{h^\circ_\theta(x)}\mathbf{1}_A\right] \\ &= \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)}\frac{h^\circ_\theta(X_t)}{h^\circ_\theta(x)}\mathbf{1}_{(A,\, t<\tau^{\{0\}})}\right] \\ &= \mathbb{P}_x(A,\, t < \tau^{\{0\}}), \qquad A \in \mathcal{F}_t, t \geq 0.\end{aligned}$$

That is to say that the resulting process agrees with the process $(X, \mathbb{P}_x)$, $x \neq 0$, up to first hitting of the origin.

This is also apparent when we consider that the effect on the Esscher transform of the underlying MAP clearly reverses the effect of the initial conditioning. Indeed,

$$\boldsymbol{\Delta}_{\boldsymbol{v}}^{-1}(-\theta+\theta)\boldsymbol{\Delta}_{\boldsymbol{v}}(\theta)\boldsymbol{F}_\theta(z-\theta)\boldsymbol{\Delta}_{\boldsymbol{v}}(\theta)^{-1}\boldsymbol{\Delta}_{\boldsymbol{v}}(-\theta+\theta) = \boldsymbol{F}(z).$$

Similar calculations show the same reversal if we had assumed $\theta < 0$.

Degenerate MAPs: We deliberately excluded the setting that $\{-1, 1\}$ is reducible for the underlying MAP. In many cases, aside from, at most, a single crossing of the origin, we note the the conditionings considered here reduce to known conditionings of Lévy processes. In particular, these are the cases of conditioning a Lévy process to stay positive, cf, [8, 9], conditioning a Lévy process to continuously absorb at the origin, cf. [8] and conditioning a subordinator to stay in a strip, [26].
Stable processes: The central family of examples which fits the settting of the two main theorems above is that of a (strictly) stable process with index $\alpha \in (0, 2)$, which is killed on first hitting the origin. Recall that the latter processes are those rssMp which do not have continuous paths and which are also in the class of Lévy processes.

As a Lévy process, a stable process has characteristic exponent $\Psi(\lambda) := -t^{-1}\log \mathbb{E}_0[\mathrm{e}^{\mathrm{i}\lambda X_t}]$, $\lambda \in \mathbb{R}$, $t > 0$, given by

$$\Psi(\lambda) = |\lambda|^\alpha(\mathrm{e}^{\pi\mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\lambda>0)} + \mathrm{e}^{-\pi\mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\lambda<0)}), \qquad \lambda \in \mathbb{R},$$

where $\rho := \mathbb{P}_0(X_1 > 0)$. For convenience, we assume throughout this section that $\alpha\rho \in (0,1)$, which is to say that the stable process has paths with discontinuities of both signs.

For such processes, the matrix exponent of the underlying MAP in the Lamperti–Kiu representation has been computed in [22], with the help of computations in [10], and takes the form

$$\boldsymbol{F}(z) = \begin{bmatrix} -\dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho}-z)\Gamma(1-\alpha\hat{\rho}+z)} & \dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\ \\ \dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho-z)\Gamma(1-\alpha\rho+z)} \end{bmatrix}, \tag{2.17}$$

for $\mathrm{Re}(z) \in (-1,\alpha)$, where $\hat{\rho} = 1-\rho$. Note, the domain $(-1,\alpha)$ is a specific to the stable process and will not be the case for all rssMps.

A straightforward computation shows that, for $\mathrm{Re}(z) \in (-1,\alpha)$,

$$\det\boldsymbol{F}(z) = \frac{\Gamma(\alpha-z)^2\Gamma(1+z)^2}{\pi^2}\left\{\sin(\pi(\alpha\rho-z))\sin(\pi(\alpha\hat{\rho}-z)) - \sin(\pi\alpha\rho)\sin(\pi\alpha\hat{\rho})\right\},$$

which has a root at $z = \alpha-1$. In turn, this implies that $\chi(\alpha-1) = 0$. One also easily checks with the help of the reflection formula for gamma functions that

$$\boldsymbol{v}(\alpha-1) \propto \begin{bmatrix} \sin(\pi\alpha\hat{\rho}) \\ \sin(\pi\alpha\rho) \end{bmatrix}.$$

In that case, we see that Theorems 2 and 3 justify the claim that the family of measures $(\mathbb{P}^\circ_x, x \in \mathbb{R})$ defined via the relation

$$\left.\frac{\mathrm{d}\mathbb{P}^\circ_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} := \frac{\sin(\pi\alpha\hat{\rho})\mathbf{1}_{(X_t>0)} + \sin(\pi\alpha\rho)\mathbf{1}_{(X_t<0)}}{\sin(\pi\alpha\hat{\rho})\mathbf{1}_{(x>0)} + \sin(\pi\alpha\rho)\mathbf{1}_{(x<0)}}\left|\frac{X_t}{x}\right|^{\alpha-1}\mathbf{1}_{(t<\tau^{\{0\}})}, \qquad t \geq 0,$$

is the the Doob h-transform corresponding to the stable process conditioned to avoid the origin when $\alpha \in (1,2)$, and the stable process conditioned to be continuously absorbed at the origin when $\alpha \in (0,1)$. The former of these two conditionings has already been observed in [10], the latter is a new observation. Note that, when $\theta = \alpha-1 = 0$, the Doob h-transform corresponds to no change of measure at all, as the density is equal to unity and $\tau^{\{0\}} = \infty$ almost surely under $\mathbb{P}_x$, $x \in \mathbb{R}$. This is precisely the case of a Cauchy process. It is less clear in this case how to condition it to hit the origin. One may prove Theorems 2 and 3 for stable processes by appealing to a direct form of reasoning using Bayes formula, scaling, dominated convergence using the fact that $\mathbb{E}_x[|X_t|^{\alpha-\varepsilon}] < \infty$,

$x \in \mathbb{R}, t > 0, 0 < \varepsilon < \alpha$, and the representation of the probabilities:

$$\mathbb{P}_x(\tau^{(-1,1)^c} < \tau^{\{0\}}) = (\alpha-1)x^{\alpha-1}\int_1^{1/x}(t-1)^{\alpha\rho-1}(t+1)^{\alpha\hat{\rho}-1}\mathrm{d}t, \qquad x \in (0,1)$$

for $\alpha \in (1,2)$, and

$$\mathbb{P}_x(\tau^{(-1,1)} < \infty) = \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha)}\int_{\frac{x-1}{x+1}}^{1} t^{\alpha\hat{\rho}-1}(1-t)^{-\alpha}\,\mathrm{d}t, \qquad x > 1$$

for $\alpha \in (0,1)$. The first of these probabilities is taken from Corollary 1 of [29] and the second from Corollary 1.2 of [25]. For the general case, no such detailed formulae are available and a different approach is needed. The main point of interest is in understanding the asymptotic probabilities of the conditioning event in Theorems 2 and 3 by appealing to a Cramér-type result for the decay of the probabilities $\mathbb{P}_x(\tau^{(-a,a)} < \infty)$ and $\mathbb{P}_x(\tau^{(-a,a)^c} < \infty)$ as $a \to \infty$.

Interpreting the Riesz–Bogdan–Żak transform: An additional point of interest in the case of stable processes pertains to the setting of the so-called Riesz–Bogdan–Żak transform, which was first proved in [7] for isotropic stable processes and [21] for anistropic stable processes; see also [24]. The understanding of $\mathbb{P}^\circ_x$, $x \in \mathbb{R}\backslash\{0\}$ as a conditioning, gives context to the transformation which states that transforming the range of a stable process through the mapping $-1/x$, and then making an additional change of time, results in a new process which is the Doob h-transform of the stable process. We now see that the latter is nothing more than one of the two conditionings discussed in Theorem 2.

Theorem 5 (Riesz–Bogdan–Żak transform)**.** Suppose that X is a stable process with $\alpha \in (0,2)$ satisfying $\alpha\rho \in (0,1)$. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha}\mathrm{d}u > t\}, \qquad t \geq 0.$$

Then, for all $x \in \mathbb{R}\backslash\{0\}$, $(-1/X_{\eta(t)})_{t\geq 0}$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^\circ_{-1/x})$. Moreover, the process $(X, \mathbb{P}^\circ_x)$, $x \in \mathbb{R}\backslash\{0\}$ is a self-similar Markov process with underlying MAP via the Lamperti–Kiu representation whose matrix exponent satisfies, for $\mathrm{Re}(z) \in (-\alpha, 1)$,

$$\boldsymbol{F}^\circ(z) = \begin{bmatrix} -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\rho-z)\Gamma(\alpha\rho+z)} & \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \\ \\ \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} & -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\hat{\rho}-z)\Gamma(\alpha\hat{\rho}+z)} \end{bmatrix}. \qquad (2.18)$$

2.4 Cramér-type results for MAPs and the proof of Theorem 2

Appealing to the Lamperti–Kiu process, we note that, for $|x| < a$

$$\mathbb{P}_x(\tau^{(-a,a)^c} < \tau^{\{0\}}) = \mathbf{P}_{\log|x|,\mathrm{sign}(x)}(T^+_{\log a} < \infty) = \mathbf{P}_{0,\mathrm{sign}(x)}(T^+_{\log(a/|x|)} < \infty)$$

where $T^+_y = \inf\{t > 0 : \xi(t) > y\}$. A similar result may be written for $\mathbb{P}_x(\tau^{(-a,a)} < \infty)$, albeit using $T^-_y := \inf\{t > 0 : \xi(t) < y\}$. This suggests that the asymptotic behaviour of the two probabilities of interest can be studied through the behaviour of the underlying MAP. In fact, it turns out that, in both cases, a Cramér-type result in the MAP context provides the desired asymptotics.

Proposition 18. Suppose that X is a rssMp under assumption (A).

(a) When $\theta > 0$, there exists a constant $C_\theta \in (0,\infty)$ such that, for $|y| > 0$

$$\lim_{a\to\infty} a^\theta \mathbb{P}_y(\tau^{(-a,a)^c} < \tau^{\{0\}}) = v_{\mathrm{sign}(y)}(\theta) C_\theta |y|^\theta.$$

In particular,

$$\lim_{a\to\infty} \frac{\mathbb{P}_y(\tau^{(-a,a)^c} < \tau^{\{0\}})}{\mathbb{P}_x(\tau^{(-a,a)^c} < \tau^{\{0\}})} = \lim_{a\to\infty} \frac{\mathbf{P}_{0,\mathrm{sign}(y)}(T^+_{\log(a/|y|)} < \infty)}{\mathbf{P}_{0,\mathrm{sign}(x)}(T^+_{\log(a/|x|)} < \infty)} = \frac{v_{\mathrm{sign}(y)}(\theta)}{v_{\mathrm{sign}(x)}(\theta)} \left|\frac{y}{x}\right|^\theta, \qquad x, y \in \mathbb{R}. \tag{2.19}$$

(b) When $\theta < 0$, there exists a constant $\tilde{C}_\theta \in (0,\infty)$ such that, for $|y| > 0$

$$\lim_{a\to 0} a^\theta \mathbb{P}_y(\tau^{(-a,a)} < \infty) = v_{\mathrm{sign}(y)}(\theta) \tilde{C}_\theta |y|^\theta.$$

In particular,

$$\lim_{a\to 0} \frac{\mathbb{P}_y(\tau^{(-a,a)} < \infty)}{\mathbb{P}_x(\tau^{(-a,a)} < \infty)} = \lim_{a\to\infty} \frac{\mathbf{P}_{0,\mathrm{sign}(x)}(T^-_{\log(a/|y|)} < \infty)}{\mathbf{P}_{0,\mathrm{sign}(x)}(T^-_{\log(a/|x|)} < \infty)} = \frac{v_{\mathrm{sign}(y)}(\theta)}{v_{\mathrm{sign}(x)}(\theta)} \left|\frac{y}{x}\right|^\theta, \qquad x, y \in \mathbb{R}. \tag{2.20}$$

This result will be proved below after some preliminary lemmas. Recalling the discussion from [21], an excursion theory for MAPs reflected in their running maxima exists with strong similarities to the case of Lévy processes. Specifically, there is a MAP, say $(H^+(t), J^+(t))_{t\geq 0}$, with the property that H^+ is non-decreasing with the same range as the running maximum process $\sup_{s\leq t}\xi(s)$, $t \geq 0$. Moreover, the trajectory of the associated Markov chain J^+ agrees with the chain J on the times of increase of the running maximum. We also refer to the Appendix in [16] for further information on classical excursion theory for MAPs.

As an increasing MAP, the process (H^+, J^+) has associated to it a number of characteristics. For convenience, we will introduce the Laplace matrix exponent $\boldsymbol{\kappa}$ in the form

$$\mathbf{E}_{0,i}[\mathrm{e}^{-\lambda H^+(t)}; J^+(t) = j] = [\mathrm{e}^{-\boldsymbol{\kappa}(\lambda)t}]_{i,j}, \qquad \lambda \geq 0.$$

In a similar fashion to 2.2, the exponent $\boldsymbol{\kappa}$ can be written as

$$\boldsymbol{\kappa}(\lambda) = \mathrm{diag}(\Phi_1(\lambda), \Phi_{-1}(\lambda)) - \boldsymbol{\Lambda} \circ \boldsymbol{K}(\lambda), \qquad \lambda \geq 0, \tag{2.21}$$

where for $i = \pm 1$, $\Phi_i(\lambda)$ is the Laplace exponent of the subordinator encoding the dynamics of H^+ when $J^+ = i$, $\boldsymbol{\Lambda}$ is the intensity matrix of J^+ and $\boldsymbol{K}(\lambda)_{i,j} = \int_{(0,\infty)} \mathrm{e}^{-\lambda x} F^+_{i,j}(\mathrm{d}x)$ with $i \neq j$ $F^+_{i,j}$ is a probability measure with non-negative support for $i, j = \pm 1$ and otherwise $\boldsymbol{K}(\lambda)_{i,i} = 1$, for $i = \pm 1$.

Referring to Chapter 5 of [23], the Laplace exponent $(\Phi_i(\lambda))_{i=\pm 1}$ can be written as

$$\Phi_i(\lambda) = q_i + \delta_i \lambda + \int_0^\infty (1 - \mathrm{e}^{-\lambda x}) \Upsilon_i(\mathrm{d}x), \ \text{ for } \ i = \pm 1, \tag{2.22}$$

where $q_i, \delta_i \geq 0$ and $\int_0^\infty \min(1, x) \Upsilon_i(\mathrm{d}x) < \infty$ for all $i = \pm 1$.

We can interpret (2.21) and(2.22) using the Lévy-Itô decomposition described in Chapter 2 of [23]. More precisely, when $J^+ = \pm 1$, the process H^+ has the increments of a subordinator with drift $\delta_{\pm 1}$ and Lévy measure $\Upsilon_{\pm 1}$ and is sent to a cemetery state $\{+\infty\}$ at rate $q_{\pm 1}$. When J^+ jumps from i to j with $i, j \in \{-1, 1\}$ and $i \neq j$, the process H^+ experiences an independent jump with distribution $F^+_{i,j}$ at rate $\Lambda_{i,j}$.

In the next lemma we write the crossing probability of interest in terms of the potential measures

$$U^+_{i,j}(\mathrm{d}x) = \int_0^\infty \mathbf{P}_{0,i}(H^+(t) \in \mathrm{d}x, J^+(t) = j)\mathrm{d}s, \qquad x \geq 0, i, j \in \{-1, 1\}.$$

Lemma 1. The probability of first passage over a threshold can be decomposed into the probability of creeping and the probability of jumping over it.

(a) For $y > 0$, the probability of jumping over a threshold can be written as

$$\begin{aligned} &\mathbf{P}_{0,i}(T^+_y < \infty, H^+(T^+_y) > y) \\ &\quad = \sum_{j,k=\pm 1} \int_0^y U^+_{i,j}(\mathrm{d}z) \left[\mathbf{1}_{(k\neq j)} \Lambda_{j,k} \overline{F}^+_{j,k}(y - z) + \mathbf{1}_{(k=j)} \overline{\Upsilon}_j(y - z)\right]. \end{aligned} \tag{2.23}$$

(b) If $\delta_j > 0$ for some $j = \pm 1$, then $U^+_{i,j}$ has a density on $[0, \infty)$ for $i = \pm 1$, which has a continuous version, say $u^+_{i,j}$. Moreover, for $y > 0$,

$$p_i(y) := \mathbf{P}_i(T^+_y < \infty, H^+(T^+_y) = y) = \sum_{j=\pm 1} \delta_j u^+_{i,j}(y), \qquad y > 0, i = \pm 1,$$

where we understand $u^+_{i,j} \equiv 0$ if $\delta_j = 0$. If $\delta_j = 0$ for both $j = \pm 1$, then $p_i(y) = 0$ for all $y > 0$.

Proof. (a) Appealing to the compensation formula for the Cox process that describes the jumps in

H^+, we may write for $y > 0$,

$$\begin{aligned}
\mathbf{P}_{0,i}(T_y^+ < \infty) &= \mathbf{E}_{0,i}\left[\sum_{0<s<\infty} \mathbf{1}_{(y-H^+(s-)>0)}\mathbf{1}_{(y-H^+(s)<0)}\right] \\
&= \sum_{j,k=\pm1} \mathbf{E}_{0,i}\left[\sum_{0<s<\infty} \mathbf{1}_{(y-H^+(s-)>0)}\mathbf{1}_{(\Delta H^+(s)>y-H^+(s-))}\mathbf{1}_{(J^+(s-)=j,J^+(s)=k)}\right] \\
&= \sum_{j,k=\pm1} \mathbf{1}_{(j\neq k)} \int_0^\infty \mathbf{P}_{0,i}(H^+(s-) < y, J^+(s-) = j)\Lambda_{j,k}\overline{F}^+_{j,k}(y - H^+(s-))\mathrm{d}s \\
&\quad + \sum_{j=\pm1} \int_0^\infty \mathbf{P}_{0,i}(H^+(s-) < y, J^+(s-) = j)\overline{\Upsilon}_j(y - H^+(s-))\mathrm{d}s, \qquad (2.24)
\end{aligned}$$

where $\Delta H^+(s) = H^+(s) - H^+(s-)$, $\overline{F}^+_{j,k}(x) = 1 - F^+_{j,k}(x)$ and $\overline{\Upsilon}_j(x) = \Upsilon_j(x,\infty)$. When we express the right-hand side of (2.24) in terms of the potential measure we get (2.23).

(b) We first define, for $a > 0$,

$$M_i(a) := \int_0^a \mathbf{P}_{0,i}(H^+(T_y^+) = y, T_y^+ < \infty)dy = \int_0^a p_i(y)dy. \qquad (2.25)$$

The analogue of the Lévy-Itô decomposition for subordinators tells us that, up to killing at rate $q_{\pm1}$, when J^+ is in state ±1,

$$H^+(t) = \int_0^t \delta_{J^+(t)}dt + \sum_{0<s<t} \Delta H^+(s), \qquad t \geq 0.$$

Then,

$$M_i(a) = \mathbf{E}_{0,i}\left[H^+(T_a^+-) - \sum_{0<s<T_a^+}(H^+(s) - H^+(s-))\,;\, T_a^+ < \infty\right] = \mathbf{E}_{0,i}\left[\int_0^{T_a^+} \delta_{J^+(t)}dt\,;\, T_a^+ < \infty\right].$$

Hence, for $a > 0$,

$$M_i(a) = \mathbf{E}_i\left[\int_0^\infty \mathbf{1}_{(0\leq H^+(t)\leq a)}\delta_{J(t)}\mathrm{d}t\right] = \sum_{j=\pm1} \delta_j U^+_{i,j}[0,a].$$

Noting from (2.25) that M_i is almost everywhere differentiable on $(0,\infty)$, the above equality tells us that, for j such that $\delta_j \neq 0$ the potential measure $U^+_{i,j}$ has a density. Otherwise, if $\delta_j = 0$ for both $j = \pm1$, then $p_i(y) = 0$ for Lebesgue almost every $y > 0$.

We define, for each $i, j = \pm1$ and $x > 0$,

$$p_{i,j}(x) = \mathbf{P}_{0,i}(T_x^+ < \infty, H^+(T_x^+) = x, J^+(T_x^+) = j) \text{ such that } p_i(x) = \sum_{j=\pm1} p_{i,j}(x).$$

Fix $i \in \{-1, 1\}$. We want to show that $p_i(x)$ is continuous. For that, we shall use the fact that

$$\lim_{\epsilon \downarrow 0} p_{i,j}(\epsilon) = \mathbf{1}(\delta_i > 0)\mathbf{1}(i = j). \tag{2.26}$$

This is due to the fact that the stopping time $T := \inf\{s > 0 : J^+(s) \neq i \text{ or } H^+(s) = +\infty\}$ is exponentially distributed while the time $T_\epsilon^+ \downarrow 0$ as $\epsilon \downarrow 0$. Hence, on $\{t < T\}$, $H^+(t)$ behaves as a (killed) Lévy subordinator and so $T_\epsilon^+ < T$ with increasing probability, tending to 1 as $\epsilon \downarrow 0$. Hence, the result follows from the classical case of Lévy subordinators; see [20].

By the Markov property we have the lower bound

$$p_i(x + \epsilon) \geq \mathbf{P}_{0,i}(H^+(T_x^+) = x, H^+(T_{x+\epsilon}^+) = x + \epsilon, T_{x+\epsilon}^+ < \infty) = \sum_{j=\pm 1} p_{i,j}(x) p_j(\epsilon). \tag{2.27}$$

If we take the limit $\epsilon \downarrow 0$ and use (2.26), then we have that

$$\lim_{\epsilon \downarrow 0} p_i(x + \epsilon) \geq \sum_{j=\pm 1} p_{i,j}(x)\mathbf{1}(\delta_j > 0) = \sum_{j=\pm 1} p_{i,j}(x) = p_i(x).$$

On the other hand, we can split the behavior of creeping over $x + \epsilon$ into two types

$$\begin{aligned} p_i(x+\epsilon) &= \mathbf{P}_{0,i}(H^+(T_x^+) = x, H^+(T_{x+\epsilon}^+) = x + \epsilon, T_{x+\epsilon}^+ < \infty) \\ &\quad + \mathbf{P}_{0,i}(H^+(T_x^+) > x, H^+(T_{x+\epsilon}^+) = x + \epsilon, T_{x+\epsilon}^+ < \infty). \end{aligned}$$

The first probability on the right-hand side above corresponds to the right-hand side of (2.27) and we can bound the second term by the event that $\{0 < O_x \leq \epsilon\}$, where we define the overshoot $O_x := H^+(T_x^+) - x$. Hence, we deduce that

$$p_i(x + \epsilon) \leq \sum_{j=\pm 1} p_{i,j}(\epsilon) p_j(x) + \mathbf{P}_{0,i}(O_x \in (0, \epsilon]).$$

The second probability on the right-hand side above goes to zero as $\epsilon \to 0$. If we now combine this inequality with (2.27) and take the limit $\epsilon \downarrow 0$, then we can then show that

$$\lim_{\epsilon \downarrow 0} p_i(x + \epsilon) = p_i(x) = \sum_{j=\pm 1} p_{i,j}(x).$$

We can also show in a similar way that $\lim_{\epsilon \downarrow 0} p_i(x - \epsilon) = p_i(x)$ and hence p_i is continuous. Note that the preceding reasoning is valid without discrimination for the case that p_i is almost everywhere equal to zero. The proof is now complete. □

Understanding the asymptotic of $\mathbf{P}_{0,i}(T_y^+ < \infty)$ is now a matter of Markov additive renewal theory. In this respect, let us say some more words about the Markov additive renewal measure $U_{i,j}$.

We will restrict the forthcoming discussion to the setting that $\theta > 0$. Recall from the discussion at the end of Section 2.1.1 that this implies $\lim_{t\to\infty} \xi(t) = -\infty$, where ξ is the MAP underlying the rssMp. A consequence of this observation is that the process H^+ experiences killing. To be more precise it has killing rates which we previously denoted by $q_{\pm 1} > 0$. This makes the measures $U^+_{i,j}$ finite. As with classical renewal theory, we can use the existence of the Cramér number θ to renormalise the measures $U^+_{i,j}$ so that they are appropriate for use with asymptotic Markov additive renewal theory.

Appealing to the exponential change of measure described in Proposition 15, we note that the law of (H^+, J^+) under $\mathbf{P}^\theta_{0,i}$ satisfies

$$\mathbf{P}^\theta_{0,i}(H^+(t) \in \mathrm{d}x, J^+(t) = j) = \frac{v_j(\theta)}{v_i(\theta)} \mathrm{e}^{\theta x} \mathbf{P}_{0,i}(H^+(t) \in \mathrm{d}x, J^+(t) = j), \qquad i, j = \pm 1, x \geq 0.$$

In particular, the role of $\boldsymbol{\kappa}$ for (H^+, J^+) under $\mathbb{P}^\theta_{0,i}$, $i = \pm 1$ is played by

$$\boldsymbol{\kappa}_\theta(\lambda) = \boldsymbol{\kappa}(\lambda - \theta), \qquad \lambda \geq 0.$$

Hence, we have that

$$U^{\theta,+}_{i,j}(\mathrm{d}x) := \int_0^\infty \mathbf{P}^\theta_{0,i}(H^+(t) \in \mathrm{d}x, J^+(t) = j)\mathrm{d}t = \frac{v_j(\theta)}{v_i(\theta)} \mathrm{e}^{\theta x} U^+_{i,j}(\mathrm{d}x), \qquad x \geq 0.$$

Again, referring to the discussion at the end of Section 2.1.1, since $\lim_{t\to\infty} \xi(t) = \infty$ almost surely under $\mathbf{P}^\theta_{0,i}$, we may now claim that the adjusted Markov additive renewal measure $U^{\theta,+}_{i,j}(\mathrm{d}x)$ is that of an unkilled subordinator MAP.

Lemma 2. Suppose that $\theta > 0$. There exists a constant $C_\theta > 0$, such that, as $y \to \infty$,

$$\mathrm{e}^{\theta y} \mathbf{P}_{0,i}(T^+_y < \infty) \to v_i(\theta) C_\theta.$$

Proof. Picking up equation (2.23), we have, for $i = \pm 1$,

$$\begin{aligned} &\mathrm{e}^{\theta y} \mathbf{P}_{0,i}(T^+_y < \infty,\, H^+(T^+_y) > y) \\ &= v_i(\theta) \sum_{j,k=\pm 1} \int_0^y \mathrm{e}^{\theta(y-z)} \frac{1}{v_j(\theta)} U^{\theta,+}_{i,j}(\mathrm{d}z) \left[\mathbf{1}_{(k\neq j)} \Lambda_{j,k} \overline{F}^+_{j,k}(y-z) + \mathbf{1}_{(k=j)} \overline{\Upsilon}_j(y-z)\right]. \end{aligned} \tag{2.28}$$

Our aim is to convert this into a form that we can apply the discrete-time Markov Additive Renewal Thoerem 24 in the Appendix.

To this end, we define the sequence of random times $\Theta_1, \Theta_2, \cdots$ such that $\Theta_{i+1} - \Theta_i$ are independent and exponentially distributed with parameter 1. For convenience, define $\Theta_0 = 0$. We want to relate (H^+, J^+) to a discrete-time Markov additive renewal process (Ξ_n, M_n), $n \geq 0$, such that

$$\Delta_n := \Xi_{n+1} - \Xi_n = H^+(\Theta_{n+1}) - H^+(\Theta_n) \text{ and } M_n = J^+(\Theta_n), \qquad n \geq 0.$$

A future quantity of interest is the stationary mean increment $\mu_\theta^+ := \mathbf{E}^\theta_{0,\boldsymbol{\pi}^\theta}[H_1(\Theta_1)]$, where $\boldsymbol{\pi}^\theta = (\pi^\theta_1, \pi^\theta_{-1})$ is the stationary distribution of J (and hence of J^+ since it is described pathwise by J sampled at a sequence of stopping times) under $\mathbf{P}^\theta$. In this respect, we note from Corollary 2.5 in Chapter XI of [4] that,

$$
\begin{aligned}
\mu_\theta^+ &= \int_0^\infty \mathrm{e}^{-t}\mathbf{E}^\theta_{0,\boldsymbol{\pi}^\theta}[H^+(t)]\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-t}[\chi_\theta^+(0)t + \boldsymbol{\pi}^\theta \cdot \boldsymbol{k}^\theta - \boldsymbol{\pi}^\theta \cdot \mathrm{e}^{\boldsymbol{\Lambda}^\theta t}\boldsymbol{k}^\theta]\mathrm{d}t \\
&= \chi_\theta^+(0) + \boldsymbol{\pi}^\theta \cdot \boldsymbol{k}^\theta - \boldsymbol{\pi}^\theta \cdot (\boldsymbol{\Lambda}^\theta - \boldsymbol{I})^{-1}\boldsymbol{k}^\theta, \qquad (2.29)
\end{aligned}
$$

where $\chi_\theta^+(0)$ is the leading eigenvalue of $\boldsymbol{\kappa}_\theta(0)$, $\boldsymbol{k}^\theta = \boldsymbol{v}'(\theta)$ and $\boldsymbol{\Lambda}^\theta = \boldsymbol{\kappa}_\theta(0)$. All of these quantities are guaranteed to exist thanks to the assumption (A); see for example Section 2 of Chapter XI in [4].

Note, moreover, that

$$
\begin{aligned}
U^{\theta,+}_{i,j}(\mathrm{d}x) &= \int_0^\infty \mathbf{P}^\theta_{0,i}(H^+_t \in \mathrm{d}x,\ J^+(t) = j)\mathrm{d}t \\
&= \sum_{n=1}^\infty \int_0^\infty \mathrm{e}^{-t}\frac{t^{n-1}}{(n-1)!}\mathbf{P}^\theta_{0,i}(H^+_t \in \mathrm{d}x,\ J^+(t) = j)\mathrm{d}t \\
&= \sum_{n=1}^\infty \mathbf{P}^\theta_{0,i}(H_{\Theta_n} \in \mathrm{d}x, J_{\Theta_n} = j) \\
&=: R^\theta_{i,j}(\mathrm{d}x) - \delta_0(\mathrm{d}x)\mathbf{1}(i=j), \qquad (2.30)
\end{aligned}
$$

where, on the right-hand side, we have used the notation of the discrete-time Markov additive renewal measure in the Appendix.

Turning back to (2.28), if we define

$$
g_j(x) = \sum_{k=\pm1}\frac{1}{v_j(\theta)}\mathrm{e}^{\theta x}\left[\mathbf{1}(k\neq j)\Lambda_{j,k}\overline{F}_{j,k}(x) + \mathbf{1}(k=j)\Upsilon_j(x)\right], \qquad x \geq 0, \qquad (2.31)
$$

for $j = \pm1$, then, as soon as we can verify that these functions are directly Riemann integrable, then we can apply the conclusion of Theorem 24 in the Appendix and conclude that

$$
\begin{aligned}
&\lim_{y\to\infty} \mathrm{e}^{\theta y}\mathbf{P}_{0,i}(T^+_y < \infty,\ H^+(T^+_y) > y) \\
&\quad = v_i(\theta)\sum_{j,k=\pm1}\frac{\pi^\theta_j}{v_j(\theta)\mu^+_\theta}\int_0^\infty \mathrm{e}^{\theta s}\left[\mathbf{1}_{(k\neq j)}\Lambda_{j,k}\overline{F}^+_{j,k}(s) + \mathbf{1}_{(k=j)}\overline{\Upsilon}_j(s)\right]\mathrm{d}s,
\end{aligned}
$$

where π^θ_j, $j = \pm1$ is the stationary distribution of the chain J^+ under $\mathbf{P}^\theta_{x,i}$, $x \in \mathbb{R}$, $i = \pm1$. Note, moreover that, from Lemma 1, together with Theorem 1.2 of [1],

$$\mathrm{e}^{\theta y}\mathbb{P}_i(T_y^+ < \infty, H^+(T_y^+) = y) = v_i(\theta)\sum_{j=\pm1}\frac{1}{v_j(\theta)}\delta_j u_{i,j}^{\theta,+}(y) \to v_i(\theta)\sum_{j=\pm1}\delta_j\frac{\pi_j^\theta}{v_j(\theta)\mu_\theta^+},$$

as $y \to \infty$.

To finish the proof we must thus verify the direct Riemann integrability of $g_j(x)$, $j = \pm1$ in (2.31). Note however, that $g_j(x)$ is the product of $\mathrm{e}^{\theta x}$ and a monotone decreasing function, hence it suffices to check that $\int_0^\infty g_j(x)\mathrm{d}x < \infty$, $j = \pm1$. To this end, remark that, for λ in the domain where $\boldsymbol{\kappa}$ is defined,

$$(\boldsymbol{\kappa}(\lambda)\mathbf{1})_j = q_j + \delta_j\lambda + \int_0^\infty (1 - \mathrm{e}^{-\lambda x})\Upsilon_j(\mathrm{d}x) + \sum_{k=\pm1}\mathbf{1}_{(j\neq k)}\Lambda_{j,k}\int_0^\infty \mathrm{e}^{-\lambda x}F_{j,k}(\mathrm{d}x), \qquad j = \pm1.$$

In particular, with an integration by parts, we have

$$\frac{q_j - (\boldsymbol{\kappa}(-\theta)\mathbf{1})_j}{\theta} = \delta_j + \int_0^\infty \mathrm{e}^{\theta s}\left[\sum_{k=\pm1}\mathbf{1}_{(k\neq j)}\Lambda_{j,k}\overline{F}_{j,k}^+(s) + \mathbf{1}_{(k=j)}\overline{\Upsilon}_j(s)\right]\mathrm{d}s, \qquad j = \pm1,$$

where the left-hand side is finite thanks to the assumption (A). This completes the proof, albeit to note that

$$\lim_{y\to\infty}\mathrm{e}^{\theta y}\mathbb{P}_i(T_y^+ < \infty) = v_i(\theta)\sum_{j=\pm1}\frac{\pi_j^\theta[q_j - (\boldsymbol{\kappa}(-\theta)\mathbf{1})_j]}{\theta v_j(\theta)\mu_\theta^+},$$

which identifies explicitly the constant C_θ in the statement of the lemma. □

Proof of Proposition 18. First assume that $\theta > 0$. A particular consequence of Lemma 2 is that

$$\lim_{a\to\infty}\frac{\mathbb{P}_y(\tau^{(-a,a)^c} < \tau^{\{0\}})}{\mathbb{P}_x(\tau^{(-a,a)^c} < \tau^{\{0\}})} = \lim_{a\to\infty}\frac{\mathbf{P}_{0,\mathrm{sign}(y)}(T^+_{\log(a/|y|)} < \infty)}{\mathbf{P}_{0,\mathrm{sign}(x)}(T^+_{\log(a/|x|)} < \infty)} = \frac{v_{\mathrm{sign}(y)}(\theta)}{v_{\mathrm{sign}(x)}(\theta)}\left|\frac{y}{x}\right|^\theta, \qquad x, y \in \mathbb{R}.$$

Now we turn our attention to the case that $\theta < 0$. We appeal to duality and write

$$\mathbb{P}_x(\tau^{(-a,a)} < \infty) = \mathbf{P}_{(\log|x|,\mathrm{sign}(x))}(T^-_{\log a} < \infty) = \tilde{\mathbf{P}}_{(-\log|x|,\mathrm{sign}(x))}(T^+_{-\log a} < \infty),$$

where under $\tilde{\mathbf{P}}_{x,i}$, $x \in \mathbb{R}$, $i = \pm1$, is the law of $(-\xi, J)$. Note, the associated matrix exponent of this process is $\tilde{\boldsymbol{F}}(z) := \boldsymbol{F}(-z)$, whenever the right-hand side is well defined. In particular, we note that $\tilde{\boldsymbol{F}}(-\theta) = 0$, which is to say that $-\theta > 0$ is the Cramér number for the process $(-\xi, J)$. Moreover, $\tilde{\boldsymbol{F}}(-\theta)\boldsymbol{v}(\theta) := \boldsymbol{F}(\theta)\boldsymbol{v}(\theta) = 0$, which is to say that $\tilde{\boldsymbol{v}}(-\theta) = \boldsymbol{v}(\theta)$. The first part of the proof can now be re-cycled to deduce the conclusions in part (b) of the statement of the proposition. □

2.5 Proof of Therem 2

Proof of Theorem 2. The (super)martingale (2.10) applies an exponential change of measure to (ξ, J), albeit on the sequence of stopping times $\varphi(t)$, for $t < \tau^{\{0\}}$. As the change of measure (2.5) keeps (ξ, J) in the class of MAPs, thanks to Proposition 15, it follows that $\mathbb{P}^\circ_x$, $x \in \mathbb{R}\backslash\{0\}$, corresponds to the law of a rssMp whose underlying MAP is that of the Esscher transform of (ξ, J).

In the case of (a), recalling the discussion preceding Section 2.1.2, the underlying MAP for $(X, \mathbb{P}^\circ_x)$, $x \in \mathbb{R}\backslash\{0\}$ drifts to $+\infty$. This means that under the change of measure, X is a rssMp that never touches the origin, i.e. it is a conservative process. In the case of (b), the underlying MAP drifts to $-\infty$ and hence, under the change of measure X is continuously absorbed at the origin, so it is non-conservative.

For the proof of (a), we follow a standard line of reasoning that can be found, for example, in [8]. Appealing to the Markov property, self-similarity, Fatou's Lemma and (2.19), we have, for $A \in \mathcal{F}_t$,

$$
\begin{aligned}
\liminf_{a\to\infty} & \mathbb{P}_x(A \cap \{t < \tau^{(-a,a)^c}\} \,|\, \tau^{(-a,a)^c} < \tau^{\{0\}}) \\
&= \liminf_{a\to\infty} \mathbb{E}_x\left[\mathbf{1}_{(A,\, t<\tau^{\{0\}}\wedge\tau^{(-a,a)^c})} \frac{\mathbb{P}_{X_t}(\tau^{(-a,a)^c} < \tau^{\{0\}})}{\mathbb{P}_x(\tau^{(-a,a)^c} < \tau^{\{0\}})}\right] \\
&\geq \mathbb{E}_x\left[\mathbf{1}_{(A,\, t<\tau^{\{0\}})} \liminf_{a\to\infty} \frac{\mathbb{P}_{a^{-1}X_t}(\tau^{(-1,1)^c} < \tau^{\{0\}})}{\mathbb{P}_{a^{-1}x}(\tau^{(-1,1)^c} < \tau^{\{0\}})}\right] \\
&= \mathbb{E}_x\left[\mathbf{1}_{(A,\, t<\tau^{\{0\}})} \frac{h_\theta(X_t)}{h_\theta(x)}\right].
\end{aligned}
$$

Recalling the martingale property from (2.10) together with the above inequality, but now applied to the event A^c, tells us that

$$
\begin{aligned}
\limsup_{a\to\infty} & \mathbb{P}_x(A \cap \{t < \tau^{(-a,a)^c}\} \,|\, \tau^{(-a,a)^c} < \tau^{\{0\}}) \\
&\leq 1 - \liminf_{a\to\infty} \mathbb{P}_x(A^c \cap \{t < \tau^{(-a,a)^c}\} \,|\, \tau^{(-a,a)^c} < \tau^{\{0\}}) \\
&\leq \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)} \mathbf{1}_{(t<\tau^{\{0\}})}\right] - \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)} \mathbf{1}_{(A^c,\, t<\tau^{\{0\}})}\right] \\
&= \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)} \mathbf{1}_{(A,\, t<\tau^{\{0\}})}\right],
\end{aligned}
$$

where the final equality follows as we have used the martingale property of the chance of measure for which recall the discussion around (2.10). The required limiting identity follows. For $x \in \mathbb{R}\backslash\{0\}$, the probabilities $\mathbb{P}^\circ_x(A)$, for $A \in \mathcal{F}_t$ with $t \geq 0$, can be extended to uniquely determine $\mathbb{P}^\circ_x(A)$, for $A \in \mathcal{F}$, see Section 1.3 from [12].

The proof of (b) is similar to that of (a) except that in this case (2.10) ensures that X_t^θ is a super-martingale only and hence the final part of the argument above does not extend to this setting. To overcome this difficulty we proceed as follows. Notice $\tau^{(-a,a)} \to \tau^{\{0\}}$ as $a \to 0$. As before for $A \in \mathcal{F}_t$, we have

$$
\begin{aligned}
&\liminf_{a\to 0}\mathbb{P}_x(A\cap\{t<\tau^{(-a,a)}\}\,|\,\tau^{(-a,a)}<\infty)\\
&\quad=\liminf_{a\to 0}\mathbb{E}_x\left[\mathbf{1}_{(A,\,t<\tau^{(-a,a)})}\frac{\mathbb{P}_{X_t}(\tau^{(-a,a)}<\infty)}{\mathbb{P}_x(\tau^{(-a,a)}<\infty)}\right]\\
&\quad\geq \mathbb{E}_x\left[\mathbf{1}_{(A,\,t<\tau^{\{0\}})}\liminf_{a\to 0}\frac{\mathbb{P}_{a^{-1}X_t}(\tau^{(-1,1)}<\infty)}{\mathbb{P}_{a^{-1}x}(\tau^{(-1,1)}<\infty)}\right]\\
&\quad=\mathbb{E}_x\left[\mathbf{1}_{(A,\,t<\tau^{\{0\}})}\frac{h_\theta(X_t)}{h_\theta(x)}\right]\\
&\quad=\mathbb{E}_x\left[\mathbf{1}_A\frac{h_\theta(X_t)}{h_\theta(x)}\right],
\end{aligned}
$$

where, recalling the discussion around (2.9), in the final equality we have used the fact that $\theta<0$ implies that $\tau^{\{0\}}=\infty$ almost surely (irrespective of the point of issue of X). Now, the second half of the argument in (a) extends to this setting if the following equation holds true

$$
\lim_{a\to 0}\mathbb{P}_x(t<\tau^{(-a,a)}\,|\,\tau^{(-a,a)}<\infty)=\mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)}\right].
$$

On the one hand, the Markov property, Fatou's lemma and the estimate (2.20) imply that

$$
\begin{aligned}
\liminf_{a\to 0}\mathbb{P}_x(t<\tau^{(-a,a)}\,|\,\tau^{(-a,a)}<\infty)&=\liminf_{a\to 0}\mathbb{P}_x\left(\mathbf{1}_{(t<\tau^{(-a,a)})}\frac{\mathbb{P}_{X_t}(\tau^{(-a,a)}<\infty)}{\mathbb{P}_x(\tau^{(-a,a)}<\infty)}\right)\\
&\geq \mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)}\mathbf{1}_{(t<\tau^{\{0\}})}\right]\\
&=\mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)}\right].
\end{aligned}
$$

Now, the estimate in (b) in Proposition (18) implies that for $y\neq 0$

$$
\lim_{a\to 0}\left(\frac{a}{|y|}\right)^\theta\mathbb{P}_y(\tau^{(-a,a)}<\infty)=\lim_{a\to 0}\left(\frac{a}{|y|}\right)^\theta\mathbb{P}_{\mathrm{sgn}(y)}(\tau^{(-\frac{a}{|y|},\frac{a}{|y|})}<\infty)=v_{\mathrm{sign}(y)}(\theta)\tilde{C}_\theta,
$$

and the convergence holds uniformly in $a/|y|$ such that $a/|y|<\epsilon$, for $\epsilon>0$.

Moreover, for $a/|y| > \epsilon$ the term $(a/|y|)^\theta \mathbb{P}_y(\tau^{(-a,a)} < \infty)$ remains bounded. Thus for $x \neq 0, \epsilon > 0$, fixed we have

$$
\begin{aligned}
&\limsup_{a\to 0} \mathbb{P}_x\left(\mathbf{1}_{(t<\tau^{(-a,a)})}\frac{\mathbb{P}_{X_t}(\tau^{(-a,a)}<\infty)}{\mathbb{P}_x(\tau^{(-a,a)}<\infty)}\right)\\
&=\limsup_{a\to 0} \mathbb{P}_x\left(\mathbf{1}_{((a/|X_t|)<\epsilon,\ t<\tau^{(-a,a)})}\frac{\mathbb{P}_{X_t}(\tau^{(-a,a)}<\infty)}{\mathbb{P}_x(\tau^{(-a,a)}<\infty)}\right)\\
&\quad+\limsup_{a\to 0} \mathbb{P}_x\left(\mathbf{1}_{((a/|X_t|)\geq\epsilon,\ t<\tau^{(-a,a)})}\frac{\mathbb{P}_{X_t}(\tau^{(-a,a)}<\infty)}{\mathbb{P}_x(\tau^{(-a,a)}<\infty)}\right)\\
&=\mathbb{E}_x\left[\frac{h_\theta(X_t)}{h_\theta(x)}\mathbf{1}_{(t<\tau^{\{0\}})}\right]+\limsup_{a\to 0} \mathbb{P}_x\left(\mathbf{1}_{((a/|X_t|)\geq\epsilon,\ t<\tau^{(-a,a)})}\frac{\mathbb{P}_{X_t}(\tau^{(-a,a)}<\infty)}{\mathbb{P}_x(\tau^{(-a,a)}<\infty)}\right).
\end{aligned}
$$

Finally the limsup in the above estimate is equal to zero because it can be bounded by above as follows

$$
\begin{aligned}
&\mathbb{P}_x\left(\mathbf{1}_{((a/|X_t|)\geq\epsilon,\ t<\tau^{(-a,a)})}\frac{\mathbb{P}_{X_t}(\tau^{(-a,a)}<\infty)}{\mathbb{P}_x(\tau^{(-a,a)}<\infty)}\right)\\
&\leq \frac{1}{a^\theta \mathbb{P}_x(\tau^{(-a,a)}<\infty)}\mathbb{P}_x\left(\mathbf{1}_{((a/|X_t|)\geq\epsilon,\ t<\tau^{(-a,a)})}|X_t|^\theta \sup_{|z|\geq\epsilon}|z|^\theta \mathbb{P}_{\mathrm{sgn}(z)}(\tau^{(-z,z)}<\infty)\right)\\
&=\frac{x^\theta \sup_{|z|\geq\epsilon}|z|^\theta \mathbb{P}_{\mathrm{sgn}(z)}(\tau^{(-z,z)}<\infty)}{a^\theta \mathbb{P}_x(\tau^{(-a,a)}<\infty)}\mathbb{P}^\circ_x\left(\mathbf{1}_{(a/|X_t|\geq\epsilon,\ t<\tau^{(-a,a)})}\frac{v_{\mathrm{sign}(x)}(\theta)}{v_{\mathrm{sign}(X_t)}(\theta)}\right),
\end{aligned}
$$

and by the monotone convergence theorem the rightmost term in the above inequality tends to zero when $a \to 0$.

With a similar argument to the case (a), for $x \in \mathbb{R}\setminus\{0\}$, we can also extend the probabilities $\mathbb{P}_x(A \cap \{t < \tau^{\{0\}}\})$, for $A \in \mathcal{F}_t$, to define $\mathbb{P}^\circ(A)$, for all $A \in \mathcal{F}_{\tau^{\{0\}}}$, which is the same as $\mathcal{F}$ since $X_t = 0$ for all $t \geq \tau^{\{0\}}$ almost surely. □

Proof of Theorem 3. This can be done with the same reasoning with the proof of Theorem 2. □

2.6 Integrated exponential MAPs, proof of Theorem 4

The asymptotic behaviour of the tail distribution of objects similar to I, when the process ξ is replaced by a Lévy process, has been considered in [31, 2]. We will borrow some of the ideas from the second of these two papers and apply them in the Markov additive setting in establishing the estimate in Theorem 4. To this end, recall that in this setting J takes values in a finite state space E, and let us introduce the potential measure

$$
V_{i,j}(\mathrm{d}x) = \int_0^\infty \mathbf{P}_{0,i}(\xi(s) \in \mathrm{d}x, J(s) = j)\mathrm{d}s, \quad i, j \in E.
$$

Proposition 19. For $t > 0$ and $i \in E$,

$$\mathbf{P}_{0,i}(I > t)\mathrm{d}t = \sum_{j \in E} \int_{\mathbb{R}} V_{i,j}(\mathrm{d}y)\mathrm{e}^{\alpha y}\mathbf{P}_{0,j}(\mathrm{e}^{\alpha y} I \in \mathrm{d}t). \tag{2.32}$$

Proof. The method of proof is to show the left- and right-hand sides of (2.32) are equal by considering their Laplace transforms. Integration by parts shows us that, for $\lambda > 0$, we have on the one hand,

$$\mathbf{E}_{0,i}(1 - \mathrm{e}^{-\lambda I}) = \lambda \int_0^\infty \mathrm{e}^{-\lambda t}\mathbf{P}_{0,i}(I > t)\mathrm{d}t. \tag{2.33}$$

We shall use the above equation for comparison later. On the other hand, we have for $\lambda > 0$,

$$\begin{aligned}
\mathbf{E}_{0,i}(1 - \mathrm{e}^{-\lambda I}) &= \mathbf{E}_{0,i}\left[\int_0^\infty \mathrm{d}(\mathrm{e}^{-\lambda \int_s^\infty \mathrm{e}^{\alpha\xi(u)}\mathrm{d}u})\right] \\
&= \lambda \mathbf{E}_{0,i}\left[\int_0^\infty \mathrm{e}^{\alpha\xi(s)}\mathrm{e}^{-\lambda \int_s^\infty \mathrm{e}^{\alpha\xi(u)}\mathrm{d}u}\mathrm{d}s\right] \\
&= \lambda \int_0^\infty \sum_{j \in E} \mathbf{E}_{0,i}\left[\mathrm{e}^{\alpha\xi(s)}\mathrm{e}^{-\lambda \mathrm{e}^{\alpha\xi(s)} \int_s^\infty \mathrm{e}^{\alpha(\xi(u)-\xi(s))}\mathrm{d}u}; J(s) = j\right]\mathrm{d}s \\
&= \lambda \sum_{j \in E} \int_0^\infty \mathbf{E}_{0,i}\left[\mathrm{e}^{\alpha\xi(s)}\mathbf{E}_{0,j}\left[\mathrm{e}^{-\lambda \mathrm{e}^{\alpha y} I}\right]\Big|_{y=\xi(s)}\right]\mathrm{d}s \\
&= \lambda \sum_{j \in E} \int_{\mathbb{R}} V_{i,j}(\mathrm{d}y)\mathrm{e}^{\alpha y}\mathbf{E}_{0,j}[\mathrm{e}^{-\lambda \mathrm{e}^{\alpha y} I}] \\
&= \lambda \sum_{j \in E} \int_{\mathbb{R}} V_{i,j}(\mathrm{d}y)\mathrm{e}^{\alpha y} \int_0^\infty \mathbf{P}_{0,j}(\mathrm{e}^{\alpha y} I \in \mathrm{d}t)\mathrm{e}^{-\lambda t},
\end{aligned} \tag{2.34}$$

where we have applied the conditional stationary independent increments of (ξ, J) in the fourth equality. Now comparing (2.34) with (2.33), we see that

$$\mathbf{P}_{0,i}(I > t)\mathrm{d}t = \sum_{j \in E} \int_{\mathbb{R}} V_{i,j}(\mathrm{d}y)\mathrm{e}^{\alpha y}\mathbf{P}_{0,j}(\mathrm{e}^{\alpha y} I \in \mathrm{d}t),$$

for $t > 0$, as required. □

Now that we have expressed the tail probabilities $\mathbf{P}_{0,i}(I > t)$ in terms of the potential measure $V_{i,j}$, we may again turn to renewal theory for Markov additive random walks in order to extract the desired asymptotics as $t \to \infty$. With a view to applying Theorem 24 in the Appendix, let us therefore introduce (M_n, Δ_n) defined as

$$M_n = J(\Theta_n) \text{ and } \Delta_n = \xi(\Theta_n), \qquad n \geq 0,$$

where, as before, $\Theta_0 = 0$ and Θ_n is the sum of an independent sequence of exponential random variables with unit mean. As in the Appendix, we write $R_{i,j}(\mathrm{d}x)$ for the renewal measure of (Ξ, M), where $\Xi_0 = 0$, $\Xi_n = \Delta_1 + \cdots \Delta_n$, $n \geq 1$. We also introduce

$$R^\theta_{i,j}(\mathrm{d}x) := \frac{v_j(\theta)}{v_i(\theta)} \mathrm{e}^{\theta x} R_{i,j}(\mathrm{d}x), \qquad x \in \mathbb{R},\ i,j \in E.$$

We note again that $V_{i,j}(\mathrm{d}x) = R_{i,j}(\mathrm{d}x) - \delta_0(\mathrm{d}x)\mathbf{1}_{(i=j)}$.

In a similar spirit to (2.30), we may use these Markov additive random walks to write for any interval $A \subseteq [0,\infty)$

$$\begin{aligned}
\mathrm{e}^{(\theta-\alpha)t}\int_{A\mathrm{e}^{\alpha t}} \mathbf{P}_{0,i}(I > s)\mathrm{d}s &= \sum_{j\in E}\int_{\mathbb{R}} V_{i,j}(\mathrm{d}y)\mathrm{e}^{\alpha y}\mathrm{e}^{(\theta-\alpha)t}\int_{A\mathrm{e}^{\alpha t}} \mathbf{P}_{0,j}(\mathrm{e}^{\alpha y} I \in \mathrm{d}s)\\
&= v_i(\theta)\sum_{j\in E}\frac{1}{v_j(\theta)}\int_{\mathbb{R}} R^\theta_{i,j}(\mathrm{d}y)\mathrm{e}^{(\theta-\alpha)(t-y)}\int_{A\mathrm{e}^{\alpha t}} \mathbf{P}_{0,j}(\mathrm{e}^{\alpha y} I \in \mathrm{d}s)\\
&\quad +\mathbf{1}_{(i=j)}\mathrm{e}^{(\theta-\alpha)t}\mathbf{P}_{0,j}(I \in A\mathrm{e}^{\alpha t})\\
&= v_i(\theta)\sum_{j\in E}\frac{1}{v_j(\theta)}\int_{\mathbb{R}} R^\theta_{i,j}(\mathrm{d}y)\mathrm{e}^{(\theta-\alpha)(t-y)}\mathbf{P}_{0,j}(I \in A\mathrm{e}^{\alpha(t-y)})\\
&\quad -\mathbf{1}_{(i=j)}\mathrm{e}^{(\theta-\alpha)t}\mathbf{P}_{0,j}(I \in A\mathrm{e}^{\alpha t}).
\end{aligned} \tag{2.35}$$

Noting that the main term on the right-hand side above is a convolution between the renewal measure $R^\theta_{i,j}$ and the function

$$g_j(z, A) := \frac{1}{v_j(\theta)}\mathrm{e}^{(\theta-\alpha)z}\mathbf{P}_{0,j}(I \in A\mathrm{e}^{\alpha z}), \qquad z \in \mathbb{R},\ j \in E,$$

we are now almost ready to apply the discrete-time Markov Additive Renewal Theorem 24 in the Appendix. It turns out that we need to choose the interval A judiciously according to whether θ is bigger or smaller than α in order to respect the directly Riemann integrability condition in the renewal theorem. We therefore digress with an additional technical lemma before returning to the limit in (2.35) and the proof of Theorem 4.

Lemma 3. When $\theta > 0$, $\mathbf{E}_{0,j}\left(I^{\theta/\alpha-1}\right) < \infty$, for all $j \in E$.

Proof. When $\theta = \alpha$ the result is trivial. The case that $\theta/\alpha < 1$ turns out to be a direct consequence of Proposition 3.6 from [22]. To be more precise, careful inspection of the proof there shows that (in our terminology) if $0 < \alpha\beta \leq \theta$ then $\mathbf{E}_{0,i}[I^{\beta-1}] < \infty$, for all $i \in E$, in which case one takes $\beta = \theta/\alpha$.

For the final case that $\theta/\alpha > 1$, we can replicate the recurrence relation from Section 1.2 of [2]. Appealing to (2.32), we have, for $\beta \in (0, \theta/\alpha)$ and $k \in E$,

$$\mathbf{E}_{0,k}[I^\beta] = \beta\int_0^\infty s^{\beta-1}\mathbf{P}_{0,k}(I > s)\mathrm{d}s = \beta\int_0^\infty s^{\beta-1}\sum_{j\in E}\int_{\mathbb{R}} V_{k,j}(\mathrm{d}y)\mathrm{e}^{\alpha y}\mathbf{P}_{0,j}(\mathrm{e}^{\alpha y} I \in \mathrm{d}s).$$

Let us momentarily assume that $\mathbb{E}_{0,k}[I^{\beta-1}] < \infty$ for $k \in E$. We can use Fubini's theorem and put

$s = t\mathrm{e}^{\alpha y}$, and get

$$\begin{aligned}
\mathbf{E}_{0,k}[I^\beta] &= \beta \sum_j \int_{\mathbb{R}} \mathrm{e}^{\alpha\beta y} V_{k,j}(\mathrm{d}y) \int_0^\infty t^{\beta-1} \mathbf{P}_{0,j}(I \in \mathrm{d}t) \\
&= \beta \sum_{j\in E} \mathbf{E}_{0,j}[I^{\beta-1}] \int_{\mathbb{R}} \mathrm{e}^{\alpha\beta y} V_{k,j}(\mathrm{d}y) \\
&= \beta \sum_{j\in E} \mathbf{E}_{0,j}[I^{\beta-1}] \int_0^\infty \mathrm{d}s \int_{\mathbb{R}} \mathrm{e}^{\alpha\beta y} \mathbf{E}_{0,k}[\xi(s) \in \mathrm{d}y, J(s) = j] \\
&= \beta \sum_{j\in E} \mathbf{E}_{0,j}[I^{\beta-1}] \int_0^\infty (\exp\{tF(\alpha\beta)\})_{k,j} \mathrm{d}t \\
&= \beta \sum_{j\in E} \mathbf{E}_{0,j}[I^{\beta-1}] (\boldsymbol{F}(\alpha\beta)^{-1})_{k,j}.
\end{aligned}$$

where the right-hand side uses the fact that $\beta \in (0, \theta/\alpha)$. We deduce that $\mathbf{E}_{0,k}[I^{\beta-1}] < \infty$ for $k \in E$ implies that $\mathbf{E}_{0,k}[I^\beta] < \infty$ for $k \in E$.

If n is the smallest non-negative integer such that $\theta/\alpha - n \in (0, 1]$, we can use Proposition 3.6 from [22] again, to deduce that $\mathbf{E}_{0,k}[I^{\theta/\alpha-n}] < \infty$. The argument in the previous paragraph can now be used inductively to conclude that $\mathbf{E}_{0,k}[I^{\theta/\alpha-1}] < \infty$, for any $k \in E$. □

Proof of Theorem 4. We break the proof into three cases. We start by assuming that $\theta < \alpha$. In that case, referring to (2.35), we have, assuming the limit exists on the right-hand side,

$$\begin{aligned}
&\lim_{t\to\infty} \mathrm{e}^{(\theta-\alpha)t} \int_0^{\mathrm{e}^{\alpha t}} \mathbf{P}_{0,i}(I > s)\mathrm{d}s \\
&\quad = \lim_{t\to\infty} v_i(\theta) \sum_{j\in E} \frac{1}{v_j(\theta)} \int_{\mathbb{R}} R^\theta_{i,j}(\mathrm{d}y) \mathrm{e}^{(\theta-\alpha)(t-y)} \mathbf{P}_{0,j}(I \in [0, \mathrm{e}^{\alpha(t-y)}]) \\
&\quad = \lim_{t\to\infty} v_i(\theta) \sum_{j\in E} \frac{1}{v_j(\theta)} \int_{\mathbb{R}} R^\theta_{i,j}(\mathrm{d}y) g_j(t-y) \qquad (2.36)
\end{aligned}$$

where

$$g_k(y) = \frac{1}{v_k(\theta)} \mathrm{e}^{(\theta-\alpha)y} \int_0^{\mathrm{e}^{\alpha y}} \mathbf{P}_{0,k}(I \in \mathrm{d}s), \qquad k \in E,\ y \in \mathbb{R}.$$

Note in particular that

$$\int_{\mathbb{R}} g_k(y)\mathrm{d}y = \frac{1}{(\alpha-\theta)v_k(\theta)} \mathbf{E}_{0,k}[I^{\theta/\alpha-1}], \qquad k \in E,$$

which is finite by Lemma 3. Moreover, since $g_k(x)$ is product of an exponential and a monotone function, it is a standard exercise to show that it is also directly Riemann integrable.

The discrete-time Markov Additive Renewal Theorem 24 in the Appendix now justifies the limit in (2.36) so that

$$\lim_{t\to\infty} \mathrm{e}^{(\theta-\alpha)t} \int_0^{\mathrm{e}^{\alpha t}} \mathbf{P}_{0,i}(I > s)\mathrm{d}s \quad = \quad v_i(\theta) \sum_{j\in E} \frac{\pi_j^\theta}{\mu_\theta|\alpha-\theta| v_j(\theta)} \mathbf{E}_{0,j}[I^{\theta/\alpha-1}], \tag{2.37}$$

provided $\mu_\theta < \infty$. This last condition is easily verified as a consequence of assumption (A). Indeed, according to Corollary 2.5 of Chapter XI in [4], we have

$$\mu_\theta = \chi'(\theta) + \boldsymbol{\pi}^\theta \cdot \boldsymbol{k}^\theta - \boldsymbol{\pi}^\theta \cdot (\boldsymbol{Q}^\theta - \boldsymbol{I})^{-1} \boldsymbol{k}^\theta,$$

where $\boldsymbol{Q}^\theta = \boldsymbol{F}_\theta(0)$ is the intensity matrix of J under $\mathbf{P}^\theta$. Writing the limit in (2.37) with a change of variables, we have

$$\lim_{u\to\infty} u^{(\theta/\alpha-1)} \int_0^u \mathbf{P}_{0,i}(I > s)\mathrm{d}s = v_i(\theta) \sum_{j\in E} \frac{\pi_j^\theta}{\mu_\theta|\alpha-\theta| v_j(\theta)} \mathbf{E}_{0,j}[I^{\theta/\alpha-1}],$$

which shows, for each i, regular variation of the integral on the left-hand side. Appealing to the monotone density theorem for regularly varying functions, we now conclude that

$$\mathbf{P}_{0,i}(I > u) \sim u^{-\theta/\alpha} v_i(\theta) \sum_{j\in E} \frac{\pi_j^\theta}{\mu_\theta|\alpha-\theta| v_j(\theta)} \mathbf{E}_{0,j}[I^{\theta/\alpha-1}], \qquad u\to\infty,$$

and the result for the case that $\theta < \alpha$ now follows from (2.16).

The proof for the case $\theta > \alpha$ is completed by starting the reasoning as with the case of $\theta < \alpha$ but with $A = (1,\infty)$ in (2.35). The desired asymptotics again comes from the first term on the right-hand side of (2.35) using a similar application of the Markov Additive Renewal Theorem 24. The details are left to the reader. The second term on the right-hand side of (2.35) becomes negligible since

$$\lim_{t\to\infty} \mathrm{e}^{(\theta-\alpha)t} \mathbf{P}_{0,j}(I > \mathrm{e}^{\alpha t}) = 0$$

on account of the fact that $\mathbf{E}_{0,i}[I^{\theta/\alpha-1}] < \infty$.

The case that $\alpha = \theta$ is dealt with similarly by starting from (2.35) but now setting $A = (1,\lambda)$ for some $\lambda > 1$. In that case, the second term on the right-hand side of (2.35) makes no contribution to the limit in question since

$$\lim_{t\to\infty} \mathbf{P}_{0,j}(I > \mathrm{e}^{\alpha t}) = 0.$$

The integral in the first term on the right-hand side of (2.35) can be written in the form

$$\int_{\mathbb{R}} R_{i,j}^\theta(\mathrm{d}y) \mathbf{P}_{0,j}(I \in A\mathrm{e}^{\alpha(t-y)}) = \int_{\mathbb{R}} \mathbf{P}_{0,j}(I \in \mathrm{d}v) R_{i,j}^\theta(t - \alpha^{-1}\log v, t - \alpha^{-1}\log v + \alpha^{-1}\log\lambda).$$

Thanks to Lemma 3.5 of [1], we have the uniform estimate

$$\sup_{x\in\mathbb{R}} R^\theta_{i,j}(x, x+\alpha^{-1}\log\lambda) \le \pi^\theta_j R^\theta_{i,i}(-\alpha^{-1}\log\lambda, \alpha^{-1}\log\lambda).$$

This result is accompanied by the classical form of the Markov Additive Renewal Theorem (c.f Theorem 3.1 of [1]), which states that

$$\lim_{x\to\infty} R^\theta_{i,j}(x, x+\alpha^{-1}\log\lambda) = \pi^\theta_j \frac{\log\lambda}{\alpha\mu_\theta}.$$

This allows us to apply the dominated convergence and note, in conjunction with the classical form of the Markov Additive Renewal Theorem (c.f Theorem 3.1 of [1]) that

$$\lim_{t\to\infty}\int_{\mathbb{R}} R^\theta_{i,j}(\mathrm{d}y)\mathbf{P}_{0,j}(I\in A\mathrm{e}^{\alpha(t-y)}) = \pi^\theta_j\frac{\log\lambda}{\alpha\mu_\theta}.$$

Plugging this limit back into the first term on the right-hand side of (2.35) provides the necessary convergence to complete the proof in the same way as the previous two cases. The details are again left to the reader. □

Appendix: Markov additive renewal theory

Consider a discrete-time stochastic process described by the pair $(\Delta, M) := ((\Delta_n, M_n))_{n\ge0}$, where Δ_n takes real (or just positive) values and M_n takes values in the set $E := \{1, 2, \ldots, N\}$. We shall specify the law of such a process as follows.

Set $\Delta_0 = 0$. For each $i, j \in E$, there is a probability distribution $P_{i,j}(x)$ such that, conditioning on the history of (Δ, M) up to time $n-1$, the distribution of (Δ_n, M_n) is given by

$$\mathrm{P}(M_n = j, \Delta_n \le x|(M_k, \Delta_k), k = 0, \ldots, n-1) = P_{M_{n-1},j}(x).$$

In this sense, we have that the the process $M = \{M_n : n \ge 0\}$ alone is a Markov chain on E with transition matrix $p_{i,j} := P_{i,j}(\infty)$, for $i, j \in E$. The possibility that $p_{ii} > 0$ is not excluded here.

The distribution of Δ_n only depends on the state at time $n-1$ which makes the discrete-time Markov additive process

$$\Xi_n := \sum_{k=0}^n \Delta_k, \qquad n \ge 0,$$

the analogue of a Markov additive random walk (or Markov additive renewal process if the increments are all positive).

To state a classical renewal result for discrete-time Markov additive processes, we need to introduce a little more notation. The mean transition is given by

$$\eta_i = \sum_{j \in E} \int_{\mathbb{R}} x P_{i,j}(\mathrm{d}x), \qquad i \in E$$

Moreover, the measure $R_{i,j}$ denotes the occupation measure

$$R_{i,j}(x) = \sum_{n=1}^{\infty} \mathrm{P}(\Xi_n \leq x,\, M_n = j | M_0 = i).$$

The following discrete-time Markov additive renewal theorem is lifted from Proposition 9.3 in [19].

Theorem 6 (Markov Additive Renewal Theorem)**.** Given a sequence of functions $g_1, g_2, \ldots, g_N$ that are directly Riemann integrable, we have, for $j \in E$,

$$\lim_{t \to \infty} \int_{\mathbb{R}} g_j(t-s) R_{i,j}(\mathrm{d}s) = \frac{\pi_j \int_0^\infty g_j(y)\mathrm{d}y}{\sum_{j=1}^N \pi_j \eta_j}, \tag{2.38}$$

as soon as $\sum_{j=1}^N \pi_j \eta_j \in (0, \infty)$, where π_i is the stationary distribution for the chain M.

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Concluding remarks

We have conditioned a real self-similar Markov process to avoid or hit the origin. This gives us a better intuition of how a self-similar Markov process behaves near the origin. We will push this idea further into a higher dimensional setting in Chapter 4. In the next chapter, we will compute explicitly some fluctuation identities in a similar fashion as Lemma 1 in the case of d-dimensional isotropic stable processes.

Chapter 3

Deep factorisation of the stable process

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Remark. This article has been submitted using the name Deep factorisation of the stable process III: Radial excursion theory and the point of closest reach.

Abstract

In this paper, we continue our understanding of the stable process from the perspective of the theory of self-similar Markov processes in the spirit of [11, 15]. In particular, we turn our attention to the case of d-dimensional isotropic stable process, for $d \geq 2$. Using a completely new approach we consider the distribution of the point of closest reach. This leads us to a number of other substantial new results for this class of stable processes. We engage with a new radial excursion theory, never before used, from which we develop the classical Blumenthal–Getoor–Ray identities for first entry/exit into a ball, cf. [3], to the setting of n-tuple laws. We identify explicitly the stationary distribution of the stable process when reflected in its running radial supremum. Moreover, we provide a representation of the Wiener–Hopf factorisation of the MAP that underlies the stable process through the Lamperti–Kiu transform.

3.1 Introduction and main results

For $d \geq 1$, let $X := (X_t : t \geq 0)$, with probabilities $\mathbb{P}_x$, $x \in \mathbb{R}^d$, be a d-dimensional isotropic stable process of index $\alpha \in (0, 2)$. That is to say that X is a $\mathbb{R}^d$-valued Lévy process having characteristic

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triplet $(0,0,\Pi)$, where

$$\Pi(B) = \frac{2^{\alpha}\Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|}\int_B \frac{1}{|y|^{\alpha+d}}\mathrm{d}y, \qquad B \in \mathcal{B}(\mathbb{R}). \tag{3.1}$$

Equivalently, this means X is a d-dimensional Lévy process with characteristic exponent $\Psi(\theta) = -\log \mathbb{E}_0(\mathrm{e}^{\mathrm{i}\theta X_1})$ which satisfies

$$\Psi(\theta) = |\theta|^{\alpha}, \qquad \theta \in \mathbb{R}.$$

Stable processes are also self-similar in the sense that they satisfy a scaling property. More precisely, for $c > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$,

$$\text{under } \mathbb{P}_x, \text{ the law of } (cX_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P}_{cx}. \tag{3.2}$$

As such, stable processes are useful prototypes for the study of the class of Lévy processes and, more recently, for the study of the class of self-similar Markov processes. The latter class of processes are regular strong Markov processes which respect the scaling relation (3.2), and accordingly are identified as having stability index $1/\alpha$.

In the last few years, the fluctuation theory of one-dimensional stable processes has benefitted from the interplay between these two theories, in particular, exploiting Lamperti-type decompositions of self-similar Markov processes. Examples of recent results include a deeper examination of the first passage problem, for the half-line, in one dimension, [13], the distribution of the first point of entry into a strip, [14], and the stationary distribution of the process reflected in its radial maximum, [15].

In this paper, we aim to push this agenda further into the setting of isotropic stable processes in dimension $d \geq 2$ (henceforth assumed). According to Section I.4 from [2], such processes are transient in the sense that

$$\lim_{t\to\infty} |X_t| = \infty \tag{3.3}$$

almost surely. Accordingly, when issued from a point $x \neq 0$, it makes sense to define the point of closest reach to the origin; that is, the coordinates of the point in the closure of the range of X with minimal radial distance from the origin. Our main results offer the exact distribution for the point of closest reach as well as a number of completely new fluctuation identities that fall out of its proof and the use of radial excursion theory.

Before describing them in more detail, let us define point of closest reach with a little more precision. We need to note a number of facts. First, isotropy and transience ensures that $|X|$ is a positive self-similar Markov process with index of self-similarity $1/\alpha$ that does not hit 0. Accordingly it can be represented via the classical Lamperti transformation

$$|X_t| = \mathrm{e}^{\xi_{\varphi(t)}}, \qquad t \geq 0, \tag{3.4}$$

where

$$\varphi(t) = \inf\{s > 0 : \int_0^s \mathrm{e}^{\alpha\xi_u}\mathrm{d}u > t\} \tag{3.5}$$

and $\xi = (\xi_s : s \geq 0)$, with probabilities $\mathbf{P}_x$, $x \in \mathbb{R}$, is a Lévy process. It was shown in [5] that the process ξ belongs to the class of so-called hypergeometric Lévy processes. In particular, its Wiener–Hopf factorisation is explicit. Indeed, suppose we write its characteristic exponent $\Psi_\xi(\theta) = -\log \mathbf{E}_0[\exp\{\mathrm{i}\theta\xi_1\}]$, $\theta \in \mathbb{R}$, then up to a multiplicative constant,

$$\Psi_\xi(\theta) = \frac{\Gamma(\frac{1}{2}(-\mathrm{i}\theta + \alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}\theta)} \times \frac{\Gamma(\frac{1}{2}(\mathrm{i}\theta + d))}{\Gamma(\frac{1}{2}(\mathrm{i}\theta + d - \alpha))}, \qquad \theta \in \mathbb{R}, \tag{3.6}$$

where the two terms either side of the multiplication sign constitute the two Wiener–Hopf factors. See e.g. Chapter VI in [2] for background. Recall that if Ψ is the characteristic exponent of any Lévy process, then there exist two Bernstein functions κ and $\hat{\kappa}$ (see [19] for a definition) such that, up to a multiplicative constant,

$$\Psi(\mathrm{i}\theta) = \kappa(-\mathrm{i}\theta)\hat{\kappa}(\mathrm{i}\theta), \qquad \theta \in \mathbb{R}. \tag{3.7}$$

Identity (3.7) is what we refer to as the Wiener–Hopf factorisation. The left-hand factor codes the range of the running maximum and the right-hand factor codes the range of the running infimum of ξ. It can be checked that both belong to the class of so-called beta subordinators (see [9], as well as some of the discussion later in this paper) and, in particular, have infinite activity. This implies that ξ is regular for both the upper and lower half-lines, which in turn, means that any sphere of radius $r > 0$ is regular for both its interior and exterior for X. This and the fact that X has càdlàg paths ensures that, denoting

$$\mathtt{G}(t) := \sup\{s \leq t : |X_s| = \inf_{u \leq s} |X_u|\}, \qquad t \geq 0,$$

the quantity $X_{\mathtt{G}(t)}$ is well defined as the point of closest reach to the origin up to time t in the sense that $X_{\mathtt{G}(t)-} = X_{\mathtt{G}(t)}$ and $|X_{\mathtt{G}(t)}| = \inf_{s \leq t}|X_s|$, see Lemma VI.6 from [2]. The process $(\mathtt{G}(t), t \geq 0)$ is monotone increasing and hence there is no problem defining $\mathtt{G}(\infty) = \lim_{t\to\infty}\mathtt{G}(t)$ almost surely. Moreover, as X is transient in the sense of (3.3), it is also clear that, almost surely, $\mathtt{G}(\infty) = \mathtt{G}(t)$ for all t sufficiently large and that

$$|X_{\mathtt{G}(\infty)}| = \inf_{s \geq 0} |X_s|.$$

Our first main result provides explicitly the law of $X_{\mathtt{G}(\infty)}$.

Theorem 7 (Point of Closest Reach to the origin)**.** The law of the point of closest reach to the origin is given by

$$\mathbb{P}_x(X_{\mathtt{G}(\infty)} \in \mathrm{d}y) = \pi^{-d/2} \frac{\Gamma\left(d/2\right)^2}{\Gamma\left((d-\alpha)/2\right)\Gamma\left(\alpha/2\right)} \frac{(|x|^2 - |y|^2)^{\alpha/2}}{|x-y|^d |y|^\alpha} \mathrm{d}y, \qquad 0 < |y| < |x|.$$

Fundamentally, the proof of Theorem 7 will be derived from two main facts. The first is a suite of exit/entrance formulae from balls for stable processes which come from the classical work of Blumenthal–Getoor–Ray [3].

To state these results, let us write

$$\tau_r^{\oplus} = \inf\{t > 0 : |X_t| < r\} \text{ and } \tau_r^{\ominus} = \inf\{t > 0 : |X_t| > r\},$$

for $r > 0$.

Theorem 8 (Blumenthal–Getoor–Ray [3])**.** For either $|x| < r < |y|$ when $\tau = \tau_r^{\ominus}$, or $|y| < r < |x|$ when $\tau = \tau_r^{\oplus}$,

$$\mathbb{P}_x(X_\tau \in \mathrm{d}y) = \pi^{-(d/2+1)} \Gamma\left(d/2\right) \sin\left(\frac{\pi\alpha}{2}\right) \frac{|r^2 - |x|^2|^{\alpha/2}}{|r^2 - |y|^2|^{\alpha/2}} |x-y|^{-d} \mathrm{d}y. \tag{3.8}$$

Moreover, for $|x| > r$,

$$\mathbb{P}_x(\tau_r^{\oplus} = \infty) = \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)} \int_0^{(|x|^2/r^2)-1} (u+1)^{-d/2} u^{\alpha/2-1} \mathrm{d}u \tag{3.9}$$

and, for $|x| < r$ and bounded measurable f on $\mathbb{R}^d$,

$$\mathbb{E}_x\left[\int_0^{\tau_r^{\ominus}} f(X_s)\mathrm{d}s\right] = \int_{|y|>r} h_r^{\ominus}(x,y) f(y)\mathrm{d}y$$

such that

$$h_r^{\ominus}(x,y) = 2^{-\alpha}\pi^{-d/2} \frac{\Gamma(d/2)}{\Gamma(\alpha/2)^2} |x-y|^{\alpha-d} \int_0^{\zeta_r(x,y)} (u+1)^{-d/2} u^{\alpha/2-1} \mathrm{d}u, \qquad |y| < r, \tag{3.10}$$

where

$$\zeta_r^{\ominus}(x,y) = (r^2 - |x|^2)(r^2 - |y|^2)/r^2|x-y|^2.$$

Remark. It is worth remarking that (3.9) can be used to derive the density of $|X_{\mathtt{G}(\infty)}|$ quite easily. Indeed, thanks to the scaling property and rotational symmetry, it suffices in this respect to consider the law of $|X_{\mathtt{G}(\infty)}|$ under $\mathbb{P}_{\mathtt{1}}$, where $\mathtt{1} = (1, 0, \cdots, 0)$ is the 'North Pole' on $\mathbb{S}_{d-1}$. In this respect, we note that $\mathbb{P}_{\mathtt{1}}(|X_{\mathtt{G}(\infty)}| \le r) = 1 - \mathbb{P}_{\mathtt{1}}(\tau_r^{\oplus} = \infty)$, hence, for $\gamma > 0$,

$$\begin{aligned}\mathbb{E}_{\mathtt{1}}[|X_{\mathtt{G}(\infty)}|^{2\gamma}] &= \int_0^1 r^{2\gamma}\mathrm{d}\mathbb{P}_{\mathtt{1}}(|X_{\mathtt{G}(\infty)}| \le r) \\ &= \frac{2\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\int_0^1 r^{2\gamma+(d-\alpha)-1}(1-r^2)^{\frac{\alpha}{2}-1}\mathrm{d}r \\ &= \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\int_0^1 u^{\gamma+\frac{(d-\alpha)}{2}-1}(1-u)^{\frac{\alpha}{2}-1}\mathrm{d}u. \qquad (3.11)\end{aligned}$$

From this it is straightforward to see that $|X_{\mathtt{G}(\infty)}|$ under $\mathbb{P}_{\mathtt{1}}$ is equal in law to $\sqrt{\mathtt{A}}$, where $\mathtt{A}$ is a Beta$((d-\alpha)/2, \alpha/2)$ distribution.

The second main fact that drives the proof of Theorem 7 is the Lamperti–Kiu representation of self-similar Markov processes. To describe it, we need to introduce the notion of a Markov Additive Process, henceforth written MAP for short.

Let $\mathbb{S}_{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$. With an abuse of previous notation, we say that $(\xi, \Theta) = ((\xi_t, \Theta_t), t \ge 0)$ is a MAP if it is a regular Strong Markov Process on $\mathbb{R}\times\mathbb{S}_{d-1}$, with probabilities $\mathbf{P}_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in \mathbb{S}_{d-1}$, such that, for any $t \ge 0$, the conditional law of the process $((\xi_{s+t}-\xi_t, \Theta_{s+t}) : s \ge 0)$, given $\{(\xi_u, \Theta_u), u \le t\}$, is that of (ξ, Θ) under $\mathbf{P}_{0,\theta}$, with $\theta = \Theta_t$. For a MAP pair (ξ, Θ), we call ξ the *ordinate* and Θ the *modulator*.

According to one of the main results in [1], there exists a MAP such that the d-dimensional isotropic stable process can be written

$$X_t = \exp\{\xi_{\varphi(t)}\}\Theta_{\varphi(t)} \qquad t \ge 0, \qquad (3.12)$$

where φ has the same definition as (3.5). Now we see the reason for our preemptive choice of notation as clearly $|X_t|$ now agrees with (3.4) and we can understand e.g. $\mathbf{P}_x(\xi_t \in A) = \int_{\mathbb{S}_{d-1}} \mathbf{P}_{x,\theta}(\xi_t \in A, \Theta_t \in \mathrm{d}\theta)$, for $t \ge 0$ and $A \in \mathcal{B}(\mathbb{R})$. Whilst the processes Θ and ξ are corellated, it is clearly the case that Θ is isotropic in the distributional sense, and hence an ergodic process on a compact domain with uniform stationary distribution.

Remark. Noting that $X_{\mathtt{G}(\infty)} = |X_{\mathtt{G}(\infty)}| \times \arg(X_{\mathtt{G}(\infty)})$, it is tempting to believe that it is a simple step to take the distributional identity in (3.11) into the law of $X_{\mathtt{G}(\infty)}$. Somewhat naively, this is a particularly attractive perspective because of the similarity between (3.8) and the *a postiori* conclusion in Theorem 7. Indeed one of our approaches was to try to derive the one from the other by a simple limiting procedure. Making this idea rigorous turned out to be much more difficult than originally anticipated on account of the very subtle nature of the correlation between radial and angular behaviour of the MAP that underlies the stable process.

Our proof of Theorem 7 will take us on a journey through an excursion theory of X from its radial maximum. In dimension $d \ge 2$, this is the first time, to our knowledge, that such a radial excursion theory has been used, see however [6]. This will also allow us to prove the n-tuple laws at first entry/exit of a ball (below), which provide a non-trivial extension to the classical identities of Blumenthal, Getoor and Ray [3] given in Theorem 8. Indeed, once the relevant radial excursion theory is made clear, the following theorem and its corollary emerge as a consequence of an application of the appropriate exit system, very much in the spirit of how analogous calculations would be made e.g. in the setting of Lévy processes. What makes them difficult, however, is that the underlying excursion theory deals with excursions of the process X_t/M_t, $t \geq 0$, away from the *set* $\mathbb{S}_{d-1}$, where $M_t := \sup_{s \leq t} |X_s|$, $t \geq 0$. As such it is significantly harder to deal with the family of associated excursion measures that appear in the exit system and which are indexed by $\mathbb{S}_{d-1}$, see below for further details.

Theorem 9 (Triple law at first entrance/exit of a ball)**.** Fix $r > 0$ and define

$$\chi_x(z, y, v) := \pi^{-3d/2} \frac{\Gamma((d+\alpha)/2)}{|\Gamma(-\alpha/2)|} \frac{\Gamma(d/2)^2}{\Gamma(\alpha/2)^2} \frac{||z|^2 - |x|^2|^{\alpha/2} ||y|^2 - |z|^2|^{\alpha/2}}{|z|^\alpha |z-x|^d |z-y|^d |v-y|^{\alpha+d}},$$

for $x, z, y, v \in \mathbb{R}^d \backslash \{0\}$ with $x \neq z$, $y \neq z$ and $v \neq z$.

(i) Write

$$\mathtt{G}(\tau_r^\oplus) = \sup\{s < \tau_r^\oplus : |X_s| = \inf_{u \leq s} |X_u|\}$$

for the instant of closest reach of the origin before first entry into $\{x \in \mathbb{R}^d : |x| < r\}$. For $|x| > |z| > r$, $|y| > |z|$ and $|v| < r$,

$$\mathbb{P}_x(X_{\mathtt{G}(\tau_r^\oplus)} \in \mathrm{d}z,\ X_{\tau_r^\oplus -} \in \mathrm{d}y,\ X_{\tau_r^\oplus} \in \mathrm{d}v;\ \tau_r^\oplus < \infty) = \chi_x(z, y, v)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}v.$$

(ii) Define $\mathcal{G}(t) = \sup\{s < t : |X_s| = \sup_{u \leq s} |X_u|\}$, $t \geq 0$, and write

$$\mathcal{G}(\tau_r^\ominus) = \sup\{s < \tau_r^\ominus : |X_s| = \sup_{u \leq s} |X_u|\}.$$

for the instant of furthest reach from the origin immediately before first exit from $r\mathbb{S}_{d-1}$. For $|x| < |z| < r$, $|y| < |z|$ and $|v| > r$,

$$\mathbb{P}_x(X_{\mathcal{G}(\tau_r^\ominus)} \in \mathrm{d}z,\ X_{\tau_r^\ominus -} \in \mathrm{d}y,\ X_{\tau_r^\ominus} \in \mathrm{d}v) = \chi_x(z, y, v)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}v.$$

Marginalising the first triple law in Theorem 7 to give the joint law of the pair $(X_{\mathtt{G}(\tau_r^\oplus)}, X_{\tau_r^\oplus})$ or the pair $(X_{\tau_r^\oplus -}, X_{\tau_r^\oplus})$ is not necessarily straightforward (although the reader familiar with the manipulation of Riesz potentials may feel more comfortable as such). Whilst an analytical computation for the marginalisation should be possible, if not tedious, we provide a proof which combines other fluctuation identities that we will uncover *en route*.

Corollary 1 (First entrance/exit and closest reach)**.** Fix $r > 0$ and define, for $x, z, v \in \mathbb{R}^d \backslash \{0\}$,

$$\chi_x(z, \bullet, v) := \frac{\Gamma(d/2)^2}{\pi^d |\Gamma(-\alpha/2)|\Gamma(\alpha/2)} \frac{||z|^2 - |x|^2|^{\alpha/2}}{||z|^2 - |v|^2|^{\alpha/2} |z - v|^d |z - x|^d}.$$

(i) For $|x| > |z| > r$, $|v| < r$,

$$\mathbb{P}_x(X_{\mathcal{G}(\tau_r^\oplus)} \in \mathrm{d}z,\ X_{\tau_r^\oplus} \in \mathrm{d}v;\ \tau_r^\oplus < \infty) = \chi_x(z, \bullet, v) \mathrm{d}z\, \mathrm{d}v.$$

(ii) For $|x| < |z| < r$ and $|v| > r$,

$$\mathbb{P}_x(X_{\mathcal{G}(\tau_r^\ominus)} \in \mathrm{d}z,\ X_{\tau_r^\ominus} \in \mathrm{d}v) = \chi_x(z, \bullet, v)\, \mathrm{d}z\, \mathrm{d}v.$$

Corollary 2 (First entrance/exit and preceding position)**.** Fix $r > 0$ and define, for $x, z, y, v \in \mathbb{R}^d \backslash \{0\}$,

$$\chi_x(\bullet, y, v) := \frac{\Gamma((d+\alpha)/2)\Gamma(d/2)}{\pi^d |\Gamma(-\alpha/2)|\Gamma(\alpha/2)^2} \left(\int_0^{\zeta_r^\oplus(x,y)} (u+1)^{-d/2} u^{\alpha/2 - 1} \mathrm{d}u \right) \frac{|x-y|^{\alpha - d}}{|v - y|^{\alpha + d}}\, \mathrm{d}v\, \mathrm{d}y,$$

where

$$\zeta_r^\oplus(x, y) := (|x|^2 - r^2)(|y|^2 - r^2)/r^2 |x - y|^2.$$

(i) For $|x|, |y| > r$, $|v| < r$,

$$\mathbb{P}_x(X_{\tau_r^\oplus -} \in \mathrm{d}y,\ X_{\tau_r^\oplus} \in \mathrm{d}v;\ \tau_r^\oplus < \infty) = \chi_x(\bullet, y, v) \mathrm{d}y\, \mathrm{d}v.$$

(ii) For $|x|, |y| < r$ and $|v| > r$,

$$\mathbb{P}_x(X_{\tau_r^\ominus -} \in \mathrm{d}y,\ X_{\tau_r^\ominus} \in \mathrm{d}v) = \chi_x(\bullet, y, v)\, \mathrm{d}y\, \mathrm{d}v.$$

In [11, 15], one-dimensional stable processes were considered (up to first hitting of the origin in the case that $\alpha \in (1, 2)$), for which the process Θ in the underlying MAP is nothing more than a two-state Markov chain on $\{1, -1\}$. Such MAPs are known to have a Wiener–Hopf-type decomposition.

To be more precise, one may describe the semigroup of (ξ, Θ) via a matrix Laplace exponent which plays a similar role to the characteristic exponent of ξ. When it exists, the matrix $\boldsymbol{\Psi}$, mapping

$\mathbb{C}$ to the space of 2×2 complex valued matrices[4], satisfies,

$$(e^{-\boldsymbol{\Psi}(z)t})_{i,j} = \mathbf{E}_{0,i}[e^{-z\xi(t)}; J_t = j], \qquad i, j = \pm 1, t \geq 0.$$

In fact, it is known to take the form

$$\boldsymbol{\Psi}(z) = \left(\begin{array}{cc} \dfrac{\Gamma(\alpha+z)\Gamma(1-z)}{\Gamma(\alpha\hat{\rho}+z)\Gamma(1-\alpha\hat{\rho}-z)} & -\dfrac{\Gamma(\alpha+z)\Gamma(1-z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\ & \\ -\dfrac{\Gamma(\alpha+z)\Gamma(1-z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & \dfrac{\Gamma(\alpha+z)\Gamma(1-z)}{\Gamma(\alpha\rho+z)\Gamma(1-\alpha\rho-z)} \end{array} \right), \tag{3.13}$$

for $\mathrm{Re}(z) \in (-1, \alpha)$; see [7] and [10]. Similar to the case of Lévy processes, we can define $\boldsymbol{\kappa}$ and $\hat{\boldsymbol{\kappa}}$ as the matrix Laplace exponents of two MAPs, each with non-decreasing ordinate, whose ordinate ranges and accompanying modulation coincide in distribution with the the range of the running maximum of ξ and that of the dual process $\hat{\xi}$, with accompanying modulation. The analogue of the Wiener–Hopf factorisation for MAPs states that, up to pre-multiplying $\boldsymbol{\kappa}$ or $\hat{\boldsymbol{\kappa}}$ (and hence equivalently up to pre-multiplying $\boldsymbol{\Psi}$) by a strictly positive diagonal matrix, we have that

$$\boldsymbol{\Psi}(-\mathrm{i}\lambda) = \boldsymbol{\Delta}_\pi^{-1}\hat{\boldsymbol{\kappa}}(\mathrm{i}\lambda)^T\boldsymbol{\Delta}_\pi\boldsymbol{\kappa}(-\mathrm{i}\lambda), \tag{3.14}$$

for $\lambda \in \mathbb{R}$, where

$$\boldsymbol{\Delta}_\pi := \left(\begin{array}{cc} \sin(\pi\alpha\rho), & 0 \\ 0 & \sin(\pi\alpha\hat{\rho}) \end{array} \right).$$

In the setting of the MAP which underlies the stable process, the so-called deep Wiener–Hopf factorisation was computed in [11], thereby providing the first explicit example of the Wiener–Hopf factorisation for a MAP. When X is a symmetric one-dimensional stable process, then, without loss of generality, we may take $\boldsymbol{\Delta}_\pi$ as the identity matrix, the underlying MAP becomes symmetric, in which case $\hat{\boldsymbol{\kappa}}^T = \hat{\boldsymbol{\kappa}}$ and, moreover, $\hat{\boldsymbol{\kappa}}(\lambda) = \boldsymbol{\kappa}(\lambda + 1 - \alpha)$, $\lambda \geq 0$. In that case, the factorisation simplifies to

$$\boldsymbol{\Psi}(-\mathrm{i}\lambda) = \boldsymbol{\kappa}(\mathrm{i}\lambda + 1 - \alpha)\boldsymbol{\kappa}(-\mathrm{i}\lambda), \qquad \lambda \in \mathbb{R}, \tag{3.15}$$

up to multiplication by a strictly positive diagonal matrix.

For dimension $d \geq 2$, by adopting the right mathematical language, we are also able to provide the deep factorisation of the d-dimensional isotropic stable process, which also generalises the situation in one dimension. To this end, let us introduce the notion of the descending ladder MAP process for (ξ, Θ).

[4]Here the matrix entries are arranged by

$$A = \begin{pmatrix} A_{1,1} & A_{1,-1} \\ A_{-1,1} & A_{-1,-1} \end{pmatrix}.$$

It is not difficult to show that the pair $((\bar{\xi}_t - \xi_t, \Theta_t), t \geq 0)$, forms a strong Markov process, where $\bar{\xi}_t := \sup_{s\leq t} \xi_s$, $t \geq 0$ is the running maximum of ξ. Naturally, on account of the fact that ξ, as a lone process, is a Lévy process, $(\bar{\xi}_t - \xi_t,\ t \geq 0)$, is also a strong Markov process, but we are more interested here on its dependency on Θ. If we denote by L the local time at zero of $\bar{\xi} - \xi$, then the strong Markov property tells us that $(L^{-1}_t, H^+_t, \Theta^+_t)$, $t \geq 0$, defines a Markov additive process, whose first two elements are ordinates that are non-decreasing, where $H^+_t = \xi_{L^{-1}_t}$ and whose modulator $\Theta^+_t = \Theta_{L^{-1}_t}$, $t \geq 0$. In this sense, L also serves as a local time on the set $\{0\} \times \mathbb{S}_{d-1}$ of the Markov process $(\bar{\xi} - \xi, \Theta)$. Because ξ, alone, is also a Lévy process then the pair (L^{-1}, H^+), without reference to the associated modulation Θ^+, are Markovian and play the role of the ascending ladder time and height subordinators of ξ. But again, we are more concerned here with their dependency on Θ^+.

If we are to state a factorisation analogous to (3.15), we must understand how we should define the quantities that are analogous to $\boldsymbol{\Psi}$ and $\boldsymbol{\kappa}$. Inspiration to this end comes from [15], where it was shown that it is more convenient to understand the relationship (3.14) in its inverse form. This is equivalent to showing how the resolvent of the underlying MAP relates to the potential measures associated to $\boldsymbol{\kappa}$ and $\hat{\boldsymbol{\kappa}}$.

Therefore, in the current setting of d-dimensional isotropic stable processes, we define the operators

$$\mathbf{R}_z[f](\theta) = \mathbf{E}_{0,\theta}\left[\int_0^\infty \mathrm{e}^{-z\xi_t} f(\Theta_t)\mathrm{d}t\right], \qquad \theta \in \mathbb{S}_{d-1}, z \in \mathbb{C}$$

and

$$\boldsymbol{\rho}_z[f](\theta) = \mathbf{E}_{0,\theta}\left[\int_0^\infty \mathrm{e}^{-zH^+_t} f(\Theta^+_t)\mathrm{d}t\right], \qquad \theta \in \mathbb{S}_{d-1}, z \in \mathbb{C},$$

for bounded measurable $f : \mathbb{S}_{d-1} \mapsto [0, \infty)$, whenever the integrals make sense.

Theorem 10 (Deep factorisation of the d-dimensional isotropic stable process)**.** Suppose that $f : \mathbb{S}_{d-1} \mapsto \mathbb{R}$ is bounded and measurable. Then

$$\mathbf{R}_{-\mathrm{i}\lambda}[f](\theta) = C_{\alpha,d}\, \boldsymbol{\rho}_{\mathrm{i}\lambda+d-\alpha}\big[\boldsymbol{\rho}_{-\mathrm{i}\lambda}[f]\big](\theta), \qquad \theta \in \mathbb{S}_{d-1}, \lambda \in \mathbb{R},$$

where $C_{\alpha,d} = 2^{-\alpha}\Gamma((d-\alpha)/2)^2/\Gamma(d/2)^2$. Moreover,

$$\boldsymbol{\rho}_z[f](\theta) = \pi^{-d/2}\frac{\Gamma(d/2)^2}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\int_{|y|>1} f(\arg(y))\frac{||y|^2 - |\theta|^2|^{\alpha/2}}{|y|^{\alpha+z}|\theta - y|^d}\mathrm{d}y, \qquad \mathrm{Re}(z) \geq 0$$

and

$$\mathbf{R}_{-\mathrm{i}\lambda}[f](\theta) = \frac{\Gamma((d-\alpha)/2)}{2^\alpha \pi^{d/2}\Gamma(\alpha/2)}\int_{\mathbb{R}^d} f(\arg(y))|y - \theta|^{\mathrm{i}\lambda - d}\mathrm{d}y, \qquad \lambda \in \mathbb{R}.$$

This, our third main result, is the first example we know of in the literature which provides in explicit detail the Wiener–Hopf factorisation of a MAP (in the same spirit as [8]) for which the modulator has an uncountable state space.

Our final main result concerns the stationary distribution of the stable process reflected in its radial supremum. Define $M_t = \sup_{s\le t}|X_s|$, $t \ge 0$. It is a straightforward computation to show that $(X_t/M_t, M_t)$, $t \ge 0$ is a Markov process which lives on $\mathbb{B}_d \times (0,\infty)$, where $\mathbb{B}_d = \{x \in \mathbb{R}^d : |x| \le 1\}$. Thanks to the transience of X, it is clear that $\lim_{t\to\infty} M_t = \infty$, however, thanks to repeated normalistaion of X by its radial maximum, we can expect that the $\lim_{t\to\infty} X_t/M_t$ exists in distribution. Indeed, in the one-dimensional setting this has already been proved to be the case in [15].

Theorem 11. For all bounded measurable $f : \mathbb{B}_d \mapsto \mathbb{R}$ and $x \in \mathbb{R}\backslash\{0\}$

$$\lim_{t\to\infty} \mathbb{E}_x[f(X_t/M_t)] = \pi^{-d/2}\frac{\Gamma((d+\alpha)/2)}{\Gamma(\alpha/2)}\int_{\mathbb{S}_{d-1}}\sigma_1(\mathrm{d}\phi)\int_{|w|<1} f(w)\frac{|1-|w|^2|^{\alpha/2}}{|\phi-w|^d}\mathrm{d}w,$$

where $\sigma_1(\mathrm{d}y)$ is the surface measure on $\mathbb{S}_{d-1}$, normalised to have unit mass.

Remark. Although we are dealing with the case $d \ge 2$, with the help of the duplication formula for gamma functions, we can verify that the above limiting identity agrees with the stationary distribution for the radially reflected process when $d = 1$ given in Theorem 1.3 in [15] if we set $d = 1$ and $\alpha \in (0, 1)$.

We also note that the stationary distribution in the previous theorem is equal in law to the independent product of random variables $\mathtt{U} \times \sqrt{\mathtt{B}}$, where $\mathtt{U}$ is uniformly distributed on $\mathbb{S}_{d-1}$ and $\mathtt{B}$ is a Beta$(d/2, \alpha/2)$ distribution. Indeed, suppose we take $f(w) = |w|^{2\gamma}g(\arg(w))$ for $\gamma > 0$, then we also see that

$$\lim_{t\to\infty} \mathbb{E}_x[f(X_t/M_t)] = \frac{2\Gamma((d+\alpha)/2)}{\Gamma(d/2)\Gamma(\alpha/2)}\int_{\mathbb{S}_{d-1}}\sigma_1(\mathrm{d}\phi)\int_0^1 r^{2\gamma+d-1}(1-r^2)^{\alpha/2}\mathrm{d}r\int_{\mathbb{S}_{d-1}}\frac{g(\theta)}{|\phi-r\theta|^d}\sigma_1(\mathrm{d}\theta).$$

A Newton potential formula tells us that $\int_{\mathbb{S}_{d-1}}|\phi - r\theta|^{-d}\sigma_1(\mathrm{d}\phi) = 1$, see for example Remark III.2.5 in [12], and hence, after an application of Fubini's theorem for the two spherical integrals and change of variable,

$$\lim_{t\to\infty} \mathbb{E}_x[f(X_t/M_t)] = \frac{\Gamma((d+\alpha)/2)}{\Gamma(d/2)\Gamma(\alpha/2)}\int_0^1 u^{\gamma+\frac{d}{2}-1}(1-u)^{\frac{\alpha}{2}-1}\mathrm{d}u \times \int_{\mathbb{S}_{d-1}} g(\theta)\sigma_1(\mathrm{d}\theta),$$

verifying the claimed distributional decomposition.

The remainder of this paper is structured as follows. In the next section we discuss the fundamental tool that allows us to conduct our analysis: an appropriate excursion theory of the underlying MAP (ξ, Θ). This may otherwise be understood as (up to a change of time and change of scale space) the excursion of X from its radial minimum. With this in hand, we progress directly to the proof of Theorem 7 in Section 3.3. Thereafter, in Section 3.4, we introduce the so-called Riesz–Bogdan–Żak transform and discuss its relation to some of the key quantities that appear in the aforesaid radial fluctuation theory. Next, in Section 3.5 we analyse in more detail some specific identities pertaining to integration with respect to the excursion measure that appears in Section 3.2. These identities are then used to prove Theorem 9 in Section 3.6 and to prove the deep factorisation in Section 3.7. Finally, we deal with the stationary distribution, which is proved in Section 3.8.

3.2 Radial excursion theory

One of the principal tools that we will use in our computations is that of radial excursion theory of X from its running minimum. In order to build such a theory, we return to the Lamperti–Kiu transformation (3.12). In the spirit of the discussion preceding Theorem 10, by considering, say, $\ell = (\ell_t, t \geq 0)$, the local time at 0 of the reflected Lévy process $(\xi_t - \underline{\xi}_t, t \geq 0)$, where $\underline{\xi}_t := \inf_{s \leq t} \xi_s$, $t \geq 0$, we can build the descending ladder MAP $((H^-_t, \Theta^-_t), t \geq 0)$, in the obvious way. As before, although the local time ℓ pertains to the reflected Lévy process $\xi - \underline{\xi}$, we will see below that it serves as an adequate choice for the local time of the Markov process $(\xi - \underline{\xi}, \Theta)$ on the set $\{0\} \times \mathbb{S}_{d-1}$ to the extent that we can use it in the context of Maisonneuve's exit formula.

More precisely, suppose we define $\mathtt{g}_t = \sup\{s < t : \xi_s = \underline{\xi}_s\}$, and recall that the regularity of ξ for $(-\infty, 0)$ and $(0, \infty)$, i.e. satisfying (1.27), ensures that it is well defined, as is $\mathtt{g}_\infty = \lim_{t\to\infty} \mathtt{g}_t$. Set

$$\mathtt{d}_t = \inf\{s > t : \xi_s = \underline{\xi}_s\}$$

and, for all $t > 0$ such that $\mathtt{d}_t > \mathtt{g}_t$ the process

$$(\epsilon_{\mathtt{g}_t}(s), \Theta^\epsilon_{\mathtt{g}_t}(s)) := (\xi_{\mathtt{g}_t+s} - \xi_{\mathtt{g}_t}, \Theta_{\mathtt{g}_t+s}), \qquad s \leq \zeta_{\mathtt{g}_t} := \mathtt{d}_t - \mathtt{g}_t,$$

codes the excursion of $(\xi - \underline{\xi}, \Theta)$ from the set $(0, \mathbb{S}_{d-1})$ which straddles time t. Such excursions live in the space of $\mathbb{U}(\mathbb{R} \times \mathbb{S}_{d-1})$, the space of càdlàg paths with lifetime $\zeta = \inf\{s > 0 : \epsilon(s) < 0\}$ such that $(\epsilon(0), \Theta^\epsilon(0)) \in \{0\} \times \mathbb{S}_{d-1}$, $(\epsilon(s), \Theta^\epsilon(s)) \in (0, \infty) \times \mathbb{S}_{d-1}$, for $0 < s < \zeta$, and $\epsilon(\zeta) \in (-\infty, 0)$.

Taking account of the Lamperti–Kiu transform (3.12), it is natural to consider how the excursion of $(\xi - \underline{\xi}, \Theta)$ from $\{0\} \times \mathbb{S}_{d-1}$ translates into a radial excursion theory for the process

$$Y_t := \mathrm{e}^{\xi_t}\Theta_t, \qquad t \geq 0.$$

Ignoring the time change in (3.12), we see that the radial minima of the process Y agree with the radial minima of the stable process X. Indeed, an excursion of $(\xi - \underline{\xi}, \Theta)$ from $\{0\} \times \mathbb{S}_{d-1}$ constitutes an excursion of $(Y_t / \inf_{s\leq t} |Y_s|, t \geq 0)$, from $\mathbb{S}_{d-1}$, or equivalently an excursion of Y from its running radial infimum. Moreover, we see that, for all $t > 0$ such that $\mathtt{d}_t > \mathtt{g}_t$,

$$Y_{\mathtt{g}_t+s} = \mathrm{e}^{\xi_{\mathtt{g}_t}} \mathrm{e}^{\epsilon_{\mathtt{g}_t}(s)} \Theta^\epsilon_{\mathtt{g}_t}(s) = |Y_{\mathtt{g}_t}| \mathrm{e}^{\epsilon_{\mathtt{g}_t}(s)} \Theta^\epsilon_{\mathtt{g}_t}(s), \qquad s \leq \zeta_{\mathtt{g}_t}.$$

This will be useful to keep in mind in the forthcoming excursion computations.

For $t > 0$, let $R_t = \mathtt{d}_t - t$, and define the set $G = \{t > 0 : R_{t-} = 0, R_t > 0\} = \{\mathtt{g}_s : s \geq 0\}$. The classical theory of exit systems, described in Section 4 of [17], now implies that there exists an additive functional $(\Lambda_t, t \geq 0)$ carried by the set of times $\{t \geq 0 : (\xi_t - \underline{\xi}_t, \Theta_t) \in \{0\} \times \mathbb{S}_{d-1}\}$, with a bounded 1-potential, and a family of *excursion measures*, $(\mathbb{N}_\theta, \theta \in \mathbb{S}_{d-1})$, such that

(i) the map $\theta \mapsto \mathbb{N}_\theta$ is a kernel from $\mathbb{S}_{d-1}$ to $\mathbb{R} \times \mathbb{S}_{d-1}$, such that $\mathbb{N}_\theta(1 - e^{-\zeta}) < \infty$ and $\mathbb{N}_\theta$ is carried by the set $\{(\epsilon(0+), \Theta^\epsilon(0) = (0, \theta)\}$ and $\{\zeta > 0\}$;

(ii) we have the *exit formula*

$$\mathbf{E}_{x,\theta}\left[\sum_{g \in G} F((\xi_s, \Theta_s) : s < g) H((\epsilon_g, \Theta_g^\epsilon))\right]$$
$$= \mathbf{E}_{x,\theta}\left[\int_0^\infty F((\xi_s, \Theta_s) : s < t)\mathbb{N}_{\Theta_t}(H(\epsilon, \Theta^\epsilon))\mathrm{d}\Lambda_t\right], \tag{3.16}$$

for $x \neq 0$, where F is any continuous function on the space of càdlàg paths $\mathbb{D}(\mathbb{R} \times \mathbb{S}_{d-1})$ and H is any measurable function on the space of càdlàg paths $\mathbb{U}(\mathbb{R} \times \mathbb{S}_{d-1})$;

(iii) under any measure $\mathbb{N}_\theta$ the process $(\epsilon, \Theta^\epsilon)$ is Markovian with the same semigroup as (ξ, Θ) stopped at its first hitting time of $(-\infty, 0] \times \mathbb{S}_{d-1}$.

The couple $(\Lambda, \mathbb{N}^\cdot)$ is called an exit system. Note that in Maisonneuve's original formulation, the pair Λ and the kernel $\mathbb{N}$ is not unique, but once Λ is chosen the measures $(\mathbb{N}_\theta, \theta \in \mathbb{S}_{d-1})$ are determined but for a Λ-neglectable set, i.e. a set $\mathcal{A}$ such that $\mathbf{E}_{x,\theta}(\int_{t \geq 0} 1_{\{(\xi_s - \underline{\xi}_s, \Theta_s) \in \mathcal{A}\}} \mathrm{d}\Lambda_s) = 0$. Since ℓ is an additive functional with a bounded 1-potential, we will henceforth work with the exit system $(\ell, \mathbb{N}^\cdot)$ corresponding to it.

The importance of (3.16) can already be seen when we consider the distribution of $X_{\mathtt{G}(\infty)}$. Indeed, we have for bounded measurable f on $\mathbb{R}^d$,

$$\begin{aligned}
\mathbb{E}_x[f(X_{\mathtt{G}(\infty)})] &= \mathbf{E}_{\log|x|, \arg(x)}\left[\sum_{t \in G} f(\mathrm{e}^{\xi_t}\Theta_t)\mathbf{1}(\zeta_t = \infty)\right] \\
&= \mathbf{E}_{\log|x|, \arg(x)}\left[\int_0^\infty f(\mathrm{e}^{\xi_t}\Theta_t)\mathbb{N}_{\Theta_t}(\zeta = \infty)\mathrm{d}\ell_t\right] \\
&= \mathbf{E}_{\log|x|, \arg(x)}\left[\int_0^{\ell_\infty} f(\mathrm{e}^{-H_t^-}\Theta_t^-)\mathbb{N}_{\Theta_t^-}(\zeta = \infty)\mathrm{d}t\right] \\
&= \int_{|z|<|x|} U_x^-(\mathrm{d}z) f(z) \mathbb{N}_{\arg(z)}(\zeta = \infty),
\end{aligned} \tag{3.17}$$

where

$$U_x^-(\mathrm{d}z) := \int_0^\infty \mathbf{P}_{\log|x|, \arg(x)}(\mathrm{e}^{-H_t^-}\Theta_t^- \in \mathrm{d}z,\, t < \ell_\infty)\mathrm{d}t, \qquad |z| \leq |x|$$

may be thought of as the expected occupation time of $(\mathrm{e}^{-H_t^-}\Theta_t^-)_{t \geq 0}$ in $\mathrm{d}z$.

Remark. It is worth noting here that the definition of U_x^- is designed specifically to look at the expected occupation measure of the radial minima in cartesian coordinates, rather than in polar coordinates which would be another natural potential associated with (H_t^-, Θ_t^-), $t \geq 0$.

On account of the fact that X is transient, in the sense of (3.3), we know that (H^-, Θ^-) experiences killing at a rate that occurs, in principle, in a state-dependent manner, specifically $\mathbb{N}_\theta(\zeta = \infty)$, $\theta \in \mathbb{S}_{d-1}$. Isotropy allows us to conclude that all such rates take a common value and thanks to the arbitrary scaling of local time ℓ, we can choose this common value to be unity. Said another way, ℓ_∞ is exponentially distributed with rate 1.

In conclusion, we reach the identity

$$\mathbb{E}_x[f(X_{\mathtt{G}(\infty)})] = \int_{|z|<|x|} U_x^-(\mathrm{d}z) f(z) \tag{3.18}$$

or equivalently, the law of $X_{\mathtt{G}(\infty)}$ under $\mathbb{P}_x$, $x \neq 0$, is nothing more than the measure $U_x^-(\mathrm{d}z)$, $|z| \leq |x|$. From this analysis, in combination with (3.9), we also get another handy identity which will soon be of use. For $r < |x|$, $\mathbb{P}_x(\tau_r^\oplus = \infty) = \mathbb{P}_x(|X_{\mathtt{G}(\infty)}| > r)$ and hence, from Theorem 8 we have

$$\begin{aligned}\mathbb{P}_x(\tau_r^\oplus = \infty) &= \int_{r<|z|<|x|} U_x^-(\mathrm{d}z) \\ &= \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)} \int_0^{(|x|^2/r^2)-1} (u+1)^{-d/2} u^{\alpha/2-1} \mathrm{d}u. \end{aligned} \tag{3.19}$$

Another identity where we gain some insight into the quantity U_x^- is the first passage result of Blumental-Getoor-Ray [3] which was already stated in (3.8). For example, the following identity emerges very quickly from (3.16). For bounded measurable functions f, g on $\mathbb{R}^d$,

$$\begin{aligned}&\mathbb{E}_x[g(X_{\mathtt{G}(\tau_1^\oplus)}) f(X_{\tau_1^\oplus}); \tau_1^\oplus < \infty] \\ &= \int_{1<|z|<|x|} U_x^-(\mathrm{d}z) \int_{|y||z|<1} \mathbb{N}_{\arg(z)}(e^{\epsilon(\zeta)}\Theta^\epsilon(\zeta) \in \mathrm{d}y; \zeta < \infty) g(z) f(|z|y). \end{aligned} \tag{3.20}$$

With judicious computations in the spirit of those given above, one might expect to be able to extract an identity for U_x^- in combination with (3.8). For example, developing (3.20) we might write

$$\begin{aligned}\mathbb{E}_x[f(|X_{\tau_1^\oplus}|); \tau_1^\oplus < \infty] &= \int_{1<|z|<|x|} U_x^-(\mathrm{d}z) \int_{y>\log|z|} \mathbb{N}_{\arg(z)}(|\epsilon(\zeta)| \in \mathrm{d}y; \zeta < \infty) f(|z|\mathrm{e}^{-y}) \\ &= \int_{1<|z|<|x|} U_x^-(\mathrm{d}z) \int_{y>\log|z|} \nu(\mathrm{d}y) f(|z|\mathrm{e}^{-y}) \end{aligned} \tag{3.21}$$

for $|x| > 1$ and bounded measurable f on $\mathbb{R}^d$, where we have appealed to isotropy to ensure that $\mathbb{N}_{\arg(z)}(|\epsilon(\zeta)| \in \mathrm{d}y)$ does not depend on $\arg(z)$ and thus can rather be written as $\nu(\mathrm{d}y)$, where ν is therefore the Lévy measure of the subordinator H^-, see e.g. [20]. On account of the fact that the Wiener–Hopf factorisation for ξ is known, c.f. (3.6), the measure ν can written explicitly; see [5]. Indeed, the normalisation of ℓ is equivalent to the requirement that $\Phi^-(0) = 1$, where Φ^- is the Laplace exponent of H^- and hence

$$\Phi^-(\lambda) = \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda y})\nu(\mathrm{d}y) = \frac{\Gamma((d-\alpha)/2)\Gamma((\lambda+d)/2)}{\Gamma(d/2)\Gamma((\lambda+d-\alpha)/2)}, \qquad \lambda \geq 0,$$

which, inverting with the help of a change of variables and the beta integral (see also [5]), tells us that

$$\nu(\mathrm{d}y) = \frac{\alpha\Gamma((d-\alpha)/2)}{\Gamma(d/2)\Gamma(1-\alpha/2)}(1-\mathrm{e}^{-2y})^{-\frac{\alpha}{2}-1}\mathrm{e}^{-dy}\mathrm{d}y. \tag{3.22}$$

Nonetheless, despite the fact that the left-hand side of (3.21) and (3.22) are explicitly available, it seems here, and in other similar computations of this type, difficult to back out an expression for the measure U_x^-.

Whilst our approach will make use of some of the identities above, fundamentally we prove Theorem 7 via a method of approximation, out of which the expression we will obtain for U_x^- can be cleverly used, in conjunction of the excursion theory above, to derive a number of other identities.

3.3 Proof of Theorem 7

We start with some notation. First define, for $x \neq 0$, $|x| > r$, $\delta > 0$ and continuous, positive and bounded f on $\mathbb{R}^d$,

$$\Delta_r^\delta f(x) := \frac{1}{\delta}\mathbb{E}_x\left[f(\arg(X_{\mathtt{G}_\infty})); |X_{\mathtt{G}_\infty}| \in [r-\delta, r]\right].$$

The crux of our proof is to establish a limit of $\Delta_r^\delta f(x)$ in concrete terms as $\delta \to 0$.

Note that, by conditioning on first entry into the ball of radius r, we have, with the help of the first entrance law (3.8) and (3.18),

$$\begin{aligned}\Delta_r^\delta f(x) &= \frac{1}{\delta}\int_{|y|\in[r-\delta,r]} \mathbb{P}_x(X_{\tau_r^\oplus} \in \mathrm{d}y;\ \tau_r^\oplus < \infty)\mathbb{E}_y\left[f(\arg(X_{\mathtt{G}_\infty}));\ |X_{\mathtt{G}_\infty}| \in (r-\delta, |y|]\right] \\ &= \frac{1}{\delta}C_{\alpha,d}\int_{|y|\in[r-\delta,r]} \mathrm{d}y \left|\frac{r^2-|x|^2}{r^2-|y|^2}\right|^{\alpha/2} |y-x|^{-d}\mathbb{E}_y\left[f(\arg(X_{\mathtt{G}_\infty}));\ |X_{\mathtt{G}_\infty}| \in (r-\delta, |y|]\right] \\ &= \frac{1}{\delta}C_{\alpha,d}|r^2-|x|^2|^{\alpha/2}\int_{|y|\in(r-\delta,r]} \mathrm{d}y\frac{|y-x|^{-d}}{|r^2-|y|^2|^{\alpha/2}}\int_{r-\delta\leq|z|\leq|y|} U_y^-(\mathrm{d}z)f(\arg(z)),\end{aligned} \tag{3.23}$$

where

$$C_{\alpha,d} = \pi^{-(d/2+1)}\Gamma(d/2)\sin\left(\frac{\pi\alpha}{2}\right).$$

Our next objective is to try and replace $\int_{r-\delta\leq|z|\leq|y|} U_y^-(\mathrm{d}z)f(\arg(z))$ by a term of simpler form which can be asymptotically estimated in the limit as $\delta \to 0$. To this end, we need some technical lemmas.

Lemma 4. Suppose that f is a continuous function on $\mathbb{R}^d$. Then

$$\lim_{\delta\to 0}\sup_{|y|\in(r-\delta,r]}\left|\frac{\int_{r-\delta\le|z|\le|y|}U_y^-(\mathrm{d}z)f(z)}{\int_{r-\delta\le|z|\le|y|}U_y^-(\mathrm{d}z)}-f(y)\right|=0.$$

Proof. Suppose that $\mathcal{C}_{r,\delta,\varepsilon}(y)$ is the geometric region which coincides with the intersection of a cone with axis along y with radial extent 2ε, say $\mathcal{C}_\varepsilon$, and the annulus $\{z\in\mathbb{R}^d : r-\delta\le|z|\le r\}$; see Figure 3-1. Chose ε,δ such that

$$\sup_{z\in\mathcal{C}_{r,\delta,\varepsilon}(y)}|f(z)-f(y)|<\varepsilon',$$

for some choice of $\varepsilon'\ll 1$.

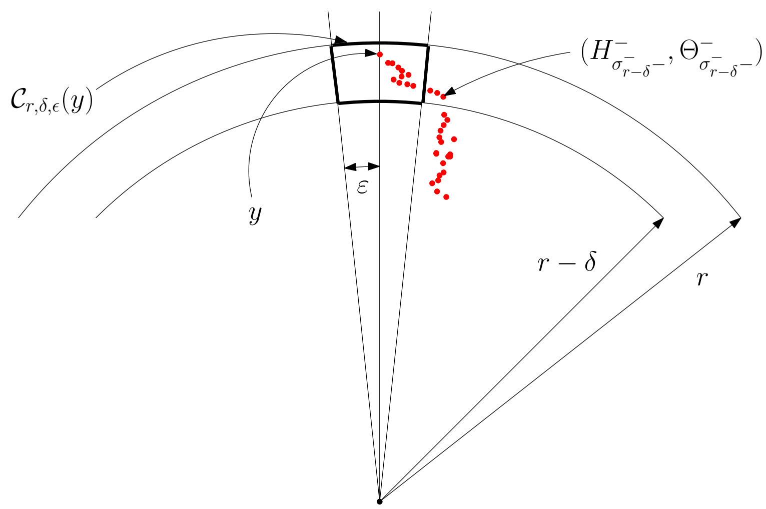

Figure 3-1: The process (H^-,Θ^-) in relation to the domain $\mathcal{C}_{r,\delta,\varepsilon}(y)$.

Since f is continuous and $\{x\in\mathbb{R}^d : |x|\le r\}$ is a compact set, we can define

$$M_r=\sup_{\{x\in\mathbb{R}^d:|x|\le r\}}|f(x)|<\infty.$$

Hence, we have that

$$\begin{aligned}&\sup_{|y|\in(r-\delta,r]}\left|\frac{\int_{r-\delta\le|z|\le|y|}U_y^-(\mathrm{d}z)f(z)}{\int_{r-\delta\le|z|\le|y|}U_y^-(\mathrm{d}z)}-f(y)\right|\\&\le\varepsilon'+M_r\sup_{|y|\in(r-\delta,r]}\frac{\int_{r-\delta\le|z|\le|y|}U_y^-(\mathrm{d}z)\mathbf{1}(z\notin\mathcal{C}_{r,\delta,\varepsilon}(y))}{\int_{r-\delta\le|z|\le|y|}U_y^-(\mathrm{d}z)}.\end{aligned}\tag{3.24}$$

In order to deal with the second term in the right-hand side above, taking the example computations of (3.20) and (3.21), note that, for $|y| \in (r-\delta, r]$,

$$
\begin{aligned}
\sup_{|y|\in(r-\delta,r]} \int_{r-\delta\le|z|\le|y|} & U_y^-(\mathrm{d}z)\mathbf{1}(z \notin \mathcal{C}_{r,\delta,\varepsilon}(y))\nu\left(\log\left(\frac{|z|}{r-\delta}\right),\infty\right) \\
&= \sup_{|y|\in(r-\delta,r]} \mathbb{P}_y(X_{\tau^\oplus_{r-\delta}-} \notin \mathcal{C}_{r,\delta,\varepsilon}(y),\, \tau^\oplus_{r-\delta}<\infty) \\
&= \sup_{\beta\in(r-\delta,r]} \mathbb{P}_{\beta\mathtt{1}}(X_{\tau^\oplus_{r-\delta}-} \notin \mathcal{C}_{r,\delta,\varepsilon}(\beta\mathtt{1}),\, \tau^\oplus_{r-\delta}<\infty) \\
&\le \sup_{\beta\in(r-\delta,r]} \mathbb{P}_{\beta\mathtt{1}}(\Theta^-_{\sigma^-_{r-\delta}-} \notin \mathcal{C}_{\varepsilon}\cap\mathbb{S}_{d-1},\, \sigma^-_{r-\delta}<\infty) \\
&\le \sup_{\beta\in(r-\delta,r]} \mathbb{P}_{\beta\mathtt{1}}(\upsilon_\varepsilon<\sigma^-_{r-\delta}) \le \mathbb{P}_{r\mathtt{1}}(\upsilon_\varepsilon<\sigma^-_{r-\delta}) \qquad (3.25)
\end{aligned}
$$

where $\mathtt{1} = (1,0,\cdots,0)$ is the 'North Pole' on $\mathbb{S}_{d-1}$, $\sigma^-_{r-\delta} = \inf\{t>0 : H^-_t < r-\delta\}$ and $\upsilon_\varepsilon = \inf\{t>0 : \Theta^-_t \notin \mathcal{C}_\varepsilon\cap\mathbb{S}_{d-1}\}$. Right-continuity of paths now ensures that the right-hand side above tends to zero as $\delta\to 0$.

On the other hand, from (3.19)

$$
\int_{r-\delta\le|z|\le|y|} U_y^-(\mathrm{d}z) = \mathbb{P}_y(\tau^\oplus_{r-\delta}=\infty) = \mathbb{P}_{\frac{|y|}{(r-\delta)}\mathtt{1}}(\tau^\oplus_1=\infty), \qquad (3.26)
$$

where we have used isotropy in the final equality and from (3.9) and (3.22) a rather elementary computation shows that

$$
\begin{aligned}
&\lim_{\eta\downarrow 1}\nu(\log\eta,\infty)\,\mathbb{P}_{\eta\mathtt{1}}(\tau^\oplus_1=\infty) \\
&= \lim_{\eta\downarrow 1}\frac{\alpha}{\Gamma(\alpha/2)\Gamma(1-\alpha/2)}\left(\int_{\log\eta}^\infty (1-\mathrm{e}^{-2v})^{-\frac{\alpha}{2}-1}\mathrm{e}^{-dv}\mathrm{d}v\right)\left(\int_0^{\eta^2-1}(u+1)^{-d/2}u^{\alpha/2-1}\mathrm{d}u\right) \\
&= \frac{1}{\Gamma(1+\alpha/2)\Gamma(1-\alpha/2)}
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\lim_{\delta\to0}\sup_{|y|\in(r-\delta,r]}\frac{\int_{r-\delta\le|z|\le|y|}U_y^-(\mathrm{d}z)\mathbf{1}(z\notin\mathcal{C}_{r,\delta,\varepsilon}(y))}{\int_{r-\delta\le|z|\le|y|}U_y^-(\mathrm{d}z)} \\
&\le \lim_{\delta\to0}\sup_{|y|\in(r-\delta,r]}\frac{\int_{r-\delta\le|z|\le|y|}U_y^-(\mathrm{d}z)\mathbf{1}(z\notin\mathcal{C}_{r,\delta,\varepsilon}(y))\frac{\nu(\log(|z|/(r-\delta)),\infty)}{\nu(\log(|z|/(r-\delta)),\infty)}}{\int_{r-\delta\le|z|\le|y|}U_y^-(\mathrm{d}z)} \\
&\le \lim_{\delta\to0}\sup_{|y|\in(r-\delta,r]}\frac{\int_{r-\delta\le|z|\le|y|}U_y^-(\mathrm{d}z)\mathbf{1}(z\notin\mathcal{C}_{r,\delta,\varepsilon}(y))\nu(\log(|z|/(r-\delta)),\infty)}{\nu(\log(|y|/(r-\delta)),\infty)\mathbb{P}_y(\tau^\oplus_{r-\delta}=\infty)} \\
&\le \lim_{\delta\to0}\sup_{1<\eta<1+\frac{\delta}{(r-\delta)}}\frac{\mathbb{P}_{r\mathtt{1}}(\upsilon_\varepsilon<\sigma^-_{r-\delta})}{\nu(\log\eta,\infty)\mathbb{P}_{\eta\mathtt{1}}(\tau^\oplus_1=\infty)} = 0
\end{aligned}
$$

and thus plugging this back into (3.24) gives the result. □

With Lemma 4 in hand, noting in particular the representation (3.26), we can now return to (3.23) and note that, for each $\varepsilon > 0$, we can choose δ sufficiently small such that

$$\Delta_r^\delta f(x) = D(\varepsilon)\Delta_r^\delta 1(x) + \frac{1}{\delta}C_{\alpha,d}|r^2 - |x|^2|^{\alpha/2}\int_{|y|\in(r-\delta,r]} \mathrm{d}y\frac{|y-x|^{-d}}{|r^2-|y|^2|^{\alpha/2}}f(\arg(y))\mathbb{P}_y(\tau^\oplus_{r-\delta}=\infty),$$

where, $|D(\varepsilon)| < \varepsilon$ and for $|x| > r$,

$$\begin{aligned}
\limsup_{\delta\to0}|\Delta_r^\delta 1(x)| &\leq \limsup_{\delta\to0}\left|\frac{1}{\delta}C_{\alpha,d}|r^2-|x|^2|^{\alpha/2}\int_{|y|\in(r-\delta,r]}\mathrm{d}y\frac{|y-x|^{-d}}{|r^2-|y|^2|^{\alpha/2}}\mathbb{P}_y(\tau^\oplus_{r-\delta}=\infty)\right|\\
&= \limsup_{\delta\to0}\left|\frac{1}{\delta}\left(\mathbb{P}_x(\tau^\oplus_{r-\delta}=\infty)-\mathbb{P}_x(\tau^\oplus_r=\infty)\right)\right|\\
&= \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\left|\frac{\mathrm{d}}{\mathrm{d}v}\int_0^{(|x|^2/v^2)-1}(u+1)^{-d/2}u^{\alpha/2-1}\mathrm{d}u\Big|_{v=r}\right|\\
&= \frac{2\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\left(|x|^2-r^2\right)^{\alpha/2-1}r^{d-1-\alpha}|x|^{2-d}
\end{aligned}$$

where in the third equality we have used (3.9).

We can now say that, if the limit exists,

$$\begin{aligned}
&\lim_{\delta\to0}\Delta_r^\delta f(x)\\
&= \lim_{\delta\to0}C_{\alpha,d}|r^2-|x|^2|^{\alpha/2}\frac{1}{\delta}\int_{|y|\in(r-\delta,r]}\mathrm{d}y\frac{|y-x|^{-d}}{|r^2-|y|^2|^{\alpha/2}}f(\arg(y))\mathbb{P}_y(\tau^\oplus_{r-\delta}=\infty)\\
&= \lim_{\delta\to0}C_{\alpha,d}|r^2-|x|^2|^{\alpha/2}\frac{1}{\delta}\int_{r-\delta}^r\rho^{d-1}\mathrm{d}\rho\int_{\rho\mathbb{S}_{d-1}}\sigma_\rho(\mathrm{d}\theta)\frac{|\rho\theta-x|^{-d}}{|r^2-\rho^2|^{\alpha/2}}f(\theta)\mathbb{P}_{\rho\theta}(\tau^\oplus_{r-\delta}=\infty)\\
&= \lim_{\delta\to0}C_{\alpha,d}|r^2-|x|^2|^{\alpha/2}\frac{1}{\delta}\int_{r-\delta}^r\rho^{d-1}\mathrm{d}\rho\frac{\mathbb{P}_{\rho\mathbf{1}}(\tau^\oplus_{r-\delta}=\infty)}{|r^2-\rho^2|^{\alpha/2}}\int_{\rho\mathbb{S}_{d-1}}\sigma_\rho(\mathrm{d}\theta)|\rho\theta-x|^{-d}f(\theta),
\end{aligned}\tag{3.27}$$

where, in the second equality, we have switched from d-dimensional Lebesgue measure to the generalised polar coordinate measure $\rho^{d-1}\mathrm{d}\rho\times\sigma_\rho(\mathrm{d}\theta)$, so that $\rho>0$ is the radial distance from the origin and $\sigma_\rho(\mathrm{d}\theta)$ is the surface measure on $\rho\mathbb{S}_{d-1}$, normalised to have unit mass. In the third equality we have used isotropy to write $\mathbb{P}_{\rho\theta}(\tau^\oplus_{r-\delta}=\infty)=\mathbb{P}_{\rho\mathbf{1}}(\tau^\oplus_{r-\delta}=\infty)$ for $\theta\in\mathbb{S}_{d-1}$.

Noting the continuity of the integral $\int_{\rho\mathbb{S}_{d-1}}\sigma_\rho(\mathrm{d}\theta)|\rho\mathbf{1}-x|^{-d}f(\theta)$ in ρ, the proof of Theorem 7 is complete as soon as we can evaluate

$$\lim_{\delta\to0}\frac{1}{\delta}\int_{r-\delta}^r\rho^{d-1}\mathrm{d}\rho\frac{\mathbb{P}_{\rho\mathbf{1}}(\tau^\oplus_{r-\delta}=\infty)}{|r^2-\rho^2|^{-\alpha/2}}.\tag{3.28}$$

To this end, we need a technical lemma.

Lemma 5. Let $D_{\alpha,d} = \Gamma(d/2)/\Gamma((d-\alpha)/2)\Gamma(\alpha/2)$. Then

$$\lim_{\delta\to 0}\sup_{\rho\in[r-\delta,r]}\left|(\rho^2-(r-\delta)^2)^{-\alpha/2}r^\alpha\mathbb{P}_{\rho\mathit{1}}(\tau^\oplus_{r-\delta}=\infty)-\frac{2D_{\alpha,d}}{\alpha}\right| = 0.$$

Proof. Appealing to (3.9), we start by noting that

$$\begin{aligned}
\sup_{\rho\in[r-\delta,r]}&\left|D_{\alpha,d}\int_0^{\rho^2/(r-\delta)^2-1}u^{\alpha/2-1}\mathrm{d}u-\mathbb{P}_{\rho\mathbf{1}}(\tau^\oplus_{r-\delta}=\infty)\right|\\
&\le \sup_{\rho\in[r-\delta,r]} D_{\alpha,d}\int_0^{\rho^2/(r-\delta)^2-1}\left|(1+u)^{-d/2}-1\right|u^{\alpha/2-1}\mathrm{d}u\\
&\le \sup_{\rho\in[r-\delta,r]} D_{\alpha,d}\int_0^{\rho^2/(r-\delta)^2-1}\left|1-\frac{(r-\delta)^d}{\rho^d}\right|u^{\alpha/2-1}\mathrm{d}u\\
&\le D_{\alpha,d}\left|1-\frac{(r-\delta)^d}{r^d}\right|\frac{2}{\alpha}\left(r^2-(r-\delta)^2\right)^{\alpha/2}(r-\delta)^{-\alpha}, \qquad (3.29)
\end{aligned}$$

which tends to zero as $\delta\to 0$. Furthermore,

$$\begin{aligned}
\sup_{\rho\in[r-\delta,r]}&\left|D_{\alpha,d}\int_0^{\rho^2/(r-\delta)^2-1}u^{\alpha/2-1}\mathrm{d}u-\frac{2D_{\alpha,d}}{\alpha}(\rho^2-(r-\delta))^{\alpha/2}r^{-\alpha}\right|\\
&=\sup_{\rho\in[r-\delta,r]}\frac{2D_{\alpha,d}}{\alpha}(\rho^2-(r-\delta)^2)^{\alpha/2}\left|(r-\delta)^{-\alpha}-r^{-\alpha}\right|\\
&\le\frac{2D_{\alpha,d}}{\alpha}(r^2-(r-\delta))^{\alpha/2}\left|(r-\delta)^{-\alpha}-r^{-\alpha}\right|, \qquad (3.30)
\end{aligned}$$

which also tends to zero as $\delta\to 0$. Summing (3.29) and (3.30) in the context of the triangle inequality and dividing by $r^{-\alpha}(r^2-(r-\delta))^{\alpha/2}$ we can also deduce that

$$\lim_{\delta\to 0}\sup_{\rho\in[r-\delta,r]}\left|(\rho^2-(r-\delta)^2)^{-\alpha/2}r^\alpha\mathbb{P}_{\rho\mathbf{1}}(\tau^\oplus_{r-\delta}=\infty)-\frac{2D_{\alpha,d}}{\alpha}\right| = 0,$$

and the lemma is proved. □

We are now ready to prove (3.28), and identify its limit, thereby completing the proof of Theorem 7. Appealing to Lemma 5, for all $\varepsilon>0$, there exists a δ sufficiently small,

$$\begin{aligned}
&\left|\frac{1}{\delta}\int_{r-\delta}^r\mathrm{d}\rho\frac{\mathbb{P}_{\rho\mathbf{1}}(\tau^\oplus_{r-\delta}=\infty)}{(r^2-\rho^2)^{\alpha/2}}-\frac{2D_{\alpha,d}r^{-\alpha}}{\alpha}\frac{1}{\delta}\int_{r-\delta}^r\mathrm{d}\rho\frac{(\rho^2-(r-\delta)^2)^{\alpha/2}}{(r^2-\rho^2)^{\alpha/2}}\right|\\
&\qquad<\frac{\varepsilon}{\delta}\int_{r-\delta}^r\mathrm{d}\rho\frac{(\rho^2-(r-\delta)^2)^{\alpha/2}}{(r^2-\rho^2)^{\alpha/2}}. \qquad (3.31)
\end{aligned}$$

Next note that

$$\begin{aligned}
\lim_{\delta\to 0}\frac{1}{\delta}\int_{r-\delta}^{r}\mathrm{d}\rho\frac{(\rho^2-(r-\delta)^2)^{\alpha/2}}{(r^2-\rho^2)^{\alpha/2}} &= \lim_{\delta\to 0}\frac{1}{\delta}\int_{r-\delta}^{r}\mathrm{d}\rho\left[\frac{\rho-(r-\delta)}{r-\rho}\right]^{\alpha/2}\left[\frac{\rho+(r-\delta)}{r+\rho}\right]^{\alpha/2}\\
&= \lim_{\delta\to 0}\int_0^1\mathrm{d}u\left[\frac{u}{1-u}\right]^{\alpha/2}\left[\frac{2r-2\delta+\delta u}{2r-\delta+\delta u}\right]^{\alpha/2}\\
&= \int_0^1\mathrm{d}u(1-u)^{-\alpha/2}u^{\alpha/2}\\
&= \Gamma(1-\alpha/2)\Gamma(1+\alpha/2), \qquad (3.32)
\end{aligned}$$

where we have used the substitution $\rho = (r-\delta)+u\delta$ in the second equality and dominated convergence in the third.

Putting the pieces together, we can take limits in (3.31), using (3.32), to deduce that

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{r-\delta}^{r}\mathrm{d}\rho\frac{\mathbb{P}_{\rho\mathbf{1}}(\tau^{\oplus}_{r-\delta}=\infty)}{(r^2-\rho^2)^{\alpha/2}} = \frac{2}{\alpha}D_{\alpha,d}\Gamma(1-\alpha/2)\Gamma(1+\alpha/2)r^{-\alpha}$$

which, in turn, can be plugged into (3.27) and we find that

$$\begin{aligned}
\lim_{\delta\to 0}\Delta^{\delta}_{r}f(x) &= \frac{2}{\alpha}D_{\alpha,d}\Gamma(1-\alpha/2)\Gamma(1+\alpha/2)C_{\alpha,d}r^{d-\alpha-1}|r^2-|x|^2|^{\alpha/2}\int_{r\mathbb{S}_{d-1}}\sigma_\rho(\mathrm{d}\theta)|r\theta-x|^{-d}f(\theta)\\
&= \pi^{-d/2}\frac{\Gamma(d/2)^2}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}r^{d-\alpha-1}|r^2-|x|^2|^{\alpha/2}\int_{r\mathbb{S}_{d-1}}\sigma_\rho(\mathrm{d}\theta)|r\theta-x|^{-d}f(\theta).
\end{aligned}$$

Now suppose that g is another bounded measurable function on $[0,\infty)$, then

$$\begin{aligned}
&\mathbb{E}_x[g(|X_{\mathtt{G}(\infty)}|)f(\arg(X_{\mathtt{G}(\infty)}))]\\
&= \pi^{-d/2}\frac{\Gamma(d/2)^2}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\int_0^{|x|}\int_{r\mathbb{S}_{d-1}}r^{d-1}\mathrm{d}r\sigma_\rho(\mathrm{d}\theta)\frac{|r^2-|x|^2|^{\alpha/2}}{r^\alpha|r\theta-x|^d}f(\theta)g(r)\\
&= \pi^{-d/2}\frac{\Gamma(d/2)^2}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\int_{|y|<|x|}\frac{||y|^2-|x|^2|^{\alpha/2}}{|y|^\alpha|y-x|^d}f(\arg(y))g(|y|)\mathrm{d}y,
\end{aligned}$$

which is equivalent to the statement of Theorem 7. □

3.4 Riesz–Bogdan–Żak transform and MAP duality

Recently, Bogdan and Żak [4] used an idea of Riesz from classical potential analysis to understand the relationship between a stable process and its transformation through a simple sphere inversion. (See also Alili et al. [1] and Kyprianou [11]). Suppose we write $Kx = x/|x|^2$, $x\in\mathbb{R}^d$ for the classical inversion of space through the sphere $\mathbb{S}_{d-1}$. Then in dimension $d\geq 2$, Bogdan and Żak [4] prove that, for $x \neq 0$, $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}_{Kx}$ is equal in law to $(X_t, t \geq 0)$ under $\mathbb{P}^\circ_x$, where

$$\left.\frac{\mathrm{d}\mathbb{P}^\circ_x}{\mathrm{d}\mathbb{P}_x}\right|_{\sigma(X_s:s\leq t)} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \qquad t \geq 0 \tag{3.33}$$

and $\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha}\mathrm{d}u > t\}$. It was shown in Kyprianou et al. [16] that $\mathbb{P}^\circ_x$, $x \in \mathbb{R}^d\backslash\{0\}$ can be understood, in the appropriate sense, as the stable process conditioned to be continuously absorbed at the origin. Indeed, as far as the underlying MAP (ξ, Θ) is concerned, we see that $-\mathrm{i}(\alpha - d)$ is a root of the exponent (3.6) and the change of measure (3.33) corresponds to an Esscher transform of the Lévy process ξ, rendering it a process which drifts to $-\infty$. Thus, an application of the optimal stopping theorem shows that (3.33) is equivalent to the change of measure for ξ

$$\left.\frac{\mathrm{d}\mathbf{P}^\circ_{x,\theta}}{\mathrm{d}\mathbf{P}_{x,\theta}}\right|_{\sigma((\xi_s,\Theta_s):s\leq t)} = \mathrm{e}^{(\alpha-d)(\xi_t - x)}, \qquad t \geq 0 \tag{3.34}$$

Following the reasoning in the one-dimensional case in [1, 11], it is not difficult to show that the space-time transformed process $(KX_{\eta(t)}, t \geq 0)$ is the Lamperti–Kiu transform of the MAP $(-\xi, \Theta)$. Therefore, at the level of MAPs, the Riesz–Bogdan–Żak transform says that (ξ, Θ) under the change of measure (3.34), when issued from $(\log|x|, \arg(x))$, $x \in \mathbb{R}$, is equal in law to $(-\xi, \Theta)$ when issued from $(-\log|x|, \arg(x))$.

An interesting consequence of this is that the Riesz–Bogdan–Żak transform provides an efficient way to analyse radial ascending properties of X, where previously we have studied its descending properties. That is to say, it offers the opportunity to study aspects of the process (H^+, Θ^+). A good case in point in this respect is the analogue of the potential $U^-_x(\mathrm{d}y)$, $|y| < |x|$.

For convenience, note from Theorem 7 and (3.18) that establishing the law of $X_{\mathtt{G}(\infty)}$ is equivalent to obtaining an explicit identity for $U^-_x(\mathrm{d}y)$, $|y| < |x|$ and this we have already done. Specifically, for all $|x| > 0$,

$$U^-_x(\mathrm{d}y) = \pi^{-d/2}\frac{\Gamma(d/2)^2}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\frac{||y|^2 - |x|^2|^{\alpha/2}}{|y|^\alpha|y-x|^d}, \qquad |y| < |x|. \tag{3.35}$$

On the other hand, recalling that $\lim_{t\to\infty}|X_t| = \infty$, which implies that $\lim_{t\to\infty}\xi_t = \infty$ and hence $L_\infty = \infty$, we define

$$U^+_x(\mathrm{d}z) = \int_0^\infty \mathbf{P}_{\log|x|,\arg(x)}(\mathrm{e}^{H^+_t}\Theta^+_t \in \mathrm{d}z)\mathrm{d}t, \qquad |z| \geq |x|.$$

Then the Riesz–Bogdan–Żak transform ensures that, for Borel $A \subseteq \{z \in \mathbb{R}^d : |z| < |x|\}$,

$$\frac{|z|^{\alpha-d}}{|x|^{\alpha-d}}\mathbf{P}_{\log|x|,\arg(x)}(\mathrm{e}^{H^+_t}\Theta^+_t \in A) = \mathbf{P}_{-\log|x|,\arg(x)}(\mathrm{e}^{-H^-_t}\Theta^-_t \in KA, t < \ell_\infty)$$

where $KA = \{Kz : z \in A\}$. Hence, for $|x| > 0$,

$$U_x^+(\mathrm{d}z) = \pi^{-d/2}\frac{\Gamma(d/2)^2}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\frac{||z|^{-2} - |x|^{-2}|^{\alpha/2}}{|z|^{-\alpha}\,|(z/|z|^2) - (x/|x|^2)|^d}\frac{|x|^{\alpha-d}}{|z|^{\alpha-d}}\frac{\mathrm{d}z}{|z|^{2d}}$$
$$= \pi^{-d/2}\frac{\Gamma(d/2)^2}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\frac{||z|^2 - |x|^2|^{\alpha/2}}{|z|^\alpha|x-z|^d}, \qquad |z| > |x|. \tag{3.36}$$

where we have used the fact that $\mathrm{d}y = |z|^{-2d}\mathrm{d}z$, when $y = Kz$, and

$$|Kx - Kz| = \frac{|x-z|}{|x||z|}. \tag{3.37}$$

One notices that the identities for the potential measures $U_x^-(\mathrm{d}z)$ and $U_x^+(\mathrm{d}z)$ are identical albeit that the former is supported on $|z| < |x|$ and the latter on $|z| > |x|$. These identities and, more generally, the duality that emerges through the Riesz–Bogdan–Żak transformation will be of use to us in due course.

3.5 Integration with respect to the excursion measure

In order to proceed with some of the other fluctuation identities and the deep factorisation, we need to devote some time to compute in explicit detail the excursion occupation functionals

$$\mathbb{N}_\theta\left(\int_0^\zeta g(\mathrm{e}^\epsilon(s)\Theta^\epsilon(s))\mathrm{d}s\right), \qquad \theta \in \mathbb{S}_{d-1}, \tag{3.38}$$

and the excursion overshoot

$$\mathbb{N}_\theta\left(f(\mathrm{e}^{\epsilon(\zeta)}\Theta^\epsilon(\zeta));\, \zeta < \infty\right), \qquad \theta \in \mathbb{S}_{d-1}, \tag{3.39}$$

for judicious choices of f and g that ensure these quantities are finite.

The way we do this is to use Lemma 4 to scale out the quantity of interest from a fluctuation identity in which it is placed together with the potential U_x^-. Let us start with the excursion overshoot in (3.39).

Proposition 20. for $\theta \in \mathbb{S}_{d-1}$, we have

$$\mathbb{N}_\theta\left(\mathrm{e}^{\epsilon(\zeta)}\Theta^\epsilon(\zeta) \in \mathrm{d}y; \zeta < \infty\right)$$
$$= \frac{\alpha\pi^{-d/2}}{2}\frac{\Gamma((d-\alpha)/2)}{\Gamma(1-\alpha/2)}|1-|y|^2|^{-\alpha/2}|\theta - y|^{-d}\mathrm{d}y, \qquad |y| \le 1.$$

Proof. Take $|x| > r > r_0 > 0$ and suppose that $f : \mathbb{R}^d \mapsto [0,\infty)$ is continuous with support which is compactly embedded in the ball of radius r_0. We have, on the one hand, from (3.8), the identity

$$\begin{aligned}
&\mathbb{E}_x[f(X_{\tau_r^\oplus}); \tau_r^\oplus < \infty] \\
&\quad = \pi^{-(d/2+1)}\Gamma(d/2)\sin\left(\frac{\pi\alpha}{2}\right)\int_{|y|<r}\frac{|r^2-|x|^2|^{\alpha/2}}{|r^2-|y|^2|^{\alpha/2}}|x-y|^{-d}f(y)\mathrm{d}y.
\end{aligned}$$

On the other hand, from (3.20), we also have

$$\begin{aligned}
&\mathbb{E}_x[f(X_{\tau_r^\oplus}); \tau_r^\oplus < \infty] \\
&\quad = \int_{r<|z|<|x|} U_x^-(\mathrm{d}z)\int_{|y||z|<r}\mathbb{N}_{\arg(z)}(f(|z|e^{\epsilon(\zeta)}\Theta^\epsilon(\zeta)); \zeta<\infty).
\end{aligned} \tag{3.40}$$

Note that, for each $z \in \mathbb{R}^d\backslash\{0\}$,

$$z \mapsto \mathbb{N}_{\arg(z)}(f(|z|e^{\epsilon(\zeta)}\Theta^\epsilon(\zeta)); \zeta<\infty)$$

is bounded thanks to the fact that f is bounded and its support is compactly embedded in the unit ball of radius r_0. Indeed, there exists an $\varepsilon > 0$, which depends only on the support of f, such that

$$\begin{aligned}
&\sup_{r<|z|<|x|}\left|\mathbb{N}_{\arg(z)}(f(|z|e^{\epsilon(\zeta)}\Theta^\epsilon(\zeta)); \zeta<\infty)\right| \\
&\qquad \le ||f||_\infty \nu(-\log(r_0-\varepsilon),\infty) < \infty.
\end{aligned}$$

Moreover, since we can write

$$\mathbb{N}_{\arg(z)}(f(|z|e^{\epsilon(\zeta)}\Theta^\epsilon(\zeta)); \zeta<\infty) = \mathbb{N}_{\mathtt{1}}(f(|z|e^{\epsilon(\zeta)}\Theta^\epsilon(\zeta)\star\arg(z)); \zeta<\infty), \tag{3.41}$$

where, for any $a \in \mathbb{S}_{d-1}$, the operation $\star a$ rotates the sphere so that the 'North Pole', $\mathtt{1} = (1,0,\cdots,0) \in \mathbb{S}_{d-1}$ moves to a. Using a straightforward dominated convergence argument, we see that

$$\mathbb{N}_{\arg(z)}(f(|z|e^{\epsilon(\zeta)}\Theta^\epsilon(\zeta)); \zeta<\infty)$$

is continuous in z thanks to the continuity of f.

Appealing to Lemma 4, we thus have that

$$\begin{aligned}
&\mathbb{N}_{\arg(x)}(f(|x|e^{\epsilon(\zeta)}\Theta^\epsilon(\zeta)); \zeta<\infty) \\
&\quad = \lim_{r\uparrow|x|}\frac{\int_{r<|z|<|x|}U_x^-(\mathrm{d}z)\int_{|y||z|<r}\mathbb{N}_{\arg(z)}(f(|z|e^{\epsilon(\zeta)}\Theta^\epsilon(\zeta)); \zeta<\infty)}{\int_{r<|z|\le|x|}U_x^-(\mathrm{d}z)} \\
&\quad = \lim_{r\uparrow|x|}\frac{\mathbb{E}_x[f(X_{\tau_r^\oplus}); \tau_r^\oplus<\infty]}{\mathbb{P}_x(\tau_r^\oplus=\infty)}.
\end{aligned}$$

Substituting in the analytical form of the ratio on the right-hand side above using (3.40) and (3.9), we may continue with

$$
\begin{aligned}
&\mathbb{N}_{\arg(x)}(f(|x|e^{\epsilon(\zeta)}\Theta^\epsilon(\zeta));\zeta<\infty)\\
&=\lim_{r\uparrow|x|}\pi^{-d/2}\frac{\Gamma((d-\alpha)/2)}{\Gamma(1-\alpha/2)}\frac{(|x|^2-r^2)^{\alpha/2}\int_{|y|<r}|r^2-|y|^2|^{-\alpha/2}|x-y|^{-d}f(y)\mathrm{d}y}{\int_0^{(|x|^2-r^2)/r^2}(u+1)^{-d/2}u^{\alpha/2-1}\mathrm{d}u}\\
&=\pi^{-d/2}\frac{\Gamma((d-\alpha)/2)}{\Gamma(1-\alpha/2)}\int_{|y|<|x|}||x|^2-|y|^2|^{-\alpha/2}|x-y|^{-d}f(y)\mathrm{d}y\\
&\qquad\qquad\times\lim_{r\uparrow|x|}\frac{r^\alpha[(|x|^2-r^2)/r^2]^{\alpha/2}}{\int_0^{(|x|^2-r^2)/r^2}(u+1)^{-d/2}u^{\alpha/2-1}\mathrm{d}u}\\
&=\frac{\alpha\pi^{-d/2}}{2}\frac{\Gamma((d-\alpha)/2)}{\Gamma(1-\alpha/2)}\int_{|y|<|x|}|x|^\alpha||x|^2-|y|^2|^{-\alpha/2}|x-y|^{-d}f(y)\mathrm{d}y,
\end{aligned}
\tag{3.42}
$$

where we have used that the support of f is compactly embedded in the ball of radius $|x|$ to justify the first term in the second equality. $\square$

Next we turn our attention to the quantity (3.38). Once again, our approach will be to scale an appropriate fluctuation identity by

$$
\mathbb{P}_x(\tau_r^\oplus=\infty)=\int_{r<|z|\le|x|}U_x^-(\mathrm{d}z).
$$

In this case, the natural object to work with is the expected occupation measure until first entry into the ball of radius $r<|x|$, where x is the point of issue of the stable process. That is, the quantity

$$
\mathbb{E}_x\left[\int_0^{\tau_r^\oplus}f(X_s)\mathrm{d}s\right]\tag{3.43}
$$

for $|x|>r>0$ and continuous $f:\mathbb{R}^d\mapsto[0,\infty)$ with compact support.

Although an identity for the aforesaid resolvent is not readily available in the literature, it is not difficult to derive it from (3.10), with the help of the Riesz–Bogdan–Żak transform.

Recall that this transform states that, for $x\neq0$, $(KX_{\eta(t)},t\ge0)$ under $\mathbb{P}_{Kx}$ is equal in law to $(X_t,t\ge0)$ under $\mathbb{P}_x^\circ$, where

$$
\eta(t)=\inf\{s>0:\int_0^s|X_u|^{-2\alpha}\mathrm{d}u>t\}.
$$

For convenience, set $r=1$. Noting that, since $\int_0^{\eta(t)}|X_u|^{-2\alpha}\mathrm{d}u=t$, if we write $s=\eta(t)$, then

$$
|X_s|^{-2\alpha}\mathrm{d}s=\mathrm{d}t,\qquad t>0,
$$

and hence we have that, for $|x| > 1$,

$$\int_{|z|>1} \frac{|z|^{\alpha-d}}{|x|^{\alpha-d}} h_r^{\oplus}(x,z) f(z) \mathrm{d}z = \mathbb{E}_x^{\circ}\left[\int_0^{\tau_1^{\oplus}} f(X_t)\mathrm{d}t\right] = \mathbb{E}_{Kx}\left[\int_0^{\tau_1^{\ominus}} f(KX_{\eta(t)})\mathrm{d}t\right]$$
$$= \mathbb{E}_{Kx}\left[\int_0^{\tau_1^{\ominus}} f(KX_s)|X_s|^{-2\alpha}\mathrm{d}s\right]$$
$$= \int_{|y|<1} h_1^{\ominus}(Kx,y) f(Ky)|y|^{-2\alpha}\mathrm{d}y$$

where we have pre-emptively assumed that the resolvent associated to (3.43) has a density, which we have denoted by $h_1^{\oplus}(x,y)$. In the integral on the left-hand side above, we can make the change of variables $y = Kz$, which is equivalent to $z = Ky$. Noting that $\mathrm{d}y/\mathrm{d}z = 1/|z|^{2d}$ and appealing to (3.10), we get

$$\int_{|y|>1} \frac{|z|^{\alpha-d}}{|x|^{\alpha-d}} h_1^{\oplus}(x,z) f(z)\mathrm{d}z = \int_{|z|>1} h_1^{\ominus}(Kx,Kz) f(z)\frac{|z|^{2\alpha}}{|z|^{2d}}\mathrm{d}z,$$

from which we can conclude that, for $|x|, |z| > 1$,

$$h_1^{\oplus}(x,z) = \frac{|x|^{\alpha-d}}{|z|^{\alpha-d}} h_1^{\ominus}(Kx,Kz)\frac{|z|^{2\alpha}}{|z|^{2d}}$$
$$= 2^{-\alpha}\pi^{-d/2}\frac{\Gamma(d/2)}{\Gamma(\alpha/2)^2}\frac{|x|^{\alpha-d}}{|z|^{\alpha-d}}\frac{|z|^{2\alpha}}{|z|^{2d}}|Kx-Kz|^{\alpha-d}\int_0^{\zeta_1^{\ominus}(Kx,Kz)}(u+1)^{-d/2}u^{\alpha/2-1}\mathrm{d}u.$$

Hence, after a little algebra, for $|x|, |z| > 1$,

$$h_1^{\oplus}(x,z) = 2^{-\alpha}\pi^{-d/2}\frac{\Gamma(d/2)}{\Gamma(\alpha/2)^2}|x-z|^{\alpha-d}\int_0^{\zeta^{\oplus}(x,z)}(u+1)^{-d/2}u^{\alpha/2-1}\mathrm{d}u$$

where we have again used the fact that $|Kx - Kz| = |x-z|/|x||z|$ so that

$$\zeta_1^{\ominus}(Kx,Kz) = (|x|^2-1)(|z|^2-1)/|x-z|^2 =: \zeta_1^{\oplus}(x,z).$$

After scaling this gives us a general formula for (3.43), which we record below as a lemma on account of the fact that it does not already appear elsewhere in the literature (albeit being implicitly derivable as we have done from [3]).

Lemma 6. For $|x| > r$, the resolvent (3.43) has a density given by

$$h_r^{\oplus}(x,z) = 2^{-\alpha}\pi^{-d/2}\frac{\Gamma(d/2)}{\Gamma(\alpha/2)^2}|x-z|^{\alpha-d}\int_0^{\zeta_r^{\oplus}(x,z)}(u+1)^{-d/2}u^{\alpha/2-1}\mathrm{d}u, \tag{3.44}$$

where $\zeta_r^{\oplus}(x,z) := (|x|^2-r^2)(|z|^2-r^2)/r^2|x-z|^2$.

We can now use the above lemma to compute occupation potential with respect to the excursion measure, defined in (3.38). As for other results in this development, the following result is reminiscent of a classical result in fluctuation theory of Lévy processes, see e.g. exercise 5 in Chapter VI in [2], but as it includes the information about the modulator there is no direct way to derive it from the classical result.

Proposition 21. For $x \in \mathbb{R}^d \backslash \{0\}$, and continuous $g : \mathbb{R}^d \mapsto \mathbb{R}^+$ whose support is compactly embedded in the exterior of the ball of radius $|x|$,

$$\mathbb{N}_{\arg(x)}\left(\int_0^\zeta g(|x|e^{\epsilon(u)}\Theta^\epsilon(u))\mathrm{d}u\right) = 2^{-\alpha}\frac{\Gamma((d-\alpha)/2)^2}{\Gamma(d/2)^2}\int_{|x|<|z|} g(z)U_x^+(\mathrm{d}z)$$

Proof. Fix $0 < r < |x|$. Recall from the Lamperti–Kiu representation (3.12) that

$$X_t = \exp\{\xi_{\varphi(t)}\}\Theta(\varphi(t)),\ t \geq 0,$$

where

$$\int_0^{\varphi(t)} \exp\{\alpha\xi_u\}\mathrm{d}u = t.$$

In particular, this implies that, if we write $s = \varphi(t)$, then

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \mathrm{e}^{-\alpha\xi_s}, \qquad t > 0, \tag{3.45}$$

Splitting the occupation over individual excursions, we have with the help of (3.16) that

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{\tau_r^\oplus} g(X_t)\mathrm{d}t\right] \\
&= \mathbb{E}_x\left[\int_0^\infty \mathbf{1}(\mathrm{e}^{\xi_s} > r)g(\mathrm{e}^{\xi_s}\Theta_s)\mathrm{e}^{\alpha\xi_s}\mathrm{d}s\right] \\
&= \int_{r<|z|<|x|} U_x^-(\mathrm{d}z)\mathbb{N}_{\arg(z)}\left(\int_0^\zeta g(|z|\mathrm{e}^{\epsilon(s)}\Theta^\epsilon(s))(|z|\mathrm{e}^{\epsilon(s)})^\alpha\mathrm{d}s\right).
\end{aligned} \tag{3.46}$$

Note that the left-hand side is necessarily finite as it can be upper bounded by $\mathbb{E}_x\left[\int_0^\infty g(X_t)\mathrm{d}t\right]$, which is known to be finite for the given assumptions on g.

Straightforward arguments, similar to those presented around (3.41), tell us that for continuous g with compact support that is compactly embedded in the exterior of the ball of radius $|x|$, we have that, for $r < |z| < |x|$,

$$\mathbb{N}_{\arg(z)}\left(\int_0^\zeta g(|z|\mathrm{e}^{\epsilon(s)}\Theta^\epsilon(s))\mathrm{e}^{\alpha\epsilon(s)}\mathrm{d}s\right) = \int_0^\infty \mathbb{N}_{\arg(z)}\left(g(|z|\mathrm{e}^{\epsilon(s)}\Theta^\epsilon(s))\mathrm{e}^{\alpha\epsilon(s)};\ s<\zeta\right)\mathrm{d}s$$

is a continuous function.

Accordingly we can again use Lemma 4 and Theorem 8 and write, for $x \in \mathbb{R}^d$,

$$
\begin{aligned}
&\mathbb{N}_{\arg(x)}\left(\int_0^\zeta g(|x|\mathrm{e}^{\epsilon(s)}\Theta^\epsilon(s))(|x|\mathrm{e}^{\epsilon(s)})^\alpha\right)\\
&= \lim_{r\uparrow|x|} \frac{\int_{r<|z|<|x|} U_x^-(\mathrm{d}z)\mathbb{N}_{\arg(z)}\left(\int_0^\zeta g(|z|\mathrm{e}^{\epsilon(s)}\Theta^\epsilon(s))(|z|\mathrm{e}^{\epsilon(s)})^\alpha\right)}{\int_{r<|z|\le|x|} U_x^-(\mathrm{d}z)}\\
&= \frac{\mathbb{E}_x\left[\int_0^{\tau_r^\oplus} g(X_s)\mathrm{d}s\right]}{\mathbb{P}_x(\tau_r^\oplus = \infty)}\\
&= 2^{-\alpha}\pi^{-d/2}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)} \lim_{r\uparrow|x|} \frac{\int_{|x|<|z|} \mathrm{d}z\, \mathbf{1}(r<|z|)g(z)|x-z|^{\alpha-d}\int_0^{\zeta_r^\oplus(x,z)}(u+1)^{-d/2}u^{\alpha/2-1}\mathrm{d}u}{\int_0^{(|x|^2-r^2)/r^2}(u+1)^{-d/2}u^{\alpha/2-1}\mathrm{d}u}\\
&= 2^{-\alpha}\pi^{-d/2}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)} \int_{|x|<|z|} \mathrm{d}z\, g(z)|x-z|^{-d}(|z|^2-|x|^2)^{\alpha/2},
\end{aligned}
$$

where in the final equality we have used dominated convergence (in particular the assumption on the support of g). By inspection, we also note that the right-hand side above is equal to

$$
2^{-\alpha}\frac{\Gamma((d-\alpha)/2)^2}{\Gamma(d/2)^2}\int_{|x|<|z|} g(z)|z|^\alpha U_x^+(\mathrm{d}z).
$$

The proof is completed by replacing $g(x)$ by $g(x)|x|^{-\alpha}$. □

3.6 On n-tuple laws

We are now ready to prove Theorems 9, Corollary 1 and Corollary 2 with the help of Section 3.5 and other identities. In essence, we can piece together the desired results using Maisonneuve's exit formula (3.16) applied in the appropriate way, together with some of the identities established in previous sections.

Proof of Theorem 9. (i) Appealing to the fact that the stable process $|X|$ does not creep downward and the Lévy system compensation formula for the jumps of X, from Section 0.5 of [2], we have, on the one hand,

$$
\mathbb{E}_x[f(X_{\mathtt{G}(\tau_r^\oplus)})g(X_{\tau_r^\oplus -})h(X_{\tau_r^\oplus});\tau_r^\oplus<\infty] = \mathbb{E}_x\left[\int_0^{\tau_r^\oplus} f(X_{\mathtt{G}(t)})g(X_t)k(X_t)\mathrm{d}t\right], \tag{3.47}
$$

where continuous positive-valued functions f, g, h are such that the first two are compactly supported in $\{z\in\mathbb{R}^d : |z|>r\}$ and the third is compactly supported in the open ball of radius r and

$$
k(y) = \int_{|y+w|<r} \Pi(\mathrm{d}w)h(y+w).
$$

On the other hand, a calculation similar in spirit to (3.46), using (3.16), followed by an application of Proposition 21, tells us that

$$
\begin{aligned}
&\mathbb{E}_x\left[\int_0^{\tau_r^\oplus} f(X_{\mathtt{G}(t)})g(X_t)k(X_t)\right]\\
&\qquad= \int_{r<|z|<|x|} U_x^-(\mathrm{d}z)f(z)\mathbb{N}_{\arg(z)}\left(\int_0^\zeta g(|z|\mathrm{e}^{\epsilon(s)}\Theta^\epsilon(s))k(|z|\mathrm{e}^{\epsilon(s)}\Theta^\epsilon(s))(|z|\mathrm{e}^{\epsilon(s)})^\alpha \mathrm{d}s\right)\\
&\qquad= 2^{-\alpha}\frac{\Gamma((d-\alpha)/2)^2}{\Gamma(d/2)^2}\int_{r<|z|<|x|} U_x^-(\mathrm{d}z)f(z)\int_{|z|<|y|} U_z^+(\mathrm{d}y)g(y)k(y)|y|^\alpha.
\end{aligned}
$$

Putting the pieces together, we get

$$
\begin{aligned}
&\mathbb{E}_x[f(X_{\mathtt{G}(\tau_r^\oplus)})g(X_{\tau_r^\oplus-})h(X_{\tau_r^\oplus});\tau_r^\oplus<\infty]\\
&= 2^{-\alpha}\frac{\Gamma((d-\alpha)/2)^2}{\Gamma(d/2)^2}\int_{r<|z|<|x|}\int_{|z|<|y|}\int_{|w-y|<r} U_x^-(\mathrm{d}z)U_z^+(\mathrm{d}y)\Pi(\mathrm{d}w)f(z)g(y)|y|^\alpha h(y+w)\\
&= c_{\alpha,d}\int_{r<|z|<|x|}\int_{|z|<|y|}\int_{|w-y|<r}\frac{||z|^2-|x|^2|^{\alpha/2}||y|^2-|z|^2|^{\alpha/2}}{|z|^\alpha|z-x|^d|z-y|^d|w|^{\alpha+d}}\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w f(z)g(y)h(y+w)\\
&= c_{\alpha,d}\int_{r<|z|<|x|}\int_{|z|<|y|}\int_{|v|<r}\frac{||z|^2-|x|^2|^{\alpha/2}||y|^2-|z|^2|^{\alpha/2}}{|z|^\alpha|z-x|^d|z-y|^d|v-y|^{\alpha+d}}\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}v f(z)g(y)h(v)
\end{aligned}
$$

where

$$
c_{\alpha,d} = \frac{\Gamma((d+\alpha)/2)}{|\Gamma(-\alpha/2)|}\frac{\Gamma(d/2)^2}{\pi^{3d/2}\Gamma(\alpha/2)^2}.
$$

This is equivalent to the statement of part (i) of the theorem.

(ii) This is a straightforward application of the Riesz–Bogdan–Żak transformation, with computations in the style of those used to prove Lemma 6. For the sake of brevity, the proof is left as an exercise for the reader. □

Proof of Corollary 1. As above, we only prove (i) as part (ii) can be derived appealing to the Riesz–Bogdan–Żak transformation.

From (3.20), (3.35) and Proposition 20, more specifically (3.42), we have that for bounded measurable functions f, g on $\mathbb{R}^d$,

$$
\begin{aligned}
&\mathbb{E}_x[g(X_{\mathtt{G}(\tau_1^\oplus)})f(X_{\tau_1^\oplus});\tau_1^\oplus<\infty]\\
&= \int_{1<|z|<|x|} U_x^-(\mathrm{d}z)\mathbb{N}_{\arg(z)}(f(|z|e^{\epsilon(\zeta)}\Theta^\epsilon(\zeta))\mathbf{1}(|z|e^{\epsilon(\zeta)}<1);\zeta<\infty)g(z)\\
&= \frac{\Gamma(d/2)^2\sin(\pi\alpha/2)}{\pi^d|\Gamma(-\alpha/2)|\Gamma(\alpha/2)}\int_{1<|z|<|x|}\int_{|v|<1}\frac{||z|^2-|x|^2|^{\alpha/2}}{||z|^2-|v|^2|^{\alpha/2}|z-v|^d|z-x|^d}f(v)g(z)\,\mathrm{d}z\,\mathrm{d}v.
\end{aligned}
$$

This gives the desired result when $r=1$. As usual, we use scaling to convert the above conclusion to the setting of first passage into a ball of radius $r>0$. □

Proof of Corollary 2. As with the previous proof, we only deal with (i) and the case that $r = 1$ for the same reasons. Setting $f \equiv 1$ in (3.47), we see with the help of Lemma 6 and (4.1) that

$$\begin{aligned}
&\mathbb{E}_x[g(X_{\tau_1^\oplus})h(X_{\tau_1^\oplus}); \tau_1^\oplus < \infty] \\
&= \mathbb{E}_x\left[\int_0^{\tau_1^\oplus} g(X_t)k(X_t)\right] \\
&= \frac{2^\alpha\Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|}\int_{|y|>1} g(y)\int_{|y+w|<1}\frac{1}{|w|^{\alpha+d}}\mathrm{d}w\, h(y+w)h_1^\oplus(x,y)\mathrm{d}y \\
&= \frac{2^\alpha\Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|}\int_{|y|>1}\int_{|v|<1} g(y)h(v)\frac{1}{|v-y|^{\alpha+d}}h_1^\oplus(x,y)\,\mathrm{d}v\,\mathrm{d}y
\end{aligned}$$

where the function $k(\cdot)$ is as before. The result now follows. □

3.7 Deep factorisation of the stable process

The manipulations we have made in Section 3.5, in particular in Proposition 21, are precisely what we need to demonstrate the Wiener–Hopf factorisation.

Recall that, for Theorem 10, we defined

$$\mathbf{R}_z[f](\theta) = \mathbf{E}_{0,\theta}\left[\int_0^\infty \mathrm{e}^{-z\xi_t}f(\Theta_t)\mathrm{d}t\right], \qquad \theta \in \mathbb{S}_{d-1}, z \in \mathbb{C}.$$

Moreover, define

$$\hat{\boldsymbol{\rho}}_z[f](\theta) = \mathbf{E}_{0,\theta}\left[\int_0^\infty \mathrm{e}^{-zH_t^-}f(\Theta_t^-)\mathrm{d}t\right] = \int_{|y|<1}|y|^z f(\arg(y))U_\theta^-(\mathrm{d}y)$$

and

$$\boldsymbol{\rho}_z[f](\theta) = \mathbf{E}_{0,\theta}\left[\int_0^\infty \mathrm{e}^{-zH_t^+}f(\Theta_t^+)\mathrm{d}t\right] = \int_{|y|>1}|y|^{-z} f(\arg(y))U_\theta^+(\mathrm{d}y)$$

for bounded measurable $f : \mathbb{S}_{d-1} \mapsto [0,\infty)$, whenever the integrals make sense.

We note that the expression for $\boldsymbol{\rho}_z[f](\theta)$ as given in the statement of Theorem 10 is clear given (3.36).

Moreover, from e.g. Section 2 of [3], it is known that the free potential measure of a stable process issued from $x \in \mathbb{R}^d$ has density given by

$$u(x,y) = \frac{\Gamma((d-\alpha)/2)}{2^\alpha\pi^{d/2}\Gamma(\alpha/2)}|y-x|^{\alpha-d}, \qquad y \in \mathbb{R}^d.$$

Accordingly, taking account of (3.45), it is straightforward to compute

$$\begin{aligned}\mathbf{R}_z[f](\theta) &= \mathbf{E}_{0,\theta}\left[\int_0^\infty \mathrm{e}^{-(z+\alpha)\xi_s} f(\Theta_s)\mathrm{e}^{\alpha\xi_s}\mathrm{d}s\right]\\ &= \mathbb{E}_\theta\left[\int_0^\infty |X_t|^{-(\alpha+z)} f(\arg(X_t))\mathrm{d}t\right]\\ &= \int_{\mathbb{R}^d} f(\arg(y))\frac{u(\theta,y)}{|y-\theta|^{\alpha+z}}\,\mathrm{d}y, \qquad \mathrm{Re}(z)\geq 0,\end{aligned}$$

where we have used stationary and independent increments in the final equality. Note also that this agrees with the expression for $\mathbf{R}_z[f](\theta)$ in the statement of Theorem 10.

Proof of Theorem 10. From the second and third equalities of equation (3.46) (taking $r \to 0$) and Proposition 21 gives us

$$\begin{aligned}\mathbf{R}_{-\mathrm{i}\lambda}[f](\theta) &= \int_{|w|<1} U^-_\theta(\mathrm{d}w)\mathbb{N}_{\arg(w)}\left(\int_0^\zeta (|w|\mathrm{e}^{\epsilon(s)})^{\mathrm{i}\lambda} f(\Theta^\epsilon(s))\right)\\ &= 2^{-\alpha}\frac{\Gamma((d-\alpha)/2)^2}{\Gamma(d/2)^2}\int_{|w|<1} U^-_\theta(\mathrm{d}w)\int_{|w|<|y|} f(\arg(y))|y|^{\mathrm{i}\lambda}U^+_w(\mathrm{d}y).\end{aligned} \tag{3.48}$$

Note that, by conditional stationary and independent increments, for any $w \in \mathbb{R}^d\backslash\{0\}$,

$$\begin{aligned}\int_{|w|<|y|} |y|^{\mathrm{i}\lambda} f(\arg(y))U^+_w(\mathrm{d}y) &= \mathbf{E}_{\log|w|,\arg(w)}\left[\int_0^\infty \mathrm{e}^{\mathrm{i}\lambda H^+_t} f(\Theta^+_t)\mathrm{d}t\right]\\ &= |w|^{\mathrm{i}\lambda}\mathbf{E}_{0,\arg(w)}\left[\int_0^\infty \mathrm{e}^{\mathrm{i}\lambda H^+_t} f(\Theta^+_t)\mathrm{d}t\right]\\ &= |w|^{\mathrm{i}\lambda}\int_{1<|y|} |y|^{\mathrm{i}\lambda} f(\arg(y))U^+_{\arg(w)}(\mathrm{d}y).\end{aligned}$$

Hence back in (3.48), we have

$$\mathbf{R}_{-\mathrm{i}\lambda}[f](\theta) = 2^{-\alpha}\frac{\Gamma((d-\alpha)/2)^2}{\Gamma(d/2)^2}\hat{\boldsymbol{\rho}}_{\mathrm{i}\lambda}[\boldsymbol{\rho}_{-\mathrm{i}\lambda}[f]](\theta), \qquad \lambda\in\mathbb{R}.$$

Finally we note from (3.36) that, making the change of variables $y = Kw$, so that $\arg(y) = \arg(w)$, $|y| = 1/|w|$ and $\mathrm{d}y = |w|^{-2d}\mathrm{d}w$ and, for $\theta \in \mathbb{S}_{d-1}$, $|\theta - Kw| = |\theta - w|/|w|$, we have

$$\begin{aligned}\hat{\boldsymbol{\rho}}_{\mathrm{i}\lambda}[f](\theta) &= \pi^{-d/2}\frac{\Gamma(d/2)^2}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\int_{|y|>1} |y|^{\mathrm{i}\lambda} f(\arg(y))\frac{||y|^2-1|^{\alpha/2}}{|y|^\alpha|\theta-y|^d}\mathrm{d}y\\ &= \pi^{-d/2}\frac{\Gamma(d/2)^2}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\int_{1<|w|} |w|^{-\mathrm{i}\lambda+\alpha-d} f(\arg(w))\frac{|1-|w|^2|^{\alpha/2}}{|\theta-w|^d|w|^\alpha}\mathrm{d}w\\ &= \boldsymbol{\rho}_{\mathrm{i}\lambda+(\alpha-d)}[f](\theta), \qquad \lambda\in\mathbb{R},\end{aligned}$$

as required. □

3.8 Proof of Theorem 11

Recall from the description of the Riesz–Bogdan–Żak transform that (ξ, Θ) under the change of measure in (3.34) is equal in law to $(-\xi, \Theta)$. Accordingly, we have for $q > 0$, $x \in \mathbb{R}^d \backslash \{0\}$ and bounded measurable g whose support is compactly embedded in the ball of unit radius,

$$
\begin{aligned}
&\mathbf{E}_{-\log|x|,\arg(x)}[g(\mathrm{e}^{-(\bar{\xi}_{\mathbf{e}_q}-\xi_{\mathbf{e}_q})}\Theta_{\mathbf{e}_q})] \\
&= \mathbf{E}_{\log|x|,\arg(x)}\left[\frac{\mathrm{e}^{(\alpha-d)\xi_{\mathbf{e}_q}}}{|x|^{\alpha-d}} g(\mathrm{e}^{-(\xi_{\mathbf{e}_q}-\underline{\xi}_{\mathbf{e}_q})}\Theta_{\mathbf{e}_q})\right] \\
&= |x|^{d-\alpha}\mathbf{E}_{\log|x|,\arg(x)}\left[\sum_{\mathtt{g}\in G}\mathbf{1}(\zeta_{\mathtt{g}'} < \mathbf{e}_q^{\mathtt{g}'},\ \forall G \ni \mathtt{g}' < \mathtt{g})\mathrm{e}^{(\alpha-d)\xi_{\mathtt{g}}}\mathrm{e}^{(\alpha-d)\epsilon_{\mathtt{g}}(\mathbf{e}_q^{\mathtt{g}})} g(\mathrm{e}^{-\epsilon_{\mathtt{g}}(\mathbf{e}_q^{\mathtt{g}})}\Theta^{\epsilon}_{\mathtt{g}}(\mathbf{e}_q^{\mathtt{g}}))\mathbf{1}(\mathbf{e}_q^{\mathtt{g}} < \zeta_{\mathtt{g}})\right] \\
&= |x|^{d-\alpha}\mathbf{E}_{\log|x|,\arg(x)}\left[\int_0^\infty \mathrm{e}^{-qt}\mathrm{e}^{(\alpha-d)\xi_t}\mathbb{N}_{\Theta_t}\left(\mathrm{e}^{(\alpha-d)\epsilon(\mathbf{e}_q)}g(\mathrm{e}^{-\epsilon(\mathbf{e}_q)}\Theta(\mathbf{e}_q)); \mathbf{e}_q < \zeta\right)\mathrm{d}L_t\right] \\
&= |x|^{d-\alpha}\mathbf{E}_{\log|x|,\arg(x)}\left[\int_0^\infty \mathrm{e}^{-q\ell_s^{-1}}\mathrm{e}^{-(\alpha-d)H_s^-}\mathbb{N}_{\Theta_s^-}\left(\mathrm{e}^{(\alpha-d)\epsilon(\mathbf{e}_q)}g(\mathrm{e}^{-\epsilon(\mathbf{e}_q)}\Theta(\mathbf{e}_q)); \mathbf{e}_q < \zeta\right)\mathrm{d}s\right],
\end{aligned}
$$

where, for each $\mathtt{g} \in G$, $\mathbf{e}_q^{\mathtt{g}}$ are additional marks on the associated excursion which are independent and exponentially distributed with rate q. Hence, if we define

$$
U_x^{(q),-}(\mathrm{d}y) = \int_0^\infty \mathrm{d}s\, \mathbf{E}_{\log|x|,\arg(x)}\left[\mathrm{e}^{-q\ell_t^{-1}};\ \mathrm{e}^{-H_s^-}\Theta_s^- \in \mathrm{d}y,\ s < \ell_\infty\right], \qquad |y| < |x|.
$$

then

$$
\begin{aligned}
&\mathbf{E}_{-\log|x|,\arg(x)}[g(\mathrm{e}^{-(\bar{\xi}_{\mathbf{e}_q}-\xi_{\mathbf{e}_q})}\Theta_{\mathbf{e}_q})] \\
&= \int_{(0,\infty)}\int_{|y|<|x|} qU_x^{(q),-}(\mathrm{d}y)\frac{|y|^{\alpha-d}}{|x|^{\alpha-d}}\mathbb{N}_{\arg(y)}\left(\int_0^\zeta \mathrm{e}^{-qt}\mathrm{e}^{(\alpha-d)\epsilon(t)}g(\mathrm{e}^{-\epsilon(t)}\Theta(t))\mathrm{d}t\right)
\end{aligned}
\tag{3.49}
$$

Recall that ℓ^{-1} is a subordinator (without reference to the accompanying modulation Θ^+). Suppose we denote its Laplace exponent by

$$
\Lambda^+(q) := -\log \mathbf{E}_{0,\theta}\left[\exp\{-qL_1^{-1}\}\right],\ q \geq 0,
$$

$\theta \in \mathbb{S}_{d-1}$ is unimportant in the definition. Appealing again to the Riesz–Bogdan–Żak transform, we also note that for a bounded and measurable function h on $\mathbb{S}_{d-1}$, using obvious notation

$$
\begin{aligned}
\int_{|y|<|x|}\frac{|y|^{\alpha-d}}{|x|^{\alpha-d}}qU_x^{(q),-}(\mathrm{d}y)h(\arg(y)) &= q\int_0^\infty \mathrm{d}s\,\mathbf{E}_{-\log|x|,\arg(x)}\left[\mathrm{e}^{-qL_t^{-1}}h(\Theta_s^+)\right] \\
&= \frac{q}{\Lambda^+(q)}\int_0^\infty \mathrm{d}s\,\Lambda^+(q)\mathrm{e}^{-\Lambda^+(q)s}\mathbf{E}^{(q)}_{-\log|x|,\arg(x)}\left[h(\Theta_s^+)\right] \\
&= \frac{q}{\Lambda^+(q)}\mathbf{E}^{(q)}_{-\log|x|,\arg(x)}\left[h\left(\Theta^+_{\mathbf{e}_{\Lambda^+(q)}}\right)\right]
\end{aligned}
\tag{3.50}
$$

where $\mathbf{P}^{(q)}_{-\log|x|,\arg(x)}$ appears as the result of a change of measure with martingale density $\exp\{-qL_s^{-1}+\Lambda^+(q)s\}$, $s\geq 0$, $\Lambda^+(q)$ is the Laplace exponent of the subordinator L^{-1} and $\mathbf{e}_{\Lambda^+(q)}$ is an independent exponential random variable with parameter $\Lambda^+(q)$.

Next, we want to take $q\downarrow 0$ in (3.49). To this end, we start by remarking that, as L is a local time for the Lévy process ξ (without reference to its modulation), it is known from classical Wiener–Hopf factorisation theory that, up to a multiplicative constant, $c>0$, which depends on the normalisation of the local time L, $q=c\Lambda^+(q)\Lambda^-(q)$, where $\Lambda^-(q)$ is the Laplace exponent of the local time at the infimum ℓ; see for example equation (3) in Chapter VI of [2].

On account of the fact that X is transient, we know that ℓ_∞ is exponentially distributed and the reader may recall that we earlier normalised our choice of ℓ such that its rate, $\Lambda^-(0)=1$. This implies, in turn, that $\lim_{q\downarrow 0} q/\Lambda^+(q)=c$.

Appealing to isotropy, the recurrence of $\{0\}\times\mathbb{S}_{d-1}$ for $(\bar{\xi}-\xi,\Theta)$ and weak convergence back in (3.50) as we take the limit with $q\downarrow 0$, to find that

$$\lim_{q\to 0}\int_{|y|<|x|}\frac{|y|^{\alpha-d}}{|x|^{\alpha-d}}qU_x^{(q),-}(\mathrm{d}y)h(\arg(y))=c\int_{\mathbb{S}_{d-1}}\sigma_1(\mathrm{d}\phi)h(\phi),$$

where we recall that $\sigma_1(\mathrm{d}\phi)$ is the surface measure on $\mathbb{S}_{d-1}$ normalised to have unit mass. Hence, back in (3.49) we have with the help of Proposition 21 and (3.36),

$$\begin{aligned}&\lim_{q\downarrow 0}\mathbf{E}_{-\log|x|,\arg(x)}[g(\mathrm{e}^{-(\bar{\xi}_{\mathbf{e}_q}-\xi_{\mathbf{e}_q})}\Theta_{\mathbf{e}_q})]\\&=\int_{\mathbb{S}_{d-1}}\sigma_1(\mathrm{d}\phi)\mathbb{N}_\phi\left(\int_0^\zeta \mathrm{e}^{(\alpha-d)\epsilon(t)}g(\mathrm{e}^{-\epsilon(t)}\Theta(t))\mathrm{d}t\right)\\&=c\pi^{-d/2}2^{-\alpha}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}\int_{\mathbb{S}_{d-1}}\sigma_1(\mathrm{d}\phi)\int_{1<|z|}g(Kz)\frac{||z|^2-1|^{\alpha/2}}{|z|^d|\phi-z|^d}\mathrm{d}z,\end{aligned}\tag{3.51}$$

where we recall that $Kz=z/|z|^2$.

Finally, let $f:\mathbb{B}_d\to\mathbb{R}^+$ be a bounded measurable function, where $\mathbb{B}_d$ was previously defined as $\{x\in\mathbb{R}^d:|x|\leq 1\}$. We use the Lamperti–Kiu transform and (3.45) to note that

$$f\left(X_t/M_t\right)\mathrm{d}t=f\left(\mathrm{e}^{-(\bar{\xi}_s-\xi_s)}\Theta_s\right)\mathrm{e}^{\alpha\xi_s}\mathrm{d}s$$

where $s=\varphi(t)$, suggesting that, for $y\in\mathbb{R}^d\backslash\{0\}$,

$$\lim_{t\to\infty}\mathbb{E}_y[f\left(X_t/M_t\right)]=\lim_{s\to\infty}\mathbf{E}_{\log|y|,\arg(y)}\left[f\left(\mathrm{e}^{-(\bar{\xi}_s-\xi_s)}\Theta_s\right)\mathrm{e}^{\alpha\xi_s}\right].$$

In fact, we can justify this rigorously appealing to the discussion at the bottom of p240 of [21].

Hence, putting this together with (3.51), for f and x as before, we conclude that,

$$
\begin{aligned}
\lim_{t\to\infty} \mathbb{E}_{Kx}[f\,(X_t/M_t)] &= \lim_{q\downarrow 0} \mathbf{E}_{-\log|x|,\arg(x)}[f(\mathrm{e}^{-(\bar{\xi}_{\mathbf{e}_q}-\xi_{\mathbf{e}_q})}\Theta_{\mathbf{e}_q})\mathrm{e}^{\alpha\xi_{\mathbf{e}_q}}] \\
&= c\pi^{-d/2}2^{-\alpha}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}\int_{\mathbb{S}_{d-1}}\sigma_1(\mathrm{d}\phi)\int_{1<|z|} f(Kz)\frac{|Kz|^{\alpha}||z|^2-1|^{\alpha/2}}{|z|^d|\phi-z|^d}\mathrm{d}z \\
&= c\pi^{-d/2}2^{-\alpha}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}\int_{\mathbb{S}_{d-1}}\sigma_1(\mathrm{d}\phi)\int_{|w|<1} f(w)\frac{|1-|w|^2|^{\alpha/2}}{|\phi-w|^d}\mathrm{d}w \qquad (3.52)
\end{aligned}
$$

where we changed variables $w = Kz$, or equivalently $z = Kw$, and we used (3.37), that $|w| = 1/|z|$ and that $\mathrm{d}z/\mathrm{d}w = 1/|w|^{2d}$.

In order to pin down the constant c, we need to ensure that, when $f \equiv 1$, the integral on the right-hand side of (3.52) is identically equal to 1. To do this, we recall a classical Poisson potential formula (see for example Theorem 4.3.1 in [18])

$$
(1-|w|^2)^{-1} = \int_{\mathbb{S}_{d-1}} |\phi-w|^{-d}\sigma_1(\mathrm{d}\phi) \qquad |w|<1. \qquad (3.53)
$$

Writing $\sigma_r(\mathrm{d}\theta)$, $\theta\in r\mathbb{S}_{d-1}$ for the uniform surface measure on $r\mathbb{S}_{d-1}$ normalised to have total mass equal to one, it follows that

$$
\begin{aligned}
\int_{\mathbb{S}_{d-1}}\sigma_1(\mathrm{d}\phi)\int_{|w|<1}\frac{|1-|w|^2|^{\alpha/2}}{|\phi-w|^d}\mathrm{d}w &= \int_{|w|<1}|1-|w|^2|^{\frac{\alpha}{2}-1}\mathrm{d}w \\
&= \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^1 r^{d-1}\mathrm{d}r\int_{r\mathbb{S}_{d-1}}\sigma_r(\mathrm{d}\theta)(1-r^2)^{\frac{\alpha}{2}-1} \\
&= \frac{\pi^{d/2}}{\Gamma(d/2)}\int_0^1 y^{\frac{d}{2}-1}(1-y)^{\frac{\alpha}{2}-1}\mathrm{d}y \\
&= \pi^{d/2}\frac{\Gamma(\alpha/2)}{\Gamma((d+\alpha)/2)}.
\end{aligned}
$$

This forces us to take $c = 2^{\alpha}\frac{\Gamma((d+\alpha)/2)}{\Gamma((d-\alpha)/2)}$ and so,we have

$$
\lim_{t\to\infty}\mathbb{E}_{Kx}[f\,(X_t/M_t)] = \pi^{-d/2}\frac{\Gamma((d+\alpha)/2)}{\Gamma(\alpha/2)}\int_{\mathbb{S}_{d-1}}\sigma_1(\mathrm{d}\phi)\int_{|w|<1}f(w)\frac{|1-|w|^2|^{\alpha/2}}{|\phi-w|^d}\mathrm{d}w.
$$

as required. □

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Concluding remarks

We have applied Maisonneuve's exit formula in the setting of Markov additive processes to develop a new radial excursion theory for $\mathbb{R}^d$ self-similar Markov processes, where we have used isotropic stable processes as examples. Further, we apply this new theory to extend the classical Blumenthal–Getoor–Ray for first entry/exit into a ball into n-tuple laws.

In the next chapter, we combine the ideas developed in this chapter and the previous chapter to study isotropic stable processes killed upon exiting a cone.

Chapter 4

Stable Lévy process in a cone

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Abstract

Bañuelos and Bogdan [6] and Bogdan et al. [20] analyse the asymptotic tail distribution of the first time a stable (Lévy) process in dimension $d \geq 2$ exists a cone. We use these results to develop the notion of a stable process conditioned to remain in a cone as well as the notion of a stable process conditioned to absorb continuously at the apex of a cone (without leaving the cone). As self-similar Markov processes we examine some of their fundamental properties through the lens of their Lamperti–Kiu decomposition. In particular we are interested to understand the underlying structure of the Markov additive process that drives such processes. As a consequence of our interrogation of the underlying MAP, we are able to provide an answer by example to the open question: If the modulator of a MAP has a stationary distribution, under what conditions does its ascending ladder MAP have a stationary distribution?

With the help of an analogue of the Riesz–Bogdan–Żak transform (cf. Bogdan and Żak [21], Kyprianou [41], Alili et al. [1]) as well as Hunt-Nagasawa duality theory, we show how the two forms of conditioning are dual to one another. Moreover, in the sense of Rivero [52, 53] and Fitzsimmons [34], we construct the null-recurrent extension of the stable process killed on exiting a cone, showing that it again remains in the class of self-similar Markov processes. Aside from the Riesz–Bogdan–Żak transform and Hunt-Nagasawa duality, an unusual combination of the Markov additive renewal theory of e.g. Alsmeyer [2] as well as the boundary Harnack principle (see e.g. [20]) play a central role to the analysis.

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In the spirit of several very recent works (see [45, 41, 44, 46, 42, 32]), the results presented here show that many previously unknown results of stable processes, which have long since been understood for Brownian motion, or are easily proved for Brownian motion, become accessible by appealing to the notion of the stable process as a self-similar Markov process, in addition to its special status as a Lévy processes with a semi-tractable potential analysis.

4.1 Introduction

For $d \geq 2$, let $X := (X_t : t \geq 0)$, with probabilities $\mathbb{P} = (\mathbb{P}_x, x \in \mathbb{R}^d)$, be a d-dimensional isotropic stable process of index $\alpha \in (0, 2)$. That is to say, $(X, \mathbb{P})$ is a $\mathbb{R}^d$-valued Lévy process having characteristic triplet $(0, 0, \Pi)$, where

$$\Pi(B) = \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|} \int_B \frac{1}{|y|^{\alpha+d}} \mathrm{d}y, \qquad B \in \mathcal{B}(\mathbb{R}). \tag{4.1}$$

Equivalently, this means $(X, \mathbb{P})$ is a d-dimensional Lévy process with characteristic exponent $\Psi(\theta) = -\log \mathbb{E}_0(\mathrm{e}^{\mathrm{i}<\theta, X_1>})$ which satisfies

$$\Psi(\theta) = |\theta|^\alpha, \qquad \theta \in \mathbb{R}^d.$$

Stable processes are also self-similar in the sense that they satisfy a scaling property. More precisely, for $c > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$,

$$\text{under } \mathbb{P}_x, \text{ the law of } (cX_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P}_{cx}. \tag{4.2}$$

As such, stable processes are useful prototypes for the study of the class of Lévy processes and, more recently, for the study of the class of self-similar Markov processes. The latter class of processes are regular strong Markov processes which respect the scaling relation (4.2), and accordingly are identified as having self-similarity (Hurst) index $1/\alpha$.

In this article, we are interested in understanding the notion of conditioning such stable processes to remain in a Lipschitz cone,

$$\Gamma = \{x \in \mathbb{R}^d : x \neq 0, \arg(x) \in \Omega\}, \tag{4.3}$$

where Ω is open on $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$. Note that Γ is an open domain which does not include its apex $\{0\}$, moreover, Ω need not be a connected domain. See [20] for the notion of Lipschitz cone and some related facts.

Our motivation comes principally from the desire to show how the rapidly evolving theory of self-similar Markov processes presents a number of new opportunities to contextualise existing theory and methodology in a completely new way to attack problems, which may have otherwise been seen as beyond reach. We note in this respect that the key tool, the Lamperti–Kiu transform for self-similar Markov processes, was formalised only recently in [25, 39, 1]. It identifies self-similar Markov processes as in one-to-one correspondence with Markov additive processes through a generalised polar decomposition with additional time change, and is the principal tool which, in the last five years or so, has unlocked a number of ways forward for classical problems such as the one we consider here; see [25, 45, 43, 40, 41, 44, 46, 30, 32, 31]. Moreover, this new perspective opens up an entire new set of challenges both in the setting of the underlying class of Markov additive processes (which have seldom received attention in the general setting since the foundational work of e.g. Çinlar [26, 27, 28] and Kaspi [36]) as well as the general class of self-similar Markov processes. Many of these challenges also emerge naturally in the setting of other stochastic processes and random structures where self-similarity plays an inherently fundamental role; see for example [57] in the setting of multi-type fragmentation processes, [11] in the setting of growth fragmentation processes and [7, 9] in the setting of planar maps. In this respect interrogating fundamental questions in the stable setting lays the foundations to springboard to problems of significantly greater generality. We mention in this respect, an outstanding problem in the setting of stable Lévy processes, which relates to the extremely deep work of e.g. [13, 33, 14] which showed how to condition a Brownian motion to stay in a Weyl Chamber and the important relationship this has to random matrix theory. This also inspired similar conditionings of other processes, such as those that appear in queueing theory; see [51].

Our journey towards conditioning stable processes to remain in the cone Γ will take us through a number of striking relations between stable processes killed on exiting Γ and stable processes conditioned to absorb continuously at the apex of Γ, which are captured by space-time transformations. Our analysis will necessitate examining new families of Markov additive processes that underly the conditioned stable processes through the Lamperti–Kiu transform. As an example we will exhibit a difficult result which identifies semi-explicitly the existence of a stationary distribution for the radially extreme points of the conditioned process (showing how the harmonic functions that drive our conditionings influence the strong mixing of the angular process). Moreover, our work will complement a number of other recent works which have examined the notion of entrance laws of self-similar Markov processes, as well as the subsequent notion of recurrent extension; see for example [12, 22, 52, 53, 34, 30]. Our approach will consist of an unusual mixture of techniques, coming from potential analysis and Harnack inequalities, Markov additive renewal theory, last exit decompositions *à la* Maisonneuve and Itô synthesis.

4.2 Harmonic functions in a cone

For simplicity, we let κ_Γ be the exit time from the cone i.e.

$$\kappa_\Gamma = \inf\{s > 0 : X_s \notin \Gamma\}.$$

Bañuelos and Bogdan [6] and Bogdan et al. [20] analyse the tail behaviour of the stopping time κ_Γ. Let us spend a moment reviewing their findings.

Suppose that we write

$$U_\Gamma(x, \mathrm{d}y) = \int_0^\infty \mathbb{P}_x(X_t \in \mathrm{d}y,\, t < \kappa_\Gamma)\mathrm{d}t, \qquad x, y \in \Gamma$$

for the potential of the stable process killed on exiting Γ.

Then it is known from Theorem 3.2 of [6] that $U_\Gamma(x, \mathrm{d}y)$ has a density, denoted by $u_\Gamma(x, y)$ and that

$$M(y) := \lim_{|x|\to\infty} \frac{u_\Gamma(x, y)}{u_\Gamma(x, y_0)}, \qquad y \in \Gamma,$$

exists and depends on $y_0 \in \Gamma$ only through a normalising constant. Note that it is a consequence of this definition that $M(x) = 0$ for all $x \notin \Gamma$. Moreover, M is locally bounded on $\mathbb{R}^d$ and homogeneous of degree $\beta = \beta(\Gamma, \alpha) \in (0, \alpha)$, meaning,

$$M(x) = |x|^\beta M(x/|x|) = |x|^\beta M(\arg(x)), \qquad x \neq 0. \tag{4.4}$$

It is also known that, up to a multiplicative constant, M is the unique function which is harmonic in the sense that

$$M(x) = \mathbb{E}_x[M(X_{\tau_B})\mathbf{1}_{(\tau_B < \kappa_\Gamma)}], \qquad x \in \mathbb{R}^d, \tag{4.5}$$

where B is any open bounded domain and $\tau_B = \inf\{t > 0 : X_t \notin B\}$.

The function M plays a prominent role in the following asymptotic result in Corollary 4 of Bogdan et al. [20], which strengthens e.g. Lemma 4.2 of Bañuelos and Bogdan [6].

Proposition 22 (Bogdan et al. [20])**.** We have

$$\lim_{a\to 0} \sup_{x\in\Gamma,\, |t^{-1/\alpha}x|\le a} \frac{\mathbb{P}_x(\kappa_\Gamma > t)}{M(x)t^{-\beta/\alpha}} = C,$$

where $C > 0$ is a constant.

4.3 Results for stable processes conditioned to stay in a cone

The above summary of the results in [6, 20] will allow us to introduce the notion of stable process conditioned to stay in Γ. Before doing that we make a slight digression to introduce some notation.

Let $\mathbb{D}$ be the space of càdlàg paths defined on $[0,\infty)$, with values in $\mathbb{R}^d \cup \Delta$, where Δ is a cemetery state. Each path $\omega \in \mathbb{D}$ is such that $\omega_t = \Delta$, for any $t \geq \inf\{s \geq 0 : \omega_s = \Delta\} =: \zeta(\omega)$. As usual we extend any function $f : \mathbb{R}^d \to \mathbb{R}$ to $\mathbb{R}^d \cup \Delta$ by taking $f(\Delta) = 0$. The space $\mathbb{D}$ is endowed with the Skorohod topology and its Borel σ-field. We will denote by $X = (X_t)_{t\geq 0}$ the coordinate process, and by $(\mathcal{F}_t,\, t \geq 0)$ the right-continuous filtration generated by X. We will also denote by $\mathbb{P}^\Gamma$ the law of the stable process $(X, \mathbb{P})$ killed when it leaves the cone Γ. Note, in particular, that $(X, \mathbb{P}^\Gamma)$ is also a self-similar Markov process.

Theorem 12.

(i) For any $t > 0$, and $x \in \Gamma$,

$$\mathbb{P}^\triangleleft_x(A) := \lim_{s\to\infty} \mathbb{P}_x\left(A \,|\kappa_\Gamma > t + s\right), \qquad A \in \mathcal{F}_t,$$

defines a family of conservative probabilities on the space of càdlàg paths such that

$$\left.\frac{\mathrm{d}\mathbb{P}^\triangleleft_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} := \mathbf{1}_{(t<\kappa_\Gamma)} \frac{M(X_t)}{M(x)}, \qquad t \geq 0, \text{ and } x \in \Gamma. \tag{4.6}$$

In particular, the right-hand side of (4.6) is a martingale.

(ii) Let $\mathbb{P}^\triangleleft := (\mathbb{P}^\triangleleft_x, x \in \Gamma)$. The process $(X, \mathbb{P}^\triangleleft)$, is a self-similar Markov process.

Next, we want to extend the definition of the process $(X, \mathbb{P}^\triangleleft)$, to include the apex of the cone Γ as a point of issue in a similar spirit to the inclusion of the origin as a point of issue for positive and real-valued self-similar Markov processes (cf. Bertoin and Yor [12], Bertoin and Caballero [8], Caballero and Chaumont [23], Bertoin and Savov [10], Chaumont et al. [24], Dereich et al. [30]). Said another way, we want to show the consistent inclusion of the state 0 to the state space Γ in the definition of $(X, \mathbb{P}^\triangleleft)$ as both a self-similar and a Feller process.

Before stating our theorem in this respect, we must first provide a candidate law for $\mathbb{P}^\triangleleft_0$, which is consistent with the family $\mathbb{P}^\triangleleft$. To this end, we need to recall the following theorem which is a copy of Theorem 5 in Bogdan et al. [20]. In order to state it, we need to introduce the notation $p^\Gamma_t(x,y)$, $x, y \in \Gamma$, $t \geq 0$, which will denote the transition density of $(X, \mathbb{P}^\Gamma)$. Note the reason why this density exists is because of the existence of a transition density for X, say $p_t(x,y)$, $x, y \in \mathbb{R}^d$, $t \geq 0$ and the relation

$$p^\Gamma_t(x,y) := p_t(x,y) - \mathbb{E}_x[\kappa_\Gamma < t; p_{t-\kappa_\Gamma}(X_{\kappa_\Gamma}, y)], \qquad x, y \in \Gamma, t > 0.$$

Theorem 13 (Bogdan et al. [20])**.** The following limit exists,

$$n_t(y) := \lim_{\Gamma \ni x \to 0} \frac{p_t^\Gamma(x,y)}{\mathbb{P}_x(\kappa_\Gamma > 1)}, \qquad x, y \in \Gamma, t > 0, \tag{4.7}$$

and $(n_t(y)\mathrm{d}y, t > 0)$, serves as an entrance law to $(X, \mathbb{P}^\Gamma)$, in the sense that

$$n_{t+s}(y) = \int_\Gamma n_t(x) p_s^\Gamma(x,y)\mathrm{d}x, \qquad y \in \Gamma, s, t \geq 0.$$

The function $n_t(\cdot)$ is a finite strictly positive jointly continuous function with the properties

$$n_t(y) = t^{-(d+\beta)/\alpha} n_1(t^{-1/\alpha}y) \;\text{ and }\; n_1(y) \approx \frac{\mathbb{P}_y(\kappa_\Gamma > 1)}{(1+|y|)^{d+\alpha}}, \qquad y \in \Gamma, t > 0, \tag{4.8}$$

and

$$\int_\Gamma n_t(y)\mathrm{d}y = t^{-\beta/\alpha}, \qquad t > 0, \tag{4.9}$$

where $f \approx g$ means the ratio of the functions f and g are bounded from above and below by two positive constants, uniformly in their domains.

The existence of the entrance law $(n_t(y)dy, t > 0)$, is sufficient to build a candidate for a probability measure, say $\mathbb{P}_0^\triangleleft$, on $\mathbb{D}$ carried by the paths with values in Γ, under which the paths of X start continuously from 0 and remain in Γ forever. To that end, note from (4.7) and Proposition 22, that for any $t > 0$ we have the following weak convergence,

$$\begin{aligned}\lim_{\Gamma \ni x \to 0} \frac{M(y)}{M(x)} \mathbb{P}_x(X_t \in \mathrm{d}y,\, t < \kappa_\Gamma) &= \lim_{\Gamma \ni x \to 0} C \frac{M(y)}{\mathbb{P}_x(t < \kappa_\Gamma) t^{\beta/\alpha}} p_t^\Gamma(x,y)\mathrm{d}y \\ &= CM(y) n_t(y)\mathrm{d}y, \qquad y \in \Gamma,\end{aligned} \tag{4.10}$$

where C is the constant in Proposition 22, and it is independent of t. With a pre-emptive choice of notation, let us denote by $\mathbb{P}_0^\triangleleft(X_t \in \mathrm{d}y)$ the measure that is obtained as a limit in the above relation, that is

$$\mathbb{P}_0^\triangleleft(X_t \in \mathrm{d}y) := CM(y) n_t(y)\mathrm{d}y, \qquad y \in \Gamma, t > 0.$$

Recalling from (4.6) that M forms a martingale for the process killed at its first exit from Γ, we have that necessarily $\int_\Gamma \mathbb{P}_0^\triangleleft(X_t \in \mathrm{d}y) = 1$. Furthermore, denote by $\mathbb{P}_0^\triangleleft$ the probability measure on $\mathbb{D}$ whose finite dimensional distributions are given by,

$$\mathbb{P}_0^\triangleleft(X_{t_i} \in A_i,\, i = 1, \cdots, n) := C \int_{A_1} M(y) n_{t_1}(y) \mathbb{P}_y^\triangleleft(X_{t_i - t_1} \in A_i,\, i = 2, \cdots, n)\,\mathrm{d}y,$$

for $n \in \mathbb{N}$, $0 < t_1 \leq \cdots t_n < \infty$ and Borel subsets of Γ, $A_1, \cdots, A_n$. The weak convergence in (4.10) extends in a straightforward way to the finite dimensional convergence $\mathbb{P}_x^\triangleleft \xrightarrow[\Gamma \ni x \to 0]{\text{f.d.}} \mathbb{P}_0^\triangleleft$. Our main result in this respect establishes that the convergence holds in the stronger sense of Skorohod's topology.

Theorem 14. The limit $\mathbb{P}_0^{\triangleleft} := \lim_{\Gamma \ni x \to 0} \mathbb{P}_x^{\triangleleft}$ is well defined on the Skorokhod space, so that, $(X, (\mathbb{P}_x^{\triangleleft}, x \in \Gamma \cup \{0\}))$ is both Feller and self-similar, and enters continuously at the origin, after which it never returns.

The proof of Theorem 14 leads us to a better understanding how stable processes enter into the wedge Γ from its apex. One may ask further if there is a self-similar process that behaves like $(X, \mathbb{P}^{\Gamma})$ while in Γ, but once it hits 0 it is not absorbed but returns into Γ in a sensible way. In the specialized literature, a process bearing those characteristics would be called a *self-similar recurrent extension* of the stable process killed on exiting the cone Γ, $(X, \mathbb{P}^{\Gamma})$, i.e. a $\Gamma \cup \{0\}$–valued process that behaves like $(X, \mathbb{P}^{\Gamma})$ up to the first hitting time of 0, for which 0 is a recurrent and regular state, and that has the scaling property (4.2). If such a process exists, say $(\overleftrightarrow{X}_t, t \geq 0)$, the fact that 0 is regular for it, implies that there exists a local time at 0, say L, and an excursion measure from 0, say $\mathbf{N}^{\Gamma}$. We will see in Section 4.13 that either $\mathbf{N}^{\Gamma}(X_{0+} \neq 0) = 0$ or $\mathbf{N}^{\Gamma}(X_{0+} = 0) = 0$. In the former case, we say that the recurrent extension leaves 0 continuously, and in the latter that it leaves 0 by a jump. General results from excursion theory (see e.g. [35, 16]), ensure that both objects together $(L, \mathbf{N}^{\Gamma})$ characterize $\overleftrightarrow{X}$. Furthermore, the measure $\mathbf{N}^{\Gamma}$ is a *self-similar excursion measure compatible* with the transition semi-group of $(X, \mathbb{P}^{\Gamma})$, that is, it is a measure on $(\mathbb{D}, \mathcal{F})$ such that

(i) it is carried by the set of Γ-valued paths that die at 0, with lifetime ζ,

$$\{\chi \in \mathbb{D} \,|\, \zeta > 0, \chi_s \in \Gamma, s < \zeta, \text{ and } \chi_t = 0, \text{ for all } t \geq \zeta\},$$

i.e. where, tautologically, $\zeta = \inf\{t > 0 : \chi_t = 0\}$;

(ii) the Markov property under $\mathbf{N}^{\Gamma}$, is satisfied, that is, for every bounded $\mathcal{F}$-measurable variable Y and $A \in \mathcal{F}_t$, $t > 0$,

$$\mathbf{N}^{\Gamma}(Y \circ \theta_t, A \cap \{t < \zeta\}) = \mathbf{N}^{\Gamma}\left(\mathbb{E}^{\Gamma}_{X_t}[Y], A \cap \{t < \zeta\}\right);$$

(iii) the quantity $\mathbf{N}^{\Gamma}(1 - \mathrm{e}^{-\zeta})$ is finite;

(iv) there exists a $\gamma \in (0,1)$ such that for any $q, c > 0$,

$$\mathbf{N}^{\Gamma}\left(\int_0^{\zeta} \mathrm{e}^{-qs} f(X_s)\mathrm{d}s\right) = c^{(1-\gamma)\alpha}\mathbf{N}^{\Gamma}\left(\int_0^{\zeta} \mathrm{e}^{-qc^{\alpha}s} f(cX_s)\mathrm{d}s\right). \tag{4.11}$$

The condition (iv) above is equivalent to requiring that

(iv') there exists $\gamma \in (0,1)$ such that, for any $c > 0$ and $f : \Gamma \to \mathbb{R}^+$ measurable,

$$\mathbf{N}^{\Gamma}(f(X_s), s < \zeta) = c^{\alpha\gamma}\mathbf{N}^{\Gamma}(f(c^{-1}X_{c^{\alpha}s}), c^{\alpha}s < \zeta), \text{ for } s > 0$$

The fact that $\mathbf{N}^\Gamma$ necessarily satisfies the above conditions is a consequence of a straightforward extension of the arguments in Section 2.2 in [52].

Conversely, Itô's synthesis theorem, from [53] Lemma 2, ensures that, given an excursion measure satisfying the conditions (i)-(iv) above, and a local time at zero, there is a self-similar recurrent extension of $(X, \mathbb{P}^\Gamma)$. Using this fact, and that the entrance law $(n_t(y)dy, t > 0)$ is intimately related to an excursion measure, we establish in the next result the existence of unique self-similar recurrent extension of $(X, \mathbb{P}^\Gamma)$ that leaves 0 (the apex of the cone) continuously. Furthermore, we will give a complete description of recurrent extensions that leave zero by a jump.

Theorem 15. Let $\mathbf{N}^\Gamma$ be a self-similar excursion measure compatible with $(X, \mathbb{P}^\Gamma)$. There exists a $\gamma \in (0, \beta/\alpha)$, a constant $a \geq 0$, and a measure π^Γ on $\Omega = \{\theta \in \Gamma : |\theta| = 1\}$ such that $a\pi^\Gamma \equiv 0$, $\int_\Omega \pi^\Gamma(d\theta)M(\theta) < \infty$, and $\mathbf{N}^\Gamma$ can be represented by, for any $t > 0$, and any $A \in \mathcal{F}_t$

$$\mathbf{N}^\Gamma(A\,,\, t < \zeta) = a\mathbb{E}_0^{\triangleleft}\left[\frac{1}{M(X_t)}\mathbf{1}_A\right] + \int_0^\infty \frac{dr}{r^{1+\alpha\gamma}} \int_\Omega \pi^\Gamma(d\theta)\mathbb{E}_{r\theta}[A\,,\, t < \kappa_\Gamma]. \tag{4.12}$$

If $a > 0$, the process $(X, \mathbb{P}^\Gamma)$ has unique recurrent extension that leaves 0 continuously, and $\gamma = \beta/\alpha$.

Conversely, for each $\gamma \in (0, \beta/\alpha)$, and π^Γ a non-trivial measure satisfying the above conditions, there is a unique recurrent extension that leaves zero by a jump and such that

$$\mathbf{N}^\Gamma\left(|X_{0+}| \in \mathrm{d}r, \arg(X_{0+}) \in \mathrm{d}\theta\right) = \frac{\mathrm{d}r}{r^{1+\alpha\gamma}}\pi^\Gamma(\mathrm{d}\theta), \qquad r > 0, \theta \in \Omega.$$

Finally, any self-similar recurrent extension with excursion measure $\mathbf{N}^\Gamma$ has an invariant measure

$$\begin{aligned}\widetilde{\pi}^\Gamma(\mathrm{d}x) &:= \mathbf{N}^\Gamma\left(\int_0^\zeta \mathbf{1}_{(X_t \in \mathrm{d}x)}\mathrm{d}t\right)\\ &= a|x|^{\alpha-d-\beta}M(\arg(x))\mathrm{d}x + \int_0^\infty \frac{dr}{r^{1+\alpha\gamma}} \int_\Omega \pi^\Gamma(d\theta)\mathbb{E}_{r\theta}\left[\int_0^{\kappa_\Gamma} \mathbf{1}_{(X_t \in \mathrm{d}x)}\mathrm{d}t\right],\end{aligned}$$

which is unique up to a multiplicative constant, and this measure is sigma-finite but not finite.

As a final remark, we note that, whilst we have provided a recurrent extension from the apex of the cone, one might also consider the possibility of a recurrent extension from the entire boundary of the cone. We know of no specific work in which there is a recurrent extension from a set rather than a point. That said, one may consider the work on censored stable processes as meeting this notion in some sense; see e.g. [19, 43]

4.4 Auxiliary results for associated MAPs

By an $\mathbb{R}\times\mathbb{S}^{d-1}$-valued Markov additive process (MAP), we mean here that $(\xi,\Theta)=((\xi_t,\Theta_t),t\geq 0)$ is a regular Strong Markov Process on $\mathbb{R}\times\mathbb{S}^{d-1}$ (possibly with a cemetery state), with probabilities $\mathbf{P}:=(\mathbf{P}_{x,\theta},x\in\mathbb{R},\theta\in\mathbb{S}^{d-1})$, such that, for any $t\geq 0$, the conditional law of the process $((\xi_{s+t}-\xi_t,\Theta_{s+t}):s\geq 0)$, given $\{(\xi_u,\Theta_u),u\leq t\}$, is that of (ξ,Θ) under $\mathbf{P}_{0,\theta}$, with $\theta=\Theta_t$. For a MAP pair (ξ,Θ), we call ξ the *ordinate* and Θ the *modulator*.

A very useful fact in the theory of self-similar Markov process is the so called Lamperti-Kiu transform, which is one of the main results in [1], extending the seminal work of Lamperti in [48], and establishes that there is a bijection between self-similar Markov processes (ssMp) in $\mathbb{R}^d$, and $\mathbb{R}\times\mathbb{S}^{d-1}$-valued MAPs. Indeed, for any ssMp in $\mathbb{R}^d$, say Z, there exists a unique $\mathbb{R}\times\mathbb{S}^{d-1}$-valued MAP (ξ,Θ) such that Z can be represented as

$$Z_t=\begin{cases}\exp\{\xi_{\varphi(t)}\}\Theta_{\varphi(t)}, & t<I_\infty,\\ \Delta, & t\geq I_\infty,\end{cases} \tag{4.13}$$

where $I_t:=\int_0^t \mathrm{e}^{\alpha\xi_t}\mathrm{d}t$, $t\geq 0$ (so that I_∞ is the almost sure monotone limit $=\lim_{t\to\infty} I_t$) and

$$\varphi(t)=\inf\{s>0:\int_0^s \mathrm{e}^{\alpha\xi_u}\mathrm{d}u>t\},\qquad t\geq 0. \tag{4.14}$$

Reciprocally, given a $\mathbb{R}\times\mathbb{S}^{d-1}$-valued MAP (ξ,Θ) the process defined in (4.13) is an $\mathbb{R}^d$-valued ssMp. This is known as the Lamperti-Kiu representation of the ssMp Z. When the lifetime of Z is infinite a.s. we say that law is conservative, which is equivalent to require that $I_\infty=\infty$ almost surely.

In [42], it was shown that the MAP underlying the stable process, for whom we will henceforth reserve the notation $\mathbf{P}=(\mathbf{P}_{x,\theta},x\in\mathbb{R}^d,\theta\in\mathbb{S}^{d-1})$ for its probabilities, is a pure jump process, such that ξ and Θ jump simultaneously. Moreover, the instantaneous jump rate with respect to Lebesgue time $\mathrm{d}t$ when $(\xi_t,\Theta_t)=(x,\vartheta)$ is given by

$$c(\alpha)\frac{\mathrm{e}^{yd}}{|\mathrm{e}^y\phi-\vartheta|^{\alpha+d}}\mathrm{d}y\sigma_1(\mathrm{d}\phi),\qquad t\geq 0, \tag{4.15}$$

where $\sigma_1(\phi)$ is the surface measure on $\mathbb{S}^{d-1}$ normalised to have unit mass and

$$c(\alpha)=2^{\alpha-1}\pi^{-d}\frac{\Gamma((d+\alpha)/2)\Gamma(d/2)}{\big|\Gamma(-\alpha/2)\big|}.$$

More precisely, suppose that f is a bounded measurable function on $(0,\infty)\times\mathbb{R}^2\times\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}$ such that $f(\cdot,\cdot,0,\cdot,\cdot)=0$, then, for all $\theta\in\mathbb{S}^{d-1}$,

$$\begin{aligned}&\mathbf{E}_{0,\theta}\left(\sum_{s>0} f(s,\xi_{s-},\Delta\xi_s,\Theta_{s-},\Theta_s)\right)\\&=\int_{(0,\infty)\times\mathbb{R}\times\mathbb{S}^{d-1}} V_\theta(\mathrm{d}s,\mathrm{d}z,\mathrm{d}\vartheta)\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}}\sigma_1(\mathrm{d}\phi)\mathrm{d}y\frac{c(\alpha)\mathrm{e}^{yd}}{|\mathrm{e}^y\phi-\vartheta|^{\alpha+d}}f(s,z,y,\vartheta,\phi),\end{aligned}\tag{4.16}$$

where $\Delta\xi_s=\xi_s-\xi_{s-}$,

$$V_\theta(\mathrm{d}t,\mathrm{d}z,\mathrm{d}\vartheta)=\mathbf{P}_{0,\theta}(\xi_t\in\mathrm{d}z,\Theta_t\in\mathrm{d}\vartheta)\mathrm{d}t,\qquad z\in\mathbb{R},\ \vartheta\in\mathbb{S}^{d-1},t>0,$$

$\sigma_1(\mathrm{d}\phi)$ is the surface measure on $\mathbb{S}^{d-1}$ normalised to have unit mass and

$$c(\alpha)=2^{\alpha-1}\pi^{-d}\frac{\Gamma((d+\alpha)/2)\Gamma(d/2)}{\left|\Gamma(-\alpha/2)\right|}.$$

Similar calculations as those used in [42] show that the Lévy system (H,L) of the MAP (ξ,Θ), associated to the stable process $(X,\mathbb{P})$, see [27] for background, is given by the additive functional $H_t:=t,\ t\geq 0$ and the kernel

$$L_\vartheta(\mathrm{d}\phi,\mathrm{d}y):=\sigma_1(\mathrm{d}\phi)\mathrm{d}yc(\alpha)\frac{\mathrm{e}^{yd}}{|\mathrm{e}^y\phi-\vartheta|^{\alpha+d}}.\tag{4.17}$$

So, for any predictable process $(G_t,\ t\geq 0)$, and any function f as above, one has

$$\begin{aligned}&\mathbf{E}_{0,\theta}\left(\sum_{s>0} G_s f(s,\xi_{s-},\Delta\xi_s,\Theta_{s-},\Theta_s)\right)\\&=\mathbf{E}_{0,\theta}\left(\int_0^\infty \mathrm{d}sG_s\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}} L_{\Theta_s}(\mathrm{d}\phi,\mathrm{d}y)f(s,\xi_s,y,\Theta_s,\phi)\right).\end{aligned}\tag{4.18}$$

We are interested in the characterisation of the Lévy system of the MAP associated to $(X,\mathbb{P}^\triangleleft)$, via the Lamperti-Kiu transform. To this end, suppose now we write $\mathbf{P}^\triangleleft:=(\mathbf{P}^\triangleleft_{x,\theta},(x,\theta)\in\mathbb{R}\times\Omega)$ for the probabilities of the MAP that underly $(X,\mathbb{P}^\triangleleft)$.

Proposition 23. For any positive predictable process $(G_t,\ t\geq 0)$, and any function $f:(0,\infty)\times\mathbb{R}^2\times\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}\to\mathbb{R}$, bounded and measurable, such that $f(\cdot,\cdot,0,\cdot,\cdot)=0$, one has

$$\begin{aligned}&\mathbf{E}^\triangleleft_{0,\theta}\left(\sum_{s>0} G_s f(s,\xi_{s-},\Delta\xi_s,\Theta_{s-},\Theta_s)\right)\\&=\mathbf{E}^\triangleleft_{0,\theta}\left(\int_0^\infty \mathrm{d}sG_s\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}} L^\triangleleft_{\xi_s,\Theta_s}(\mathrm{d}\phi,\mathrm{d}y)f(s,\xi_s,y,\Theta_s,\phi)\right),\quad\forall\theta\in\mathbb{S}^{d-1},\end{aligned}\tag{4.19}$$

where

$$L^{\triangleleft}_{x,\theta}(\mathrm{d}\phi, \mathrm{d}y) := \mathrm{e}^{\beta(y-x)}\frac{M(\phi)}{M(\theta)}L_\theta(\mathrm{d}\phi, \mathrm{d}y), \qquad \phi \in \mathbb{S}^{d-1}, y \in \mathbb{R}.$$

That is to say that, under $\mathbf{P}^{\triangleleft}$, the instantaneous jump rate when $(\xi_t, \Theta_t) = (x, \vartheta)$ is

$$c(\alpha)\frac{\mathrm{e}^{yd}\mathrm{e}^{\beta(y-x)}}{|\mathrm{e}^y\phi - \theta|^{\alpha+d}}\frac{M(\phi)}{M(\vartheta)}\mathrm{d}y\sigma_1(\mathrm{d}\phi)\mathrm{d}t, \qquad t \geq 0, \theta, \phi \in \Omega$$

A better understanding of the MAP that underlies $(X, \mathbb{P}^{\triangleleft})$, allows us to deduce e.g. the following result. For $a > 0$, define by $\overline{m}(\tau^{\ominus}_a-) = \sup\{t < \tau^{\ominus}_a : |X_t| = \sup_{s<t}|X_s|\}$ the last radial maximum before exiting the ball of radius a.

Theorem 16. There exists a probability measure, υ^* on Ω, which is invariant in the sense that

$$\mathbb{P}^{\triangleleft}_{\upsilon^*}\left(\arg(X_{\tau^{\ominus}_{\mathrm{e}}}) \in \mathrm{d}\theta\right) := \int_\Omega \upsilon^*(\mathrm{d}\phi)\mathbb{P}^{\triangleleft}_\phi\left(\arg(X_{\tau^{\ominus}_{\mathrm{e}}}) \in \mathrm{d}\theta\right) = \upsilon^*(\mathrm{d}\theta), \qquad \theta \in \Omega,$$

such that, for all $x \in \Gamma$, under $\mathbb{P}^{\triangleleft}_{r\theta}$, the triple

$$\left(\frac{X_{\overline{m}(\tau^{\ominus}_a-)}}{a}, \frac{X_{\tau^{\ominus}_a-}}{a}, \frac{X_{\tau^{\ominus}_a}}{a}\right)$$

converges in distribution as $a \to \infty$ to a limit which is independent of r and θ and non-degenerate. Equivalently, by scaling, the triple

$$(X_{\tau^{\ominus}_1}, X_{\tau^{\ominus}_1-}, X_{m(\tau^{\ominus}_1-)})$$

converges in distribution under $\mathbb{P}_x$, as $\Gamma \ni x \to 0$, to the same limit. More precisely, for any continuous and bounded $f : \Gamma^3 \to [0, \infty)$,

$$\lim_{\Gamma \ni x \to 0} \mathbb{E}^{\triangleleft}_x\left[f(X_{\tau^{\ominus}_1}, X_{\tau^{\ominus}_1-}, X_{m(\tau^{\ominus}_1-)})\right] = \frac{1}{\mathbb{E}^{\triangleleft}_{\upsilon^*}[\log|X_{\tau^{\ominus}_{\mathrm{e}}}|]}\int_\Omega\int_0^\infty \upsilon^*(\mathrm{d}\phi)\mathrm{d}r\, G(r, \phi),$$

where

$$G(r, \phi) = \mathbb{E}^{\triangleleft}_{\mathrm{e}^{-r}\phi}\left[f(X_{\tau^{\ominus}_1}, X_{\tau^{\ominus}_1-}, X_{m(\tau^{\ominus}_1-)})\mathbf{1}_{(\tau^{\ominus}_1 \leq \tau^{\ominus}_{\mathrm{e}^{1-r}})}\right].$$

The above theorem, although seemingly innocent and intuitively clear, offers us access to a very important result. In order to understand why, we must take a small diversion into radial excursion theory, as described in Kyprianou et al. [46].

Theorem 12 shows that $(X, \mathbb{P}^{\triangleleft})$, is a self-similar Markov process. As mentioned above, it follows that it has a Lamperti-Kiu representation of the form (4.13), with an underlying MAP, say (ξ, Θ), with probabilities $\mathbf{P}^{\triangleleft}_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in \mathbb{S}^{d-1}$. For each $t > 0$, let $\overline{\xi}_t = \sup_{u \leq t}\xi_u$ and define

$$\mathtt{g}_t = \sup\{s < t : \xi_s = \overline{\xi}_t\} \text{ and } \mathtt{d}_t = \inf\{s > t : \xi_s = \overline{\xi}_t\},$$

which code the left and right end points of excursions of ξ from its maximum, respectively. Then, for all $t > 0$, with $\mathtt{d}_t > \mathtt{g}_t$, we define the excursion process

$$(\epsilon_{\mathtt{g}_t}(s), \Theta^{\epsilon}_{\mathtt{g}_t}(s)) := (\xi_{\mathtt{g}_t+s} - \xi_{\mathtt{g}_t}, \Theta_{\mathtt{g}_t+s}), \qquad s \leq \zeta_{\mathtt{g}_t} := \mathtt{d}_t - \mathtt{g}_t;$$

it codes the excursion of $(\overline{\xi} - \xi, \Theta)$ from the set $(0, \mathbb{S}^{d-1})$ which straddles time t. Such excursions live in the space $\mathbb{U}(\mathbb{R} \times \mathbb{S}^{d-1})$, the space of càdlàg paths with lifetime $\zeta = \inf\{s > 0 : \epsilon(s) < 0\}$ such that $(\epsilon(0), \Theta^{\epsilon}(0)) \in \{0\} \times \mathbb{S}^{d-1}$, $(\epsilon(s), \Theta^{\epsilon}(s)) \in (0, \infty) \times \mathbb{S}^{d-1}$, for $0 < s < \zeta$, and $\epsilon(\zeta) \in (-\infty, 0]$.

For $t > 0$, let $R_t = \mathtt{d}_t - t$, and define the set $G = \{t > 0 : R_{t-} = 0, R_t > 0\} = \{\mathtt{g}_s : s \geq 0\}$ of the left extrema of excursions from 0 for $\overline{\xi} - \xi$. The classical theory of exit systems in Maisonneuve [49] now implies that there exist an additive functional $(\ell_t, t \geq 0)$ carried by the set of times $\{t \geq 0 : (\overline{\xi}_t - \xi_t, \Theta_t) \in \{0\} \times \mathbb{S}^{d-1}\}$, with a bounded 1-potential, and a family of *excursion measures*, $(\mathbb{N}^{\triangleleft}_{\theta}, \theta \in \mathbb{S}^{d-1})$, such that

(i) $(\mathbb{N}^{\triangleleft}_{\theta}, \theta \in \mathbb{S}^{d-1})$ is a kernel from $\mathbb{S}^{d-1}$ to $\mathbb{R} \times \mathbb{S}^{d-1}$, such that $\mathbb{N}^{\triangleleft}_{\theta}(1 - \mathrm{e}^{-\zeta}) < \infty$ and $\mathbb{N}^{\triangleleft}_{\theta}$ is carried by the set $\{(\epsilon(0), \Theta^{\epsilon}(0) = (0, \theta)\}$ and $\{\zeta > 0\}$ for all $\theta \in \mathbb{S}^{d-1}$;

(ii) we have the *exit formula*

$$\mathbf{E}^{\triangleleft}_{x,\theta}\left[\sum_{\mathtt{g} \in G} F((\xi_s, \Theta_s) : s < \mathtt{g}) H((\epsilon_{\mathtt{g}}, \Theta^{\epsilon}_{\mathtt{g}}))\right]$$
$$= \mathbf{E}^{\triangleleft}_{x,\theta}\left[\int_0^{\infty} F((\xi_s, \Theta_s) : s < t) \mathbb{N}^{\triangleleft}_{\Theta_t}(H(\epsilon, \Theta^{\epsilon})) \mathrm{d}\ell_t\right], \tag{4.20}$$

for $x \neq 0$, where F is continuous on the space of càdlàg paths $\mathbb{D}(\mathbb{R} \times \mathbb{S}^{d-1})$ and H is measurable on the space of càdlàg paths $\mathbb{U}(\mathbb{R} \times \mathbb{S}^{d-1})$;

(iii) for any $\theta \in \mathbb{S}^{d-1}$, under the measure $\mathbb{N}^{\triangleleft}_{\theta}$, the process $(\epsilon, \Theta^{\epsilon})$ is Markovian with the same semigroup as (ξ, Θ) stopped at its first hitting time of $(-\infty, 0] \times \mathbb{S}^{d-1}$.

The couple $(\ell, (\mathbb{N}^{\triangleleft}_{\theta}, \theta \in \mathbb{S}^{d-1}))$ is called an exit system. In Maisonneuve's original formulation, the pair ℓ and the kernel $(\mathbb{N}^{\triangleleft}_{\theta}, \theta \in \mathbb{S}^{d-1})$ is not unique, but once ℓ is chosen, the $(\mathbb{N}^{\triangleleft}_{\theta}, \theta \in \mathbb{S}^{d-1})$ is determined but for a ℓ-neglectable set, i.e. a set $\mathcal{A}$ such that

$$\mathbf{E}^{\triangleleft}_{x,\theta}(\int_{t \geq 0} \mathbf{1}_{((\overline{\xi}_s - \xi_s, \Theta_s) \in \mathcal{A})} \mathrm{d}\ell_s) = 0.$$

Let $(\ell^{-1}_t, t \geq 0)$ denote the right continuous inverse of ℓ, $H^+_t := \xi_{\ell^{-1}_t}$ and $\Theta^+_t = \Theta_{\ell^{-1}_t}$, $t \geq 0$. The strong Markov property tells us that $(\ell^{-1}_t, H^+_t, \Theta^+_t)$, $t \geq 0$, defines a Markov additive process, whose first two elements are ordinates that are non-decreasing. Rotational invariance of X implies that ξ, alone, is also a Lévy process, then the pair (ℓ^{-1}, H^+), without reference to the associated modulator Θ^+, are Markovian and play the role of the ascending ladder time and height subordinators of ξ. But here, we are more concerned with their dependency on Θ^+.

Taking account of the Lamperti–Kiu transform (4.13), it is natural to consider how the excursion of $(\overline{\xi}-\xi,\Theta)$ from $\{0\}\times\mathbb{S}^{d-1}$ translates into a radial excursion theory for the process

$$Y_t := \mathrm{e}^{\xi_t}\Theta_t, \qquad t\geq 0.$$

Ignoring the time change in (4.13), we see that the radial maxima of the process Y agree with the radial maxima of the stable process X. Indeed, an excursion of $(\overline{\xi}-\xi,\Theta)$ from $\{0\}\times\mathbb{S}^{d-1}$ constitutes an excursion of $(Y_t/\sup_{s\leq t}|Y_s|, t\geq 0)$, from $\mathbb{S}^{d-1}$, or equivalently an excursion of Y from its running radial supremum. Moreover, we see that, for all $t>0$ such that $\mathtt{d}_t > \mathtt{g}_t$,

$$Y_{\mathtt{g}_t+s} = \mathrm{e}^{\xi_{\mathtt{g}_t}}\mathrm{e}^{\epsilon_{\mathtt{g}_t}(s)}\Theta^{\epsilon}_{\mathtt{g}_t}(s) = |Y_{\mathtt{g}_t}|\mathrm{e}^{\epsilon_{\mathtt{g}_t}(s)}\Theta^{\epsilon}_{\mathtt{g}_t}(s) =: |Y_{\mathtt{g}_t}|\mathcal{E}_{\mathtt{g}_t}(s), \qquad s\leq \zeta_{\mathtt{g}_t}.$$

Whilst a cluster of papers on the general theory of Markov additive processes exists in the literature from the 1970s and 1980s, see e.g. Çinlar [26, 28, 27] and Kaspi [36], as well as in the setting that Θ is a discrete process, see Asmussen [4] and Albrecher and Asmussen [5], as well as some recent advances, see the Appendix in Dereich at al. [30], relatively little is known about the fluctuations of MAPs in comparison to e.g. Lévy processes. Note the latter are a degenerate class of MAPs, in the sense that a Lévy process can be seen as MAP with constant driving process.

A good example of an open problem pertaining to the fluctuation theory of MAPs is touched upon in Theorem 16: Suppose that Θ has a stationary distribution, under what conditions does Θ^+ have a stationary distribution? This is a question that has been raised in general in Section 4 of the paper [37]. Below we give a complete answer in the present setting.

Theorem 17. Under $\mathbf{P}^{\triangleleft}$, the modulator process Θ^+ has a stationary distribution, that is

$$\pi^{\triangleleft,+}(\mathrm{d}\theta) := \lim_{t\to\infty}\mathbf{P}^{\triangleleft}_{x,\theta}(\Theta^+_t\in\mathrm{d}\theta), \qquad \theta\in\Omega, x\in\mathbb{R},$$

exists as a non-degenerate distributional weak limit.

Remark. The reader will also note that, thanks to the change of measure (4.31) combined with the fact that ℓ^{-1}_t is an almost surely finite stopping time and the optimal sampling theorem, the ascending ladder MAP process (H^+,Θ^+), under $\mathbf{P}^{\triangleleft}$, has the property

$$\left.\frac{\mathrm{d}\mathbf{P}^{\triangleleft}_{0,\theta}}{\mathrm{d}\mathbf{P}_{0,\theta}}\right|_{\sigma((H^+_s,\Theta^+_s),s\leq t)} = \mathrm{e}^{\beta H^+_t}\frac{M(\Theta^+_t)}{M(\theta)}\mathbf{1}_{(t<\mathbf{k}^{\Omega,+})}, \qquad t\geq 0. \tag{4.21}$$

As a consequence of the existence of $\pi^{\triangleleft,+}$, we note that

$$\pi^{\Gamma,+}(\mathrm{d}\theta) := \frac{1}{M(\theta)}\pi^{\triangleleft,+}(\mathrm{d}\theta), \qquad \theta\in\Omega,$$

is an invariant distribution for $(\Theta^+_t\mathbf{1}_{(t<\mathbf{k}^{\Omega,+})}, t\geq 0)$ under $\mathbf{P}$.

4.5 Auxiliary results for dual processes in the cone

In order to prove some of the results listed above, we will need to understand another type of conditioned process, namely the stable process conditioned to continuously absorb at the origin.

Theorem 18. For $A \in \mathcal{F}_t$, on the space of càdlàg paths with a cemetery state,

$$\mathbb{P}^{\triangleright}_x(A,\, t < \boldsymbol{k}^{\{0\}}) := \lim_{a\to 0} \mathbb{P}_x(A,\, t < \kappa_\Gamma \wedge \tau^{\oplus}_a | \tau^{\oplus}_a < \kappa_\Gamma),$$

is well defined as a stochastic process which is continuously absorbed at the apex of Γ, where $\boldsymbol{k}^{\{0\}} = \inf\{t > 0 : |X_t| = 0\}$ and $\tau^{\oplus}_r = \inf\{s > 0 : |X_s| < r\}$. Moreover, for $A \in \mathcal{F}_t$,

$$\mathbb{P}^{\triangleright}_x(A,\, t < \boldsymbol{k}) = \mathbb{E}_x\left[\mathbf{1}_{(A,\, t<\kappa_\Gamma)} \frac{H(X_t)}{H(x)}\right], \qquad t \geq 0, \tag{4.22}$$

where

$$H(x) = |x|^{\alpha-d} M(x/|x|^2) = |x|^{\alpha-\beta-d} M(\arg(x)).$$

Next, we write $\mathbf{P}^{\triangleright} := (\mathbf{P}^{\triangleright}_{x,\theta}, x \in \mathbb{R}, \Omega)$ for the probability law of the MAP that underly $(X, \mathbb{P}^{\triangleright})$. For any positive predictable process $(G_t,\ t \geq 0)$, and any function $f : (0,\infty) \times \mathbb{R}^2 \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \to \mathbb{R}$, bounded and measurable, such that $f(\cdot,\cdot,0,\cdot,\cdot) = 0$, one has

$$\begin{aligned}
&\mathbf{E}^{\triangleright}_{0,\theta}\left(\sum_{s>0} G_s f(s, \xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s)\right) \\
&= \mathbf{E}^{\triangleright}_{0,\theta}\left(\int_0^\infty \mathrm{d}s G_s \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} L^{\triangleright}_{\xi_s,\Theta_s}(\mathrm{d}\phi, \mathrm{d}y) f(s, \xi_s, y, \Theta_s, \phi)\right), \quad \forall \theta \in \mathbb{S}^{d-1},
\end{aligned} \tag{4.23}$$

where

$$L^{\triangleright}_{x,\theta}(\mathrm{d}\phi, \mathrm{d}y) := \mathrm{e}^{\beta(y-x)} \frac{H(\phi)}{H(\theta)} L_\theta(\mathrm{d}\phi, \mathrm{d}y), \qquad \phi \in \mathbb{S}^{d-1}, y \in \mathbb{R}.$$

That is to say, that under $\mathbf{P}^{\triangleright}$, the instantaneous jump rate when $(\xi_t, \Theta_t) = (x, \vartheta)$ is

$$c(\alpha) \frac{\mathrm{e}^{yd} \mathrm{e}^{\beta(y-x)}}{|\mathrm{e}^y \phi - \theta|^{\alpha+d}} \frac{H(\phi)}{H(\vartheta)} \mathrm{d}y \sigma_1(\mathrm{d}\phi)\mathrm{d}t, \qquad t > 0, \theta, \phi \in \Omega$$

As alluded to, the process $(X, \mathbb{P}^{\triangleright})$, is intimately related to the process $(X, \mathbb{P}^{\triangleleft})$. This is made clear in our final main result which has the flavour of the Riesz–Bogdan–Żak transform; cf. Bogdan and Żak [21]. for the sake of reflection it is worth stating the Riesz–Bogdan–Żak transform immediately below first, recalling the definition of L-time and then our first main result in this section.

Theorem 19 (Riesz–Bogdan–Żak transform)**.** Suppose we write $Kx = x/|x|^2$, $x \in \mathbb{R}^d$ for the classical inversion of space through the sphere $\mathbb{S}^{d-1}$. Then, in dimension $d \geq 2$, for $x \neq 0$, $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}_{Kx}$ is equal in law to $(X_t, t \geq 0)$ under $\mathbb{P}^\circ_x$, where

$$\left.\frac{\mathrm{d}\mathbb{P}^\circ_x}{\mathrm{d}\mathbb{P}_x}\right|_{\sigma(X_s:s\leq t)} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \qquad t \geq 0 \tag{4.24}$$

and $\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha}\mathrm{d}u > t\}$.

Hereafter, by an L-time we mean the following. Suppose that $\mathcal{G}$ is the sigma-algebra generated by X and write $\mathcal{G}(\mathbb{P}^\triangleleft_\nu)$ for its completion by the null sets of $\mathbb{P}^\triangleleft_\nu$, where ν is a randomised initial distribution. Moreover, write $\overline{\mathcal{G}} = \bigcap_\nu \mathcal{G}(\mathbb{P}^\triangleleft_\nu)$, where the intersection is taken over all probability measures on the state space of X. A finite random time $\mathtt{k}$ is called an L-time (generalized last exit time) if $\{s < \mathtt{k}(\omega) - t\} = \{s < \mathtt{k}(\omega_t)\}$ for all $t, s \geq 0$. (Normally, we must include in the definition of an L-time that $\mathtt{k} \leq \zeta$, where ζ is the first entry of the process to a cemetery state. However, this is not applicable for $(X, \mathbb{P}^\triangleleft)$.) The two most important examples of L-times are killing times and last exit times.

Theorem 20. Consider again the transformation of space via the sphere inversion $Kx = x/|x|^2$, $x \in \mathbb{R}^d$.

(i) The process $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}^\triangleleft_x$, $x \in \Gamma$, is equal in law to $(X_t, t < \boldsymbol{k}^{\{0\}})$ under $\mathbb{P}^\triangleright_x$, $x \in \Gamma$, where

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha}\mathrm{d}u > t\}, \qquad t \geq 0. \tag{4.25}$$

and $\boldsymbol{k}^{\{0\}} = \inf\{t > 0 : X_t = 0\}$.

(ii) Under $\mathbb{P}^\triangleleft_0$, the time reversed process

$$\overleftarrow{X}_t := X_{(\boldsymbol{k}-t)-}, \qquad t \leq \boldsymbol{k},$$

is a homogenous strong Markov process whose transitions agree with those of $(X, \mathbb{P}^\triangleright_x)$, $x \in \Gamma$, where $\boldsymbol{k}$ is an L-time of $(X, \mathbb{P}^\triangleleft_x)$, $x \in \Gamma \cup \{0\}$.

Our third main theorem considers the possibility of a recurrent extension from the origin of of $(X, \mathbb{P}^\triangleright)$, similar in spirit to Theorem 15.

Theorem 21. Assume $0 < d + 2\beta - \alpha$. Let $\mathbf{N}^\triangleright$ be a self-similar excursion measure compatible with $(X, \mathbb{P}^\triangleright)$. We have that there exists a $\gamma \in (0, \alpha^{-1}(d + 2\beta - \alpha) \wedge 1)$, a constant $a \geq 0$, and a measure $\pi^\triangleright$ on Ω such that $a\pi^\triangleright \equiv 0$, $\int_\Omega \pi^\triangleright(d\theta)H(\theta) < \infty$, and $\mathbf{N}^\triangleright$ can be represented by, for any $t > 0$, and any $A \in \mathcal{F}_t$

$$\mathbf{N}^\triangleright(A\,,\, t < \zeta) = a\mathbb{E}^\triangleleft_0\left[\frac{H(X_t)}{M(X_t)}\mathbf{1}_A\right] + \int_0^\infty \frac{dr}{r^{1+\alpha\gamma}} \int_\Omega \pi^\triangleright(d\theta)\, \mathbb{E}^\triangleright_{r\theta}[A\,,\, t < \kappa_\Gamma]. \tag{4.26}$$

If $0 < \beta < (2\alpha - d)/2$, and $a > 0$, the process $(X, \mathbb{P}^{\triangleright})$ has unique recurrent extension that leaves 0 continuously. If $\beta \geq (2\alpha - d)/2$, then $a = 0$, and there is no recurrent extension that leaves 0 continuously.

Conversely, for each $\gamma \in (0, \alpha^{-1}(d + 2\beta - \alpha) \wedge 1)$, and $\pi^{\triangleright}$ a non-trivial measure satisfying the above conditions, there is a unique recurrent extension that leaves zero by a jump and such that

$$\mathbf{N}^{\triangleright}(|X_{0+}| \in \mathrm{d}r, \arg(X_{0+}) \in \mathrm{d}\theta) = \frac{\mathrm{d}r}{r^{1+\alpha\gamma}}\pi^{\triangleright}(\mathrm{d}\theta), \qquad r > 0,\, \theta \in \Omega.$$

Finally, any self-similar recurrent extension of $(X, \mathbb{P}^{\triangleright})$ with excursion measure $\mathbf{N}^{\triangleright}$, has an invariant measure

$$\begin{aligned} \overrightarrow{\pi}^{\triangleright}(\mathrm{d}x) &:= \mathbf{N}^{\triangleright}\left(\int_0^{\zeta} \mathbf{1}_{(X_t \in \mathrm{d}x)}\mathrm{d}t\right) \\ &= a|x|^{2(\alpha-d-\beta)}M(x)^2\mathrm{d}x + \int_0^{\infty} \frac{dr}{r^{1+\alpha\gamma}} \int_{\Omega} \pi^{\triangleright}(d\theta)\, \mathbb{E}^{\triangleright}_{r\theta}\left[\int_0^{\kappa_\Gamma} \mathbf{1}_{(X_t \in \mathrm{d}x)}\mathrm{d}t\right], \end{aligned}$$

which is unique up to a multiplicative constant, and this measure is sigma-finite but not finite.

It is interesting to remark here that if the cone is such that $\beta \geq (2\alpha - d)/2$, or equivalently $(d + 2\beta - \alpha)/\alpha \geq 1$, there is no recurrent extension that leaves zero continuously. This is due to the fact that the closer β is to α the smaller the cone is. Because the process is conditioned to hit zero continuously, a process starting from zero should return too quickly to zero, forcing there to be many small excursions, whose lengths become increasingly difficult to glue end to end in any finite interval of time. We could understand this phenomena with heuristic language by saying that 'the conditioned stable process is unable to escape the gravitational attraction to the origin because of the lack of space needed to do so'.

The rest of this paper is organised as follows. In the next section we give the proof of Theorem 12. Thereafter, we prove the above stated results in an order which differs from their presentation. We prove Theorem 23 in Section 4.7 and then turn to the proof of Theorem 16 and Corollary 17 in Sections 4.8 and 4.9, respectively. This gives us what we need to construct the process conditioned to continuously absorb at the apex of Γ, i.e. Theorem 18, in Section 4.10. With all these tools in hand, we can establish the duality properties of Theorem 20 in Section 4.11. Duality in hand, in Section 4.12, we can return to the Skorokhod convergence of the conditioned process $(X, \mathbb{P}^{\triangleleft}_x)$, $x \in \Gamma$, to the candidate for $(X, \mathbb{P}^{\triangleleft}_0)$, described in (4.10), and prove Theorem 14. Finally, in Sections 4.13 and 4.14, we complete the paper by looking at the recurrent extension of the conditioned processes inTheorem 15 and 21 respectively.

4.6 Proof of Theorem 12

We break the proof into the constituent parts of the statement of the theorem.

4.6.1 Proof of part (i)

For $A \in \mathcal{F}_t$ and $0 \neq x \in \Gamma$,

$$\mathbb{P}_x^{\triangleleft}(A,\, t < \zeta) = \lim_{s\to\infty} \mathbb{E}_x\left[\mathbf{1}_{(A\cap\{t<\kappa_\Gamma\})}\frac{\mathbb{P}_{X_t}(\kappa_\Gamma > s)}{\mathbb{P}_x(\kappa_\Gamma > t+s)}\right]. \tag{4.27}$$

From Lemma 4.2 of [6], for $t^{-1/\alpha}|x| < 1$, we have the bound

$$\frac{\mathbb{P}_x(\kappa_\Gamma > t)}{M(x)t^{-\beta/\alpha}} \in [C^{-1}, C],$$

for some $C > 1$. Otherwise, if $t^{-1/\alpha}|x| > 1$ then, $\mathbb{P}_x(\kappa_\Gamma > t) \leq 1 < |x|^\beta t^{-\beta/\alpha}$. Hence, noting that M is uniformly bounded from above, we have that, for all $x \in \Gamma$ and $s > 0$, there is a constant C' such that $\mathbb{P}_x(\kappa_\Gamma > s) \leq C'|x|^\beta s^{-\beta/\alpha}$. Hence, for s sufficiently large, there is another constant C'' (which depends on x) such that

$$\frac{\mathbb{P}_{X_t}(\kappa_\Gamma > s)}{\mathbb{P}_x(\kappa_\Gamma > t+s)} \leq \frac{C'|X_t|^\beta M(X_t)s^{-\beta/\alpha}}{C^{-1}M(x)(t+s)^{-\beta/\alpha}} < C''|X_t|^\beta.$$

It is well known that X_t has all absolute moments of any order in $(0,\alpha)$; cf. Section 25 of Sato [56]. The identity (4.6) now follows from Proposition 22 and the Dominated Convergence Theorem. Furthermore, by construction, for any $x \in \Gamma$,

$$\mathbb{P}_x^{\triangleleft}(t < \mathtt{k}) = 1, \qquad \forall t \geq 0.$$

It thus follows that under $\mathbb{P}^{\triangleleft}$, X has an infinite lifetime.

4.6.2 Proof of part (ii)

That $(X, \mathbb{P}^{\triangleleft})$ is a ssMp is a consequence of $(X, \mathbb{P})$ having the scaling property and the strong Markov property. Indeed, $(X, \mathbb{P}^{\triangleleft})$ is a strong Markov process, since it is obtained via an h-transform of $(X, \mathbb{P})$. To verify that it has the scaling property, let $c > 0$ and define $\tilde{X}_t := cX_{c^{-\alpha}t}$, $t \geq 0$. We have that

$$\tilde{\kappa}_\Gamma := \inf\{t > 0 : \tilde{X}_t \notin \Gamma\} = c^\alpha \kappa_\Gamma, \tag{4.28}$$

and by the scaling property

$$(\tilde{X}, \mathbb{P}_x) \overset{\text{Law}}{=} (X, \mathbb{P}_{cx}), \qquad x \in \Gamma. \tag{4.29}$$

Considering the transition probabilities of $(X, \mathbb{P}^{\triangleleft})$, we note with the help of (4.29) and (4.2) that, for bounded and measurable f,

$$
\begin{aligned}
\mathbb{E}^{\triangleleft}_{x}[f(\tilde{X}_t)] &= \mathbb{E}_x\left[\mathbf{1}_{(c^{-\alpha}t<\kappa_\Gamma)} f(cX_{c^{-\alpha}t})\frac{M(X_{c^{-\alpha}t})}{M(x)}\right] \\
&= \mathbb{E}_x\left[\mathbf{1}_{(t<\tilde{\kappa}_\Gamma)} f(\tilde{X}_t)\frac{|\tilde{X}_t|^\beta M(\tilde{X}_t/|\tilde{X}_t|)}{|cx|^\beta M(cx/|cx|)}\right] \\
&= \mathbb{E}_{cx}\left[\mathbf{1}_{(t<\kappa_\Gamma)} f(X_t)\frac{|X_t|^\beta M(X_t/|X_t|)}{|cx|^\beta M(cx/|cx|)}\right] \\
&= \mathbb{E}^{\triangleleft}_{cx}[f(X_t)], \qquad x\in\Gamma.
\end{aligned}
$$

This last observation together with the Markov property ensures the required self-similarity of $(X, \mathbb{P}^{\triangleleft})$. ☐

4.7 Proof of Theorem 23

We use a method taken from Theorem I.3.14 of [42]. From the Lamperti–Kiu transformation (4.13), we have

$$
\xi_t = \log(|X_{A(t)}|/|X_0|), \qquad \Theta_t = \frac{X_{A(t)}}{|X_{A(t)}|}, \qquad t\geq 0, \tag{4.30}
$$

where

$$
A(t) = \inf\{s>0 : \int_0^s |X_u|^{-\alpha}\mathrm{d}u > t\}.
$$

To show that (4.19) holds, we first note that, from the martingale property in (4.6), on account of the fact that $A(t)$ in (4.30) is an almost surely finite stopping time, we have by the optimal sampling theorem that

$$
\left.\frac{\mathrm{d}\mathbf{P}^{\triangleleft}_{x,\theta}}{\mathrm{d}\mathbf{P}_{x,\theta}}\right|_{\mathcal{G}_t} = \mathrm{e}^{\beta(\xi_t - x)}\frac{M(\Theta_t)}{M(\theta)}\mathbf{1}_{(t<\mathbf{k}^\Omega)}, \qquad t\geq 0, \tag{4.31}
$$

where $\mathcal{G}_t = \sigma((\xi_s,\Theta_s), s\leq t)$, $t\geq 0$.

Now write

$$
\mathbf{E}^{\triangleleft}_{0,\theta}\left(\sum_{s>0} G_s f(s,\xi_{s-},\Delta\xi_s,\Theta_{s-},\Theta_s)\right) = \lim_{t\to\infty}\mathbf{E}_{0,\theta}\left(\mathcal{M}_t\sum_{0<s\leq t} G_s f(s,\xi_{s-},\Delta\xi_s,\Theta_{s-},\Theta_s)\right)
$$

where

$$
\mathcal{M}_t = \mathbf{1}_{(t<\mathbf{k}^\Omega)}\mathrm{e}^{\beta\xi_t}\frac{M(\Theta_t)}{M(\theta)}, \qquad t\geq 0,
$$

is the martingale density from the change of measure (4.6).

Suppose we write Σ_t for the sum term in the final expectation above. The semi-martingale change of variable formula tells us that

$$\mathcal{M}_t\Sigma_t = \mathcal{M}_0(\theta)\Sigma_0 + \int_0^t \Sigma_{s-}\mathrm{d}\mathcal{M}_s + \int_0^t \mathcal{M}_{s-}\mathrm{d}\Sigma_s + [\mathcal{M},\Sigma]_t, \qquad t\geq 0,$$

where $[\mathcal{M},\Sigma]_t$ is the quadratic co-variation term. On account of the fact that $(\Sigma_t, t\geq 0)$, has bounded variation, the latter term takes the form $[\mathcal{M},\Sigma]_t = \sum_{s\leq t}\Delta M_t \Delta\Sigma_t$. As a consequence

$$\mathcal{M}_t\Sigma_t = \mathcal{M}_0(\theta)\Sigma_0 + \int_0^t \Sigma_{s-}\mathrm{d}\mathcal{M}_s + \int_0^t \mathcal{M}_{s}\mathrm{d}\Sigma_s, \qquad t\geq 0, \tag{4.32}$$

Moreover, after taking expectations, as the first in integral in (4.32) is a martingale and $\Sigma_0 = 0$, the only surviving terms give us

$$\begin{aligned}
&\mathbf{E}^{\triangleleft}_{0,\theta}\left(\sum_{s>0} G_s f(s,\xi_{s-},\Delta\xi_s,\Theta_{s-},\Theta_s)\right)\\
&\quad= \mathbf{E}_{0,\theta}\left(\sum_{s>0}\mathbf{1}_{(t<\mathtt{k}^\Gamma)}\mathrm{e}^{\beta\xi_s}\frac{M(\Theta_s)}{M(\theta)}G_s f(s,\xi_{s-},\Delta\xi_s,\Theta_{s-},\Theta_s)\right)\\
&\quad= \mathbf{E}_{0,\theta}\left[\int_0^\infty \mathrm{d}s\, G_s\mathbf{1}_{(s<\mathtt{k}^\Gamma)}\mathrm{e}^{\beta\xi_s}\frac{M(\Theta_s)}{M(\theta)}\int_\Omega \sigma_1(\mathrm{d}\phi)\int_{\mathbb{R}}\mathrm{d}y\frac{c(\alpha)\mathrm{e}^{y(\beta+d)}}{|\mathrm{e}^y\phi-\Theta_s|^{\alpha+d}}\frac{M(\phi)}{M(\Theta_s)}f(s,\xi_s,y,\Theta_s,\phi)\right]\\
&\quad= \mathbf{E}^{\triangleleft}_{0,\theta}\left[\int_0^\infty \mathrm{d}s\, G_s\int_\Omega \sigma_1(\mathrm{d}\phi)\int_{\mathbb{R}}\mathrm{d}y\frac{c(\alpha)\mathrm{e}^{y(\beta+d)}}{|\mathrm{e}^y\phi-\Theta_s|^{\alpha+d}}\frac{M(\phi)}{M(\Theta_s)}f(s,\xi_s,y,\Theta_s,\phi)\right]
\end{aligned}$$

where in the second equality we have used the jump rate (4.18) and in the third Fubini's theorem together with (4.31). □

4.8 Proof of Theorem 16

At the root of our proof of Theorem 16, we will appeal to Markov additive renewal theory in the spirit of Alsmeyer [2, 3], Kesten [38] and Lalley [47]. The radial excursion theory we have outlined in Section 4.4 is a natural mathematical pre-cursor to Markov additive renewal theory, however, one of the problems we have at this point in our reasoning, as it will be seen later, is that it is not yet clear whether there is a stationary behaviour for the process $(\Theta^+, \mathbf{P}^{\triangleleft})$, of the radial ascending ladder MAP. Indeed, as already discussed, we will deduce from our calculations here that a stationary distribution does indeed exist in Corollary 17.

We build instead an alternative Markov additive renewal theory around a naturally chosen discrete subset set ladder points which behave well into the hands of the scaling property of $(X, \mathbb{P}^{\triangleleft})$. As the proof is quite long, we break the remainder of this section into a number of steps, marked by subsections. The proof of Theorem 16 will thus be our.

4.8.1 A discrete ladder MAP

Under $\mathbb{P}^{\triangleleft}$, define the following sequence of stopping times,

$$T_n := \inf\{t > T_{n-1} : |X_t| > \mathrm{e}|X_{T_{n-1}}|\}, \qquad n \geq 1,$$

with $T_0 = 0$, and

$$S_n = \sum_{k=1}^{n} A_k \qquad A_n = \log \frac{|X_{T_n}|}{|X_{T_{n-1}}|} \qquad \text{and} \qquad \Xi_n = \arg(X_{T_n}), \qquad n \geq 1.$$

Note in particular that

$$X_{T_n} = |x|\mathrm{e}^{S_n}\Xi_n, \qquad n \geq 1.$$

Then, we claim that $((S_n, \Xi_n), n \geq 0)$, is a Markov additive renewal process. To verify this claim, we appeal principally to the strong Markov and scaling property of $(X, \mathbb{P}^{\triangleleft})$. Indeed, for any $x \in \Gamma$, we have that for any finite stopping time T, under $\mathbb{P}^{\triangleleft}_x$, the conditional law of $(X_{T+s}, s > 0)$ given $(X_u, u \leq T)$ equals that of $(|y|X_{s/|y|^\alpha}, s > 0)$ under $\mathbb{P}^{\triangleleft}_{\arg(y)}$, with $y = X_T$. Hence, for any $n \geq 0$, by construction, conditionally on $(X_u, u \leq T_n)$ we have that

$$\begin{aligned}
T_{n+1} &= \inf\{t > T_n : |X_t| > \mathrm{e}|X_{T_n}|\} \\
&= T_n + \inf\{s > 0 : |X_{s+T_n}| > \mathrm{e}|X_{T_n}|\} \\
&\overset{\text{Law}}{=} T_n + \inf\{s > 0 : |X_{T_n}||\tilde{X}_{s/|X_{T_n}|^\alpha}| > \mathrm{e}|X_{T_n}|\} \\
&= T_n + |X_{T_n}|^\alpha \tilde{T}_1,
\end{aligned}$$

where $\tilde{X}$ depends on $(X_u, u \leq T_n)$ only through $\arg(X_{T_n})$, has the same law as $(X, \mathbb{P}^{\triangleleft}_{\arg(X_{T_n})})$, and $\tilde{T}_1 = \inf\{t > 0 : |\tilde{X}_t| > \mathrm{e}\}$. From these facts, it follows that for any bounded and measurable f on $\mathbb{R} \times \Omega$,

$$\begin{aligned}
&\mathbb{E}^{\triangleleft}_x \left[f(S_{n+1} - S_n, \Xi_{n+1}) | (S_i, \Xi_i) : i \leq n \right] \\
&= \mathbb{E}^{\triangleleft}_x \left[f\left(\log \frac{|X_{T_{n+1}}|}{|X_{T_n}|}, \arg(X_{T_{n+1}}) \right) |X_{T_i} : i \leq n \right] \\
&= \mathbb{E}^{\triangleleft}_y \left[f(\log |X_{T_1}|, \arg(X_{T_1})) \right] |_{y=\Xi_n}.
\end{aligned}$$

These calculations ensure that $((S_n, \Xi_n), n \geq 0)$, is a Markov additive renewal process. Note, this computation also shows that, under $\mathbb{P}^{\triangleleft}$, the modulator $\Xi := (\Xi_n, n \geq 0)$ is also a Markov process.

4.8.2 Application of Markov additive renewal theory

Let us introduce the MAP renewal function associated to (S, Ξ), for Ω,

$$V_\theta(\mathrm{d}r, \mathrm{d}\phi) := \sum_{n=0}^{\infty} \mathbb{P}^{\triangleleft}_\theta(S_n \in \mathrm{d}r, \Xi_n \in \mathrm{d}\phi), \qquad r \in \mathbb{R}, \phi \in \Omega. \tag{4.33}$$

We will next show that the joint law in Theorem 16 can be expressed in term of a renewal like equation involving V_θ.

Lemma 7. For a bounded, measurable function $f : \Gamma^3 \to [0, \infty)$, we have, for $x \in \Gamma \cap B_1$,

$$\mathbb{E}^{\triangleleft}_x \left[f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus -}, X_{m(\tau_1^\ominus -)}) \right] = \int_0^{-\log|x|} \int_\Omega V_{\arg(x)}(\mathrm{d}r, \mathrm{d}\phi) G(-\log|x| - r, \phi), \tag{4.34}$$

where, for $\phi \in \Omega$,

$$G(y, \phi) := \mathbb{E}^{\triangleleft}_{\mathrm{e}^{-y}\phi} \left[f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus -}, X_{m(\tau_1^\ominus -)}) \mathbf{1}_{(\tau_1^\ominus \leq \tau^\ominus_{\mathrm{e}^{1-y}})} \right]. \tag{4.35}$$

Proof. Noting that $|X_{T_n}| = |x|\mathrm{e}^{S_n}$ and $\arg(X_{T_n}) = \Xi_n$. Appealing to the strong Markov property we get

$$\begin{aligned}
&\mathbb{E}^{\triangleleft}_x \left[f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus -}, X_{m(\tau_1^\ominus -)}) \mathbf{1}_{(\tau_1^\ominus < \infty)} \right] \\
&= \mathbb{E}^{\triangleleft}_x \left[\sum_{n \geq 0} \mathbf{1}_{(T_n < \tau_1^\ominus \leq T_{n+1})} f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus -}, X_{m(\tau_1^\ominus -)}) \right] \\
&= \mathbb{E}^{\triangleleft}_x \left[\sum_{n \geq 0} \mathbf{1}_{(T_n < \tau_1^\ominus)} \mathbb{E}_y \left[f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus -}, X_{m(\tau_1^\ominus -)}) \mathbf{1}_{(\tau_1^\ominus \leq T_1)} \right]_{y = X_{T_n}} \right] \\
&= \mathbb{E}^{\triangleleft}_x \left[\sum_{n=0}^{\infty} \mathbf{1}_{(|x|\mathrm{e}^{S_n} < 1)} \mathbb{E}_y \left[f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus -}, X_{m(\tau_1^\ominus -)}) \mathbf{1}_{(\tau_1^\ominus \leq T_1)} \right]_{y = |x|\mathrm{e}^{S_n}\Xi_n} \right],
\end{aligned}$$

where in the first equality the indicator implies $T_n \leq m(\tau_1^\ominus -)$. We can thus write

$$\begin{aligned}
&\mathbb{E}^{\triangleleft}_x \left[f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus -}, X_{m(\tau_1^\ominus -)}) \right] \\
&= \int_0^{-\log|x|} \int_\Omega V_{\arg(x)}(\mathrm{d}r, \mathrm{d}\phi) \mathbb{E}^{\triangleleft}_{|x|\mathrm{e}^r\phi} \left[f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus -}, X_{m(\tau_1^\ominus -)}) \mathbf{1}_{(\tau_1^\ominus \leq \tau^\ominus_{|x|\mathrm{e}^{r+1}})} \right],
\end{aligned}$$

which agrees with the statement of the lemma. $\square$

We now have all the elements to explain the strategy we will follow to prove Theorem 16. In light of the conclusion of Lemma 7, we will apply the Markov Additive Renewal Theorem, see for example Theorem 2.1 of Alsmeyer [2].

Subject to conditions, this result would tell us that, there is a probability measure υ^*, such that for υ^*-a.e. Ω, for any $f : \Gamma^3 \to [0,\infty)$ continuous and bounded,

$$\lim_{a\to 0} \mathbb{E}^{\triangleleft}_{a\theta}\left[f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)})\right] = \frac{1}{\mathbb{E}^{\triangleleft}_{\upsilon^*}[S_1]}\int_\Omega\int_0^\infty \upsilon^*(\mathrm{d}\phi)\mathrm{d}r\, G(r,\phi), \tag{4.36}$$

where G is as defined in (4.35). The required conditions for this to hold are:

(I) The process Ξ is an *aperiodic Harris recurrent* Markov chain, in a sense that there exists a probability measure, $\rho(\cdot)$ on $\mathcal{B}(\Omega)$ (Borel sets in Ω) such that, for some $\lambda > 0$,

$$\mathbb{P}^{\triangleleft}_{\theta}(\Xi_1 \in B) \geq \lambda\rho(E), \text{ for all } \theta \in \Omega, E \in \mathcal{B}(\Omega). \tag{4.37}$$

(II) Under $\mathbf{P}^{\triangleleft}$, $(\Xi_n, n \geq 0)$, has a stationary distribution, that is

$$\upsilon^*(\mathrm{d}\theta) := \lim_{n\to\infty} \mathbf{P}^{\triangleleft}_{0,\phi}(\Xi_n \in \mathrm{d}\theta), \qquad \theta \in \Omega, \phi \in \Omega,$$

exists as a non-degenerate distributional weak limit.

(III) With υ^* as above

$$\mathbb{E}^{\triangleleft}_{\upsilon^*}[S_1] := \int_\Omega \upsilon^*(\mathrm{d}\theta)\mathbb{E}^{\triangleleft}_{\theta}[S_1] < \infty. \tag{4.38}$$

(IV) For any continuous and bounded $f : \Gamma^3 \to [0,\infty)$, the mapping $r \mapsto G(r,\phi)$ is a.e. continuous, for any ϕ fixed, and

$$\int_\Omega\int_0^\infty \upsilon^*(\mathrm{d}\phi)\sum_{n\geq 0}\sup_{nh<r\leq(n+1)h} G(r,\phi) < \infty, \tag{4.39}$$

for some $h > 0$.

4.8.3 Harris recurrence

We will here prove that the condition (I) holds. To this end, we must first prove the following lemma and its corollary which deals with the Boundary Harnack Principle. In the current setting, it can be formulated as follows (see e.g. Bogdan et al. (BHP) in [20] and Bogdan [18]).

Lemma 8. Write $B_c := \{x \in \mathbb{R} : |x| < c\}$ for the ball of radius $c > 0$. Suppose that $u, v : \Gamma \to [0,\infty)$ are functions satisfying $u(x) = v(x) = 0$ whenever $x \in \Gamma^c \cap B_1$, and are regular harmonic on $\Gamma \cap B_1$, meaning that, for each $x \in \Gamma \cap B_1$,

$$\mathbb{E}_x\left[u(X_{\tau_1^{\ominus}\wedge\kappa_\Gamma})\right] = u(x) \quad \text{and} \quad \mathbb{E}_x\left[v(X_{\tau_1^{\ominus}\wedge\kappa_\Gamma})\right] = v(x).$$

Suppose, moreover, that $u(x_0) = v(x_0)$ for some $x_0 \in \Gamma \cap B_{1/2}$. Then, there exists a constant $C_1 = C_1(\Gamma, \alpha)$ (which does not depend on the choice of u or v) such that,

$$C_1^{-1} v(x) \leq u(x) \leq C_1 v(x), \qquad x \in \Gamma \cap B_{1/2}. \tag{4.40}$$

It is worth noting immediately that M is a regular harmonic function on $\Gamma \cap B_1$ according to the above definition. Indeed, from (4.6), the Optional Sampling Theorem and dominated converge (e.g. Theorem A in Blumenthal et al. [17] tells us that $\mathbb{E}_x[|X_{\tau_1^\ominus}|^\beta] < \infty$, we know that

$$\begin{aligned} M(x) &= \lim_{t\to\infty} \mathbb{E}_x \left[|X_{t\wedge\tau_1^\ominus}|^\beta M(\arg(X_{t\wedge\tau_1^\ominus})) \mathbf{1}_{(t\wedge\tau_1^\ominus < \kappa_\Gamma)} \right] \\ &= \mathbb{E}_x \left[M(X_{\tau_1^\ominus}) \mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)} \right] \\ &= \mathbb{E}_x \left[M(X_{\tau_1^\ominus \wedge \kappa_\Gamma}) \right]. \end{aligned}$$

As M can only be defined up to a multiplicative constant, without loss of generality, we henceforth assume there is a $x_0 \in \Gamma \cap B_{1/2}$, such that $M(x_0) = 1$.

Corollary 3. Let x_0 be as above. For each $f \geq 0$ on $\mathbb{R}^d$ such that

$$0 < \mathbb{E}_{x_0} \left[f(X_{\tau_1^\ominus}) \mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)} \right] < \infty,$$

there is a constant $C_1 = C_1(\Gamma, \alpha)$ (which does not depend on the choice of f) such that

$$C_1^{-1} M(x) \leq \frac{\mathbb{E}_x \left[f(X_{\tau_1^\ominus}) \mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)} \right]}{\mathbb{E}_{x_0} \left[f(X_{\tau_1^\ominus}) \mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)} \right]} \leq C_1 M(x), \text{ for all } x \in \Gamma \cap B_{1/2}.$$

Proof. The result follows from Lemma 3, in particular from the inequalities (4.40), as soon as we can verify that

$$g(x) := \frac{\mathbb{E}_x \left[f(X_{\tau_1^\ominus}) \mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)} \right]}{\mathbb{E}_{x_0} \left[f(X_{\tau_1^\ominus}) \mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)} \right]}, \qquad x \in \Gamma \cap B_1,$$

is regular harmonic on $\Gamma \cap B_1$. To this end, note that the function g clearly vanishes on $\Gamma^C \cap B_1$ and is equal to f on $\Gamma \cap B_1^c$ by construction and $g(x_0) = M(x_0) = 1$. Finally, note $g(X_{\kappa_\Gamma}) = 0$ and

$$g(X_{\tau_1^\ominus}) \mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)} = \frac{f(X_{\tau_1^\ominus}) \mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)}}{\mathbb{E}_{x_0} \left[f(X_{\tau_1^\ominus}) \mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)} \right]}$$

almost surely and hence, for $x \in \Gamma \cap B_1$,

$$g(x) = \frac{\mathbb{E}_x \left[f(X_{\tau_1^\ominus}) \mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)} \right]}{\mathbb{E}_{x_0} \left[f(X_{\tau_1^\ominus}) \mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)} \right]} = \mathbb{E}_x \left[g(X_{\tau_1^\ominus}) \mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)} \right] = \mathbb{E}_x \left[g(X_{\tau_1^\ominus \wedge \kappa_\Gamma}) \right],$$

as required. $\square$

Returning to the verification of (4.37), we consider a measure $\mu(\cdot,\cdot)$ given by $\mu(A,E) := \mathbb{P}^{\triangleleft}_{x_0}(S_1 \in A, \Xi_1 \in E)$, for $A \in \mathcal{B}(\mathbb{R}^+)$. Let us define the measure ρ by the relation $\rho(B) := \mathbb{P}^{\triangleleft}_{x_0}(\Xi_1 \in E)$ for $E \in \mathcal{B}(\Omega)$, which is clearly a probability measure as $\rho(\Omega) = 1$. The inequality (4.37) can be verified as a direct consequence of the following lemma.

Lemma 9. For all $\theta \in \Omega$, $A \in \mathcal{B}(\mathbb{R}^+)$ and $E \in \mathcal{B}(\Omega)$, we have that

$$C_1^{-1}\mu(A,E) \le \mathbb{P}^{\triangleleft}_{\theta}(S_1 \in A, \Xi_1 \in E) \le C_1\mu(A,E). \tag{4.41}$$

Proof. If $\mu(A,E) = 0$ the inequality (4.41) is trivially satisfied. Therefore, let us assume that $\mu(A,E) > 0$. From the previous lemma, we know that

$$g(x; A, E) := \mathbb{E}_x\left[M(X_{\tau_1^{\ominus}})\mathbf{1}_{(\log|X_{\tau_1^{\ominus}}|+1\in A, \arg(X_{\tau_1^{\ominus}})\in E,\, \tau_1^{\ominus}<\kappa_\Gamma)}\right], \qquad x \in \Gamma \cap B_1, A\times E \in \mathcal{B}(\mathbb{R}^+ \times \Omega),$$

is a regular harmonic function in $\Gamma \cap B_1$ (in the sense defined in Lemma 3). By virtue of the fact that we have normalised M so that $M(x_0) = 1$, we also have that

$$\mu(A,E) = \mathbb{P}^{\triangleleft}_{x_0}(S_1 \in A, \Xi_1 \in E) = g(x_0; A\times E).$$

Now take Ω, and note, with the help of the scaling property, and $M(x) = |x|^{\beta}M(x/|x|)$, for any $x \in \Gamma$, that

$$\begin{aligned}
\mathbb{P}^{\triangleleft}_{\theta}(S_1 \in A, \Xi_1 \in E) &= \frac{1}{M(\theta)}\mathbb{E}_\theta\left[M(X_{\tau_{\mathrm{e}}^{\ominus}})\mathbf{1}_{(\log|X_{\tau_{\mathrm{e}}^{\ominus}}|\in A, \arg(X_{\tau_{\mathrm{e}}^{\ominus}})\in E,\, \tau_{\mathrm{e}}^{\ominus}<\kappa_\Gamma)}\right]\\
&= \frac{1}{M(\theta/\mathrm{e})}\mathbb{E}_{\theta/\mathrm{e}}\left[M(X_{\tau_1^{\ominus}})\mathbf{1}_{(\log|X_{\tau_1^{\ominus}}|+1\in A, \arg(X_{\tau_1^{\ominus}})\in E,\, \tau_1^{\ominus}<\kappa_\Gamma)}\right]\\
&= \frac{g(\theta/\mathrm{e}; A, E)}{M(\theta/\mathrm{e})}
\end{aligned}$$

Thanks to the assumption $g(x_0; A, E) > 0$, the Boundary Hanack Principle (Corollary 3) tells us that, for all $|x| < 1/2$,

$$C_1^{-1}M(x) \le \frac{g(x; A, E)}{g(x_0; A, E)} \le C_1 M(x). \tag{4.42}$$

Hence, with $x = \theta/\mathrm{e}$,

$$\mathbb{P}^{\triangleleft}_{\theta}(S_1 \in A, \Xi_1 \in E) = \frac{g(\theta/\mathrm{e}; A, E)}{M(\theta/\mathrm{e})}. \tag{4.43}$$

Note also that $|x| < 1/\mathrm{e} < 1/2$, so we can put this into ineqaulity (4.42) to finish the proof. $\square$

We can put $A = \mathbb{R}^+$ and the inequality (4.37) is verified. Now that we have verified (4.37), we have the following corollary, which follows from e.g. Theorems VII.3.2 and VII.3.6 of Asmussen.

Corollary 4. Under $\mathbf{P}^{\triangleleft}$, $(\Xi_n, n \geq 0)$, has a stationary distribution, that is

$$\upsilon^*(\mathrm{d}\theta) := \lim_{n\to\infty} \mathbf{P}^{\triangleleft}_{0,\phi}(\Xi_n \in \mathrm{d}\theta), \qquad \theta \in \Omega, \phi \in \Omega,$$

exists as a non-degenerate distributional weak limit. So, the condition (II) is satisfied.

Remark. Note that

$$\int_\Omega \upsilon^*(\mathrm{d}\theta)\mathbb{P}_\theta(\Xi_n \in \mathrm{d}\phi,\, T_n < \kappa_\Gamma)\frac{M(\phi)}{M(\theta)} = \upsilon^*(\mathrm{d}\phi),$$

which makes $\upsilon^\Gamma(\mathrm{d}\phi) = \upsilon^*(\mathrm{d}\phi)/M(\phi)$, $\phi \in \Omega$, an invariant measure for the killed semigroup

$$\mathbb{P}_\theta(\Xi_n \in \mathrm{d}\phi,\, T_n < \kappa_\Gamma),\, n \geq 0.$$

Note that, under the assumptions (I) and (II), the limiting distribution (4.36) is proper, which can be seen by taking $f = 1$, in which case

$$\begin{aligned}
\int_\Omega \int_0^\infty \upsilon^*(\mathrm{d}\phi)\mathrm{d}r\, G(r,\phi) &= \int_\Omega \int_0^\infty \upsilon^*(\mathrm{d}\phi)\mathrm{d}r\mathbb{P}^{\triangleleft}_{\mathrm{e}^{-r}\phi}(\tau^{\ominus}_1 \leq \tau^{\ominus}_{\mathrm{e}^{1-r}}) \\
&= \int_\Omega \int_0^1 \upsilon^*(\mathrm{d}\phi)\mathrm{d}r + \int_\Omega \int_1^\infty \upsilon^*(\mathrm{d}\phi)\mathrm{d}r\mathbb{P}^{\triangleleft}_{\mathrm{e}^{-r}\phi}(\tau^{\ominus}_1 = \tau^{\ominus}_{\mathrm{e}^{1-r}}) \\
&= 1 + \int_\Omega \int_1^\infty \upsilon^*(\mathrm{d}\phi)\mathrm{d}r\, \mathbb{P}^{\triangleleft}_{\mathrm{e}^{-r}\phi}(\log|X_{\tau^{\ominus}_{\mathrm{e}^{1-r}}}| > 0) \\
&= 1 + \int_\Omega \int_1^\infty \upsilon^*(\mathrm{d}\phi)\mathrm{d}r\, \mathbb{P}^{\triangleleft}_{\phi}(\log|\mathrm{e}^{-r}X_{\tau^{\ominus}_{\mathrm{e}}}| > 0) \\
&= 1 + \int_\Omega \int_1^\infty \upsilon^*(\mathrm{d}\phi)\mathrm{d}r\, \mathbb{P}^{\triangleleft}_{\phi}(S_1 > r) \\
&= 1 + \mathbb{E}^{\triangleleft}_{\upsilon^*}[S_1 - 1] \\
&= \mathbb{E}^{\triangleleft}_{\upsilon^*}[S_1]
\end{aligned} \tag{4.44}$$

and hence the limit of (4.36) is equal to unity.

4.8.4 Verification of conditions (III) and (IV)

We do this with two individual lemmas.

Lemma 10. Condition (III) holds, i.e. $\mathbb{E}^{\triangleleft}_{\upsilon^*}[S_1] < \infty$.

Proof. We can appeal to the law of first exit from a sphere given in Theorem A of Blumenthal et al. [17] to deduce that, up to constant C, which is irrelevant for our computations, and may take different values in each line of the below computation, we have the following inequalities

$$
\begin{aligned}
&\sup_{|x|<1/2} \mathbb{E}_x\left[M(X_{\tau_1^\ominus})(1+\log|X_{\tau_1^\ominus}|)\mathbf{1}_{(\tau_1^\ominus<\kappa_\Gamma)}\right] \\
&\leq \sup_{|x|<1/2} \mathbb{E}_x[M(X_{\tau_1^\ominus})(1+\log|X_{\tau_1^\ominus}|)] \\
&= C\sup_{|x|<1/2}\int_{|y|>1} \mathrm{d}y(|1-|x|^2)^{\alpha/2}(|y|^2-1)^{-\alpha/2}M(y)\frac{1+\log|y|}{|y-x|^d} \\
&\leq C\int_\Omega \mathrm{d}\theta M(\theta)\int_1^\infty \mathrm{d}r(r^2-1)^{-\alpha/2}r^{d-1+\beta}\frac{1+\log r}{|r\theta-x|^d} \\
&= C\int_\Omega \mathrm{d}\theta M(\theta)\int_1^2 \mathrm{d}r(r^2-1)^{-\alpha/2}r^{d-1+\beta}\frac{1+\log r}{|r\theta-x|^d} \\
&\qquad + C\int_\Omega \mathrm{d}\theta M(\theta)\int_2^\infty \mathrm{d}r(r^2-1)^{-\alpha/2}r^{d-1+\beta}\frac{1+\log r}{|r\theta-x|^d} \\
&=: B_1+B_2.
\end{aligned} \tag{4.45}
$$

Using that $|r\theta - x| \geq r - |x| \geq 1/2$ we can bound the first term as follows:

$$
B_1 \leq 2^{2d-1+\beta}(1+\log 2)C\int_\Omega \mathrm{d}\theta M(\theta)\int_1^2 \mathrm{d}r(r^2-1)^{-\alpha/2} < \infty.
$$

To verify that the second term is finite also, we use that $|r\theta - x| \geq 3r/4$, by the triangle inequality, and that necessarily $\beta < \alpha$, to obtain that

$$
B_2 \leq \left(\frac{4}{3}\right)^{d+\frac{\alpha}{2}} C\int_\Omega \mathrm{d}\theta M(\theta)\int_2^\infty \mathrm{d}r\, r^{\beta-\alpha-1}(1+\log r) < \infty,
$$

We can now apply the Boundary Harnack Principle in Corollary 3 and the scaling property to deduce that

$$
\begin{aligned}
\int_\Omega \upsilon^*(\mathrm{d}\theta)\mathbb{E}^\triangleleft_\theta[S_1] &= \int_\Omega \upsilon^*(\mathrm{d}\theta)\mathbb{E}^\triangleleft_\theta[\log|X_{\tau_\mathrm{e}^\ominus}|] \\
&= \int_\Omega \upsilon^*(\mathrm{d}\theta)\mathbb{E}^\triangleleft_{\theta/\mathrm{e}}[\log|X_{\tau_1^\ominus}|+1] \\
&= \int_\Omega \upsilon^*(\mathrm{d}\theta)\frac{\mathbb{E}_{\theta/\mathrm{e}}\left[\mathbf{1}_{(\tau_1^\ominus<\kappa_\Gamma)}M(X_{\tau_1^\ominus})(\log|X_{\tau_1^\ominus}|+1)\right]}{M(\theta/\mathrm{e})} \\
&< \int_\Omega \upsilon^*(\mathrm{d}\theta)C_1\sup_{|x|<1/2}\mathbb{E}_x\left[\mathbf{1}_{(\tau_1^\ominus<\kappa_\Gamma)}M(X_{\tau_1^\ominus})(1+\log|X_{\tau_1^\ominus}|)\right] \\
&< \infty,
\end{aligned}
$$

where finiteness follows from (4.45). □

Lemma 11. The conditions in (IV) hold.

Proof. Let $f : \Gamma^3 \to [0,\infty)$ be a continuous and bounded function. On account of continuity of M and standard Skorokhod continuity properties of the stable process with killing at first passage times, together with the dominated convergence theorem, imply that for any $\phi \in \Gamma$ fixed, the function

$$y \mapsto G(y,\phi) := \mathbb{E}^{\triangleleft}_{\mathrm{e}^{-y}\phi}\left[f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)})\mathbf{1}_{(\tau_1^{\ominus}\leq\tau_{\mathrm{e}1-y}^{\ominus})}\right], \qquad y>0,$$

is continuous and bounded. Since f is assumed to be bounded it is enough to check that (4.39) holds with $f \equiv 1$. But this follows from a straightforward modification of the computation in (4.44), using that for any $\theta \in \Gamma$ fixed, the function $r \mapsto \mathbb{P}^{\triangleleft}_{\phi}(S_1 > r)$ is non-increasing, together with the conclusion of Lemma 10. □

We can remove the requirement that the limit is taken along the sequence of points $a\theta$, for $a \to 0$ and υ^*-a.e. θ and replaced by taking limits along $\Gamma \ni x \to 0$. Which, assuming (I)-(IV) would end the proof of Theorem 16.

Lemma 12. Suppose that (4.36) and the conditions (I)-(II) hold, then the limit also holds when the limit occurs as $\Gamma \ni x \to 0$.

Proof. Let us assume that A is a null set of υ^*. From (II), we know that $\mathbb{P}_{\upsilon^*}(\Xi_1 \in A) = \upsilon^*(A) = 0$. From (4.41), this implies that $0 = \mathbb{P}_{\upsilon^*}(\Xi_1 \in A) \geq C_1^{-1}\rho(A)$, and hence that $\rho(A) = 0$. On the other hand, we know from (4.41) and the latter fact, that, for all $\theta \in \Omega$, $\mathbb{P}_\theta(\Xi_1 \in A) \leq C_1\rho(A) = 0$.

We have thus shown that the very first step of the process Ξ positions it randomly so that it is in the concentration set of υ^*. We introduce a constant C_f given by

$$C_f = \lim_{a\to 0} \mathbb{E}^{\triangleleft}_{a\theta}\left[f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)})\right].$$

This constant only depends of f and not on θ. We have that

$$\begin{aligned}
&\left|\mathbb{E}^{\triangleleft}_{x}\left[f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)})\right] - C_f\right| \\
&\leq \mathbb{E}^{\triangleleft}_{x}\left[\left|f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)}) - C_f\right|\right] \\
&\leq \mathbb{E}^{\triangleleft}_{x}\left[\mathbf{1}_{(|x|\mathrm{e}^{S_1}<1)}\left|f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)}) - C_f\right| + \mathbf{1}_{(|x|\mathrm{e}^{S_1}>1)}\left|f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)}) - C_f\right|\right] \\
&\leq \mathbb{E}^{\triangleleft}_{x}\left[\mathbf{1}_{(|x|\mathrm{e}^{S_1}<1)}\left(||f||_\infty + C_f\right) + \mathbf{1}_{(|x|\mathrm{e}^{S_1}>1)}\left|f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)}) - C_f\right|\right].
\end{aligned} \tag{4.46}$$

Next note that, as $|x| \to 0$,

$$
\begin{aligned}
\mathbb{E}_x^{\triangleleft}&\left[\mathbf{1}_{(|x|\mathrm{e}^{S_1}<1)}\left(||f||_\infty + C_f\right)\right] \\
&\leq (||f||_\infty + C_f)\mathbb{P}_x^{\triangleleft}(\tau_1^{\ominus} \leq \tau_{\mathrm{e}|x|}^{\ominus}) \\
&= (||f||_\infty + C_f)\mathbb{P}_{\arg(x)}^{\triangleleft}(X_{\tau_1^{\ominus}} > \mathrm{e}/|x|) \\
&\leq (||f||_\infty + C_f)\sup_{\Omega}\mathbb{E}_\theta\left[\frac{M(X_{\tau_{\mathrm{e}}^{\ominus}})}{M(\arg(x))}\mathbf{1}_{(X_{\tau_1^{\ominus}}>\mathrm{e}/|x|)}\right] \to 0
\end{aligned}
$$

where the second equality follows by scaling and the final limit follows from Corollary 3. This deals with the first term on the right-hand side of (4.46).

For the second term on the right-hand side of (4.46), we appeal to (4.41) to have that

$$
\begin{aligned}
&\mathbb{E}_x^{\triangleleft}\left[\mathbf{1}_{(|x|\mathrm{e}^{S_1}>1)}\left|f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)}) - C_f\right|\right] \\
&= \int_\Omega\int_0^\infty \mathbb{P}_\theta^{\triangleleft}(S_1\in \mathrm{d}r, \Xi_1\in\mathrm{d}\phi)\mathbb{E}^{\triangleleft}_{|x|\mathrm{e}^r\phi_1}\left[\left|f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)}) - C\right|\right] \\
&\leq \int_\Omega\int_0^\infty C_1\mu(\mathrm{d}r,\mathrm{d}\phi)\mathbb{E}^{\triangleleft}_{|x|\mathrm{e}^r\phi}\left[\left|f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)}) - C_f\right|\right].
\end{aligned}
\tag{4.47}
$$

We use (4.36) and the fact that

$$
\mathbb{P}_\theta(\Xi_1\in\cdot) \ll \upsilon^*(\cdot)
$$

imply, with the help of (4.33), (4.34) and the Dominated Convergence Theorem, that

$$
\lim_{\Gamma\ni x\to 0}\mathbb{E}_x^{\triangleleft}\left[f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)})\right] = \frac{1}{\mathbb{E}^{\triangleleft}_{\upsilon^*}[S_1]}\int_\Omega\int_0^\infty \upsilon^*(\mathrm{d}\phi)\mathrm{d}r\, G(r,\phi),
$$

without restriction on $\arg(x)$ in relation to υ^*, as $\Gamma\ni x\to 0$. □

4.9 Proof of Theorem 17

Let us define a new family of stopping times with respect to the filtration generated by $((H_t^+,\theta_t^+),\ t\geq 0)$. Set $\chi_0 = 0$ and

$$
\chi_{n+1} = \inf\{s>\chi_n : H_s^+ - H_{\chi_n}^+ > 1\}, \qquad n\geq 0.
$$

We should also note that these stopping times have the property that the sequence of pairs $((S_n,\Xi_n),\ n\geq 0)$, agrees precisely with $((H_{\chi_n},\Theta^+_{\chi_n}),\ n\geq 0)$. Moreover, it is easy to show that $((\chi_n,\Xi_n),\ n\geq 0)$, is a Markov additive process, and we known $(\Xi_n, n\geq 0)$ is Harris recurrent, in the sense of (I) above.

Let,

$$
\mathscr{U}_\theta^{\triangleleft}(\mathrm{d}s,\mathrm{d}\phi) := \sum_{n\geq 0}\mathbf{P}_\theta^{\triangleleft}(\chi_n\in\mathrm{d}s, \Xi_n\in\mathrm{d}\phi), \qquad s\geq 0, \Omega.
$$

Appealing to the Markov property, we have, for Ω and bounded measurable f on Ω,

$$\begin{aligned}
\mathbf{E}^{\triangleleft}_{0,\theta}[f(\Theta^{+}_t)] &= \mathbf{E}^{\triangleleft}_{0,\theta}\left[\sum_{n\geq 0}\mathbf{1}_{(\chi_n\leq t<\chi_{n+1})}f(\Theta^{+}_t)\right]\\
&= \mathbf{E}^{\triangleleft}_{0,\theta}\left[\sum_{n\geq 0}\mathbf{1}_{(\chi_n\leq t)}\mathbf{E}^{\triangleleft}_{0,\phi}\left[\mathbf{1}_{(u<\chi_1)}f(\Theta^{+}_u)\right]_{\phi=\Theta^{+}_{\chi_n},u=t-\chi_n}\right]\\
&= \int_0^t\int_\Omega \mathscr{U}^{\triangleleft}_{\theta}(\mathrm{d}s,\mathrm{d}\phi)F(t-s,\phi),
\end{aligned}$$

with $F(s,\phi)=\mathbf{E}^{\triangleleft}_{0,\phi}[\mathbf{1}_{(s\leq\chi_1)}f(\Theta^{+}_s)]$ which is bounded and continuous in both its arguments. Note, moreover, that

$$\begin{aligned}
\int_0^\infty\int_\Omega \upsilon^*(\mathrm{d}\phi)\mathrm{d}s\,F(s,\phi) &= \int_0^\infty\int_\Omega \upsilon^*(\mathrm{d}\phi)\mathrm{d}s\,\mathbf{E}^{\triangleleft}_{0,\phi}[\mathbf{1}_{(s\leq\chi_1)}f(\Theta^{+}_s)]\\
&= \int_0^\infty\int_\Omega \upsilon^*(\mathrm{d}\phi)\mathrm{d}s\,\mathbf{E}^{\triangleleft}_{0,\phi}[\mathbf{1}_{(H^{+}_s<1)}f(\Theta^{+}_s)]\mathrm{d}s\\
&= \int_\Omega\int_\Omega \upsilon^*(\mathrm{d}\phi)U^{\triangleleft}_{\phi}([0,1),\mathrm{d}\theta)f(\theta),
\end{aligned}$$

where $U^{\triangleleft}_{\phi}(\mathrm{d}x,\mathrm{d}\theta)$, $x\geq 0$, Ω, is the ascending ladder MAP potential

$$U^{\triangleleft}_{\phi}(\mathrm{d}x,\mathrm{d}\theta)=\int_0^\infty \mathbf{P}^{\triangleleft}_{0,\phi}(H^{+}_s\in\mathrm{d}x,\Theta^{+}_s\in\mathrm{d}\theta)\mathrm{d}s.$$

As such, whenever f is bounded, we have that $\int_0^\infty\int_\Omega \upsilon^*(\mathrm{d}\phi)\mathrm{d}s\,F(s,\phi)<\infty$.

We also note that

$$\begin{aligned}
\mathbf{E}^{\triangleleft}_{0,\upsilon^*}[\chi_1] &:= \int_\Omega \upsilon^*(\mathrm{d}\phi)\mathbf{E}^{\triangleleft}_{0,\phi}[\chi_1]\\
&= \int_\Omega \upsilon^*(\mathrm{d}\phi)\int_0^\infty \mathbf{P}^{\triangleleft}_{0,\phi}(\chi_1>t)\mathrm{d}t\\
&= \int_\Omega \upsilon^*(\mathrm{d}\phi)\int_0^\infty \mathbf{P}^{\triangleleft}_{0,\phi}(H^{+}_t<1)\mathrm{d}t\\
&= \int_\Omega \upsilon^*(\mathrm{d}\phi)U^{\triangleleft}_{\phi}([0,1),\Omega)<\infty.
\end{aligned}$$

Arguing as in the proof of Lemma 11, it follows that whenever f is continuous and bounded, the mapping $(s,\phi)\mapsto F(s,\phi)$, satisfies the conditions in Theorem 2.1 in [2].

As such, and on account of the fact that $(\Xi_n, n\geq 0)$ has been proved to have a stationary distribution, υ^*, we can again invoke the Markov additive renewal theorem [2] and conclude that, for υ^*-almost every Ω,

$$\lim_{t\to\infty} \mathbf{E}^{\triangleleft}_{0,\theta}[f(\Theta^+_t)] = \frac{1}{\mathbf{E}^{\triangleleft}_{0,\upsilon^*}[\chi_1]} \int_\Omega \int_\Omega \upsilon^*(\mathrm{d}\phi) U^{\triangleleft}_\phi([0,1),\mathrm{d}\theta) f(\theta).$$

We can upgrade the previous statement to allow for all Ω by appealing to reasoning similar in fashion to the proof of Lemma 12. For the sake of brevity, we thus leave this as an exercise for the reader.

In conclusion, $(\Theta^+_t, t \geq 0)$ has a non-degenerate stationary distribution, which is given by

$$\pi^{\triangleleft,+}(\mathrm{d}\theta) = \frac{\int_\Omega \int_\Omega \upsilon^*(\mathrm{d}\phi) U^{\triangleleft}_\phi([0,1),\mathrm{d}\theta)}{\int_\Omega \upsilon^*(\mathrm{d}\phi) U^{\triangleleft}_\phi([0,1),\Omega)}, \qquad \theta \in \Omega.$$

as required. □

4.10 Proof of Theorem 18

We first need a technical Lemma. Recall that $\tau^\oplus_a := \inf\{t > 0 : |X_t| < a\}$, $a > 0$.

Lemma 13. We have the following convergence,

$$\lim_{\Gamma \ni aKx \to 0} \frac{\mathbb{P}_x(\tau^\oplus_a < \kappa_\Gamma)}{H(x) a^{d+\beta-\alpha}} = \frac{1}{\mathbb{E}^{\triangleleft}_{\upsilon^*}[\log |X_{\tau^\ominus_{\mathrm{e}}}|]} \int_\Omega \int_0^\infty \upsilon^*(\mathrm{d}\phi) \mathrm{d}r\, \mathbb{E}^{\triangleleft}_{\mathrm{e}^{-r}\phi}\left[\frac{|X_{\tau^\ominus_1}|^{\alpha-d}}{M(X_{\tau^\ominus_1})}\right] < \infty,$$

where, $H(x) = |x|^{\alpha-\beta-d} M(\arg(x))$.

Proof. We first use properties from the Riesz–Bogdan–Żak transform in Theorem 19,

$$\mathbb{P}_x(\tau^\oplus_a < \kappa_\Gamma) = \mathbb{E}^\circ_x\left[\frac{|x|^{\alpha-d}}{|X_{\tau^\oplus_a}|^{\alpha-d}}; \tau^\oplus_a < \kappa_\Gamma\right] = \mathbb{E}_{Kx}\left[\frac{|x|^{\alpha-d}}{|KX_{\tau^\ominus_{1/a}}|^{\alpha-d}}; \tau^\ominus_{1/a} < \kappa_\Gamma\right].$$

where $\tau^\ominus_{1/a} = \inf\{s > 0 : |X_s| > 1/a\}$. The scaling properties (4.2) and (4.28) (employed similarly for $\tau^\ominus_{1/a}$) tells us that

$$\begin{aligned}
\mathbb{P}_x(\tau^\oplus_a < \kappa_\Gamma) &= |x/a|^{\alpha-d} \mathbb{E}^\circ_{aKx}\left[|X_{\tau^\ominus_1}|^{\alpha-d}; \tau^\ominus_1 < \kappa_\Gamma\right] \\
&= M(aKx)|x/a|^{\alpha-d} \mathbb{E}^{\triangleleft}_{aKx}\left[\frac{|X_{\tau^\ominus_1}|^{\alpha-d}}{M(X_{\tau^\ominus_1})}\right] \\
&= M(aKx)|x/a|^{\alpha-d} \mathbb{E}^{\triangleleft}_{aKx}\left[\frac{|X_{\tau^\ominus_1}|^{\alpha-\beta-d}}{M(\arg(X_{\tau^\ominus_1}))}\right] \\
&= M(\arg(x))(|x|/a)^{\alpha-\beta-d} \mathbb{E}^{\triangleleft}_{aKx}\left[\frac{|X_{\tau^\ominus_1}|^{\alpha-d-\beta}}{M(\arg(X_{\tau^\ominus_1}))}\right].
\end{aligned}$$

Using Theorem 16, we have that

$$\lim_{\Gamma\ni aKx\to 0}\frac{\mathbb{P}_x(\tau_a^\oplus<\kappa_\Gamma)}{M(\arg(x))(|x|/a)^{\alpha-\beta-d}}$$
$$=\lim_{\Gamma\ni aKx\to 0}\mathbb{E}^\triangleleft_{aKx}\left[\frac{|X_{\tau_1^\ominus}|^{\alpha-\beta-d}}{M(\arg(X_{\tau_1^\ominus}))}\right]$$
$$=\frac{1}{\mathbb{E}^\triangleleft_{\upsilon^*}[\log|X_{\tau_{\mathrm{e}}^\ominus}|]}\int_\Omega\int_0^\infty \upsilon^*(\mathrm{d}\phi)\mathrm{d}r\,\mathbb{E}^\triangleleft_{\mathrm{e}^{-r}\phi}\left[\frac{|X_{\tau_1^\ominus}|^{\alpha-d}}{M(X_{\tau_1^\ominus})}\right]<\infty \tag{4.48}$$

where we have used that $\alpha-d<0$ and hence $|X_{\tau_1^\ominus}|\geq 1$ in the first inequality, and Theorem 16 with $f(x)=|x|^{\alpha-d}/M(x)\mathbf{1}_{(x\geq 1)}$, for $x\in\Gamma$, in the final inequality. Note, in particular, that the limit does not depend on the starting point.

The limit in (4.48) is thus finite and hence, noting that

$$|x/a|^{\alpha-d}M(aKx)=M(\arg(x))(|x|/a)^{\alpha-\beta-d}=H(x)a^{d+\beta-\alpha},$$

the result follows. □

Returning now to the proof of Theorem 18, the usual application of the strong Markov property means we need to evaluate, for $x\in\Gamma$,

$$\mathbb{P}^\triangleright_x(A)=\lim_{a\to 0}\mathbb{E}_x\left[\mathbf{1}_{(A,\,t<\kappa_\Gamma\wedge\tau_a^\oplus)}\frac{\mathbb{P}_{X_t}(\tau_a^\oplus<\kappa_\Gamma)}{\mathbb{P}_x(\tau_a^\oplus<\kappa_\Gamma)}\right], \tag{4.49}$$

where $A\in\mathcal{F}_t$. In order to do so, we first note from Lemma 13 that,

$$\lim_{a\to 0}\frac{\mathbb{P}_{X_t}(\tau_a^\oplus<\kappa_\Gamma)}{\mathbb{P}_x(\tau_a^\oplus<\kappa_\Gamma)}=\frac{H(X_t)}{H(x)}.$$

Moreover, from (4.48), we also see that, for each $\varepsilon>0$, there exists a constant $\Delta>0$ such that, when $|aKx|=(a/|x|)<\Delta$,

$$(1-\varepsilon)C_1\leq\frac{\mathbb{P}_x(\tau_a^\oplus<\kappa_\Gamma)}{H(x)a^{d+\beta-\alpha}}=\frac{\mathbb{P}_x(\tau_a^\oplus<\kappa_\Gamma)}{(a/|x|)^{\beta+d-\alpha}M(\arg(x))}\leq(1+\varepsilon)C_1.$$

With C_1 as in the previous Lemma. On the other hand, if $(a/|x|)\geq\Delta$, then

$$\mathbb{P}_x(\tau_a^\oplus<\kappa_\Gamma)\leq 1\leq(a/|x|)^{\beta+d-\alpha}\Delta^{\alpha-\beta-d}.$$

From this we conclude that there is an appropriate choice of constant C such that for $a\ll 1$,

$$\frac{\mathbb{P}_{X_t}(\tau_a^\oplus<\kappa_\Gamma)}{\mathbb{P}_x(\tau_a^\oplus<\kappa_\Gamma)}\leq C|X_t|^{\alpha-d-\beta}. \tag{4.50}$$

We want to show that

$$\mathbb{E}_x[|X_t|^{\alpha-d-\beta}\mathbf{1}_{(t<\kappa_\Gamma)}] < \infty. \tag{4.51}$$

To this end, we note that, since $\alpha-\beta-d<0$, $\mathbb{E}_x[|X_t|^{\alpha-d-\beta}\mathbf{1}_{(|X_t|>1,\,t<\kappa_\Gamma)}] \leq 1$. The problem thus lies in showing that $\mathbb{E}_x[|X_t|^{\alpha-d-\beta}\mathbf{1}_{(|X_t|\geq 1,\,t<\kappa_\Gamma)}] < \infty$. To this end, let us recall from Bogdan et al. [20] that $(X^\Gamma_t, t\geq 0)$, the stable process killed on exiting Γ, has a semigroup density, say $p^\Gamma_t(x,y)$, $x,y\in\Gamma$. Moreover, in equation (10) and (53) of the aforesaid reference, they showed that

$$p^\Gamma_t(x,t) \approx \mathbb{P}^\Gamma_x(\kappa_\Gamma > t)\mathbb{P}^\Gamma_y(\kappa_\Gamma > t)\left(t^{1/\alpha} \wedge \frac{t}{|y|^{\alpha+d}}\right), \qquad x,y\in\Gamma,$$

where $p^\Gamma_t(x,y)$ is the transition density of $(X,\mathbb{P}^\Gamma)$ and $f(x,t)\approx g(x,t)$ means that, uniformly in the domains of f and g, there exists a constant $c>0$ such that $c^{-1}\leq f/g\leq c$. It thus follows that

$$\begin{aligned}
&\mathbb{E}_x[|X_t|^{\alpha-d-\beta}\mathbf{1}_{(|X_t|\geq 1,\,t<\kappa_\Gamma)}]\\
&\quad\leq C\mathbb{P}^\Gamma_x(\kappa_\Gamma > t)\int_{|y|\leq 1}\mathbb{P}^\Gamma_y(\kappa_\Gamma > t)\left(t^{1/\alpha}\wedge\frac{t}{|y|^{\alpha+d}}\right)|y|^{\alpha-d-\beta}\mathrm{d}y\\
&\quad\leq Ct^{1/\alpha}\mathbb{P}^\Gamma_x(\kappa_\Gamma > t)\int_0^1 r^{\alpha-\beta-1}\mathrm{d}r < \infty
\end{aligned}$$

where the constant C has a different value in each line of the calculation above, but otherwise is unimportant.

The bound (4.50) and the finite moment (4.51) can now be used in conjunction with the Dominated Convergence Theorem in (4.49) to deduce (4.22).

We must also show that this process is continuously absorbed at 0. Applying the Riesz–Bogdan–Żak transform (cf. Theorem 19), for continuous and bounded $f:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$ and $0<a<|x|$,

$$\begin{aligned}
\mathbb{E}^\triangleright_x[f(X_{\underline{m}(\tau^\oplus_a-)}, X_{\tau^\oplus_a})] &= \mathbb{E}^\circ_x\left[f(X_{\underline{m}(\tau^\oplus_a-)}, X_{\tau^\oplus_a})\mathbf{1}_{(\tau^\oplus_a<\kappa_\Gamma)}\frac{M(KX_{\tau^\oplus_a})}{M(Kx)}\right]\\
&= \mathbb{E}_{Kx}\left[f(KX_{\overline{m}(\tau^\ominus_{1/a}-)}, KX_{\tau^\ominus_{1/a}})\mathbf{1}_{(\tau^\ominus_{1/a}<\kappa_\Gamma)}\frac{M(X_{\tau^\ominus_{1/a}})}{M(Kx)}\right]\\
&= \mathbb{E}^\triangleleft_{Kx}\left[f(KX_{\overline{m}(\tau^\ominus_{1/a}-)}, KX_{\tau^\ominus_{1/a}})\right],
\end{aligned}$$

where, for $a>0$, $\underline{m}(\tau^\ominus_a-) = \sup\{t<\tau^\ominus_a : |X_t| = \inf_{s<t}|X_s|\}$. From Theorem 16 it follows that the limit on the right-hand side above is equal to $f(0,0)$. This shows $(X,\mathbb{P}^\triangleright_x)$, $x\in\Gamma$ is almost surely absorbed continuously at 0.

Finally, reconsidering the proof of Proposition 23, the remaining statement is straightforward to prove in the same way. $\square$

4.11 Proof of Theorem 20

It turns out more convenient to prove Theorems 20 before we deal with Theorem 14. Indeed, it will play a crucial role in its proof.

4.11.1 Proof of Theorem 20 (i)

We can verify the statement of this part of the theorem by first noting that the transformation $(KX_{\eta(t)}, t \geq 0)$ maps $(X, \mathbb{P}^{\triangleleft})$, to a new self-similar process. Then we verify it has the transitions of $(X, \mathbb{P}^{\triangleright})$.

For the first of the aforesaid, we refer back to the Lamperti–Kiu transform. As already observed in Alili et al. [1] and Kyprianou [42], from the Lamperti–Kiu representation of $(X, \mathbb{P}^{\triangleleft})$,

$$KX_{\eta(t)} = \mathrm{e}^{-\xi_{\varphi\circ\eta(t)}}\Theta_{\varphi\circ\eta(t)} \qquad t \geq 0.$$

Note however that

$$\int_0^{\varphi(t)} \mathrm{e}^{\alpha\xi_s}\mathrm{d}s = t \text{ and } \int_0^{\eta(t)} \mathrm{e}^{-2\alpha\xi_{\varphi(u)}}\mathrm{d}u = t, \qquad t \geq 0.$$

A straightforward differentiation of the last two integrals shows that, respectively,

$$\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = \mathrm{e}^{-\alpha\xi_{\varphi(t)}} \text{ and } \frac{\mathrm{d}\eta(t)}{\mathrm{d}t} = \mathrm{e}^{2\alpha\xi_{\varphi\circ\eta(t)}}, \qquad t \geq 0,$$

and so the chain rule now tells us

$$\frac{\mathrm{d}(\varphi\circ\eta)(t)}{\mathrm{d}t} = \left.\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s}\right|_{s=\eta(t)} \frac{\mathrm{d}\eta(t)}{\mathrm{d}t} = \mathrm{e}^{\alpha\xi_{\varphi\circ\eta(t)}}, \tag{4.52}$$

and hence, $\varphi\circ\eta(t) = \inf\left\{s > 0 : \int_0^s \mathrm{e}^{-\alpha\xi_u}\mathrm{d}u > t\right\}$. It is thus clear that $(KX_{\eta(t)}, t \geq 0)$ is a self-similar Markov process with underlying MAP equal to $(-\xi, \Theta)$.

To verify it has the same transitions as $(X, \mathbb{P}^{\triangleright})$, we note that $(\eta(t),\ t \geq 0)$, is a sequence of stopping times

$$
\begin{aligned}
&\mathbb{E}^{\triangleleft}_{Kx}[f(KX_{\eta(t)})] \\
&\quad= \mathbb{E}_{Kx}\left[\frac{M(X_{\eta(t)})}{M(Kx)} f(KX_{\eta(t)});\, \eta(t) < \kappa_\Gamma\right] \\
&\quad= \mathbb{E}_{Kx}\left[\frac{|KX_{\eta(t)}|^{-\beta} M(\arg(KX_{\eta(t)}))}{|x|^{-\beta} M(\arg(x))} f(KX_{\eta(t)});\, \eta(t) < \kappa_\Gamma\right] \\
&\quad= \mathbb{E}_{x}\left[\frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}} \frac{|X_t|^{-\beta} M(\arg(X_t))}{|x|^{-\beta} M(\arg(x))} f(X_t); t < \kappa_\Gamma\right] \\
&\quad= \mathbb{E}_{x}\left[\frac{|X_t|^{\alpha-d-\beta} M(\arg(X_t))}{|x|^{\alpha-d-\beta} M(\arg(x))} f(X_t); t < \kappa_\Gamma\right] \\
&\quad= \mathbb{E}^{\triangleright}_{x}\left[f(X_t)\right]
\end{aligned}
$$

where in the third equality we have applied the regular Riesz–Bogdan–Żak transform (cf. Theorem 19) and in the final equality we have used Theorem 18. □

4.11.2 Proof of Theorem 20 (ii)

The proof of this part appeals to Theorem 3.5 of Nagasawa [50]. The aforesaid classical result gives directly the conclusion of part (ii) as soon as a number of conditions are satisfied. Most of the conditions are trivially satisfied thanks to the fact that $(X, \mathbb{P}^{\triangleleft})$, is a regular Markov process (see for example the use of this Theorem in Bertoin and Savov [10] or Döring and Kyprianou [32]). However the two most important conditions stand out as non-trivial and require verification here.

In the current context, the first condition requires the existence of a sigma-finite measure μ such that the duality relation is satisfied, for any $f, g : \Gamma \to \mathbb{R}$ measurable and bounded, one has

$$
\int_\Gamma \mu(\mathrm{d}x) f(x) \int_\Gamma \mathrm{d}y\, p^{\triangleleft}_t(x,y) g(y) = \int_\Gamma \mu(\mathrm{d}x) g(x) \int_\Gamma \mathrm{d}y\, p^{\triangleright}_t(x,y) f(y), \qquad \forall t \geq 0, \tag{4.53}
$$

and the second requires that

$$
\mu(\mathrm{d}x) = G^{\triangleleft}(0, \mathrm{d}x) := \int_0^\infty \mathbb{P}^{\triangleleft}_0(X_t \in \mathrm{d}x)\mathrm{d}t, \qquad x \in \Gamma. \tag{4.54}
$$

Our immediate job is thus to understand the analytical shape of the measure μ. To this end, we prove the following intermediary result, the conclusion of which automatically deals with (4.54).

Lemma 14. We have for bounded and measurable $f : \Gamma \to [0, \infty)$, which is compactly supported in Γ, up to a multiplicative constant,

$$
\int_\Gamma f(x) G^{\triangleleft}(0, \mathrm{d}x) = \int_\Gamma f(x) M(x) H(x) \mathrm{d}x.
$$

Proof. Referring to some of the facts displayed in Theorem 13, we have with the help of Fubini's

Theorem, the scaling properties of the transition density p^Γ, and (4.10) that

$$\begin{aligned}
&\int_\Gamma f(y) \int_0^\infty \mathbb{P}_0^{\triangleleft}(X_t \in \mathrm{d}y)\mathrm{d}t \\
&= \int_0^\infty \mathrm{d}t \int_\Gamma f(y)M(y)n_t(y)\mathrm{d}y \\
&= \int_0^\infty \mathrm{d}t \int_\Gamma f(y)M(y) \lim_{x\to 0} \frac{p_t^\Gamma(x,y)}{\mathbb{P}_x(\kappa_\Gamma > 1)}\mathrm{d}y \\
&= \int_0^1 \mathrm{d}t \int_\Gamma f(y)M(y) \lim_{x\to 0} \frac{p_t^\Gamma(x,y)}{\mathbb{P}_x(\kappa_\Gamma > 1)}\mathrm{d}y + \int_1^\infty \mathrm{d}t \int_\Gamma f(y)M(y) \lim_{x\to 0} \frac{p_t^\Gamma(x,y)}{\mathbb{P}_x(\kappa_\Gamma > 1)}\mathrm{d}y \\
&= \int_0^1 \mathrm{d}t \int_\Gamma f(y)M(y) \lim_{x\to 0} \frac{p_t^\Gamma(x,y)}{\mathbb{P}_x(\kappa_\Gamma > 1)}\mathrm{d}y \\
&\qquad + \int_1^\infty \mathrm{d}t \int_\Gamma f(y)M(y) \lim_{x\to 0} \frac{t^{-d/\alpha}p_1^\Gamma(t^{-1/\alpha}x, t^{-1/\alpha}y)}{\mathbb{P}_x(\kappa_\Gamma > 1)}\mathrm{d}y
\end{aligned} \tag{4.55}$$

We wish to use Dominated Convergence theorem to pull the limit out of each of the integrals. Referring again to Theorem 13, and recalling the compactness of the support of f, the integrand in the first term on the righthand side of (4.55) is uniformly bounded.

For the second term on the right-hand side of (4.55), we can assume without loss of generality that the support of f lies in $\Gamma \cap \{x \in \mathbb{R}^d : |x| < 1\}$. Recall again from the bound in Lemma 4.2 of [6], which states that, for $t > 1$ and $|x| < 1$, there exists a constant $C > 0$ such that

$$C^{-1}t^{-\beta/\alpha}M(x) < \mathbb{P}_{t^{-1/\alpha}x}(\kappa_\Gamma > 1) < Ct^{-\beta/\alpha}M(x).$$

Using the above, and appealing in particular to equation (56) of Bogdan et al. [20], for $t, |x| > 1$ and $y \in \Gamma$,

$$\begin{aligned}
\frac{t^{-d/\alpha}p_1^\Gamma(t^{-1/\alpha}x, t^{-1/\alpha}y)}{\mathbb{P}_x(\kappa_\Gamma > 1)} &< \frac{t^{-d/\alpha}p_1^\Gamma(t^{-1/\alpha}x, t^{-1/\alpha}y)}{M(x)} \\
&< t^{-(d+\beta)/\alpha}C\frac{\mathbb{P}_{t^{-1/\alpha}y}(\kappa_\Gamma > 1)}{(1+t^{-1/\alpha}|y|)^{d+\alpha}} \\
&< t^{-(d+2\beta)/\alpha}.
\end{aligned}$$

The right-hand side above can now be used as part of a dominated convergence argument for the second term in (4.55), noting in particular that f is compactly supported.

In conclusion, we have

$$
\begin{aligned}
&\int_\Gamma f(y)\int_0^\infty \mathbb{P}_0^{\triangleleft}(X_t\in \mathrm{d}y)\mathrm{d}t\\
&=\lim_{x\to 0}\int_\Gamma f(y)M(y)\int_0^1 \mathrm{d}t\frac{p_t^\Gamma(x,y)}{\mathbb{P}_x(\kappa_\Gamma>1)}\mathrm{d}y\\
&\qquad\qquad+\lim_{x\to 0}\int_\Gamma f(y)M(y)\int_1^\infty \mathrm{d}t\frac{p_t^\Gamma(x,y)}{\mathbb{P}_x(\kappa_\Gamma>1)}\mathrm{d}y\\
&=\lim_{x\to 0}\frac{\int_\Gamma f(y)M(y)G^\Gamma(x,y)}{M(x)}.
\end{aligned}
\tag{4.56}
$$

The above limit has already been computed in Lemma 7 of Bogdan et al. [20] and agrees with the conclusion of this Lemma. □

To complete the proof of part (ii) of Theorem 20, we must show (4.53). To this end, let us start by recalling Hunt's switching identity for X as a symmetric process and κ_Γ as a hitting time of an open domain. It ensures that for any $f,g:\Gamma\to\mathbb{R}$ measurable and bounded one has

$$
\int_\Gamma \mathrm{d}x f(x)\int_\Gamma \mathrm{d}y\, p_t^\Gamma(x,y)g(y)=\int_\Gamma \mathrm{d}x g(x)\int_\Gamma \mathrm{d}y\, p_t^\Gamma(x,y)f(y),\qquad \forall t\geq 0.
$$

With this in hand, it is easy to check that

$$
\begin{aligned}
\int_\Gamma \mu(\mathrm{d}x)f(x)\int_\Gamma \mathrm{d}y\, p_t^{\triangleleft}(x,y)g(y)&=\int_\Gamma \mathrm{d}x f(x)M(x)H(x)\int_\Gamma p_t^\Gamma(x,y)g(y)\frac{M(y)}{M(x)}\\
&=\int_\Gamma \mathrm{d}x g(x)M(x)H(x)\int_\Gamma p_t^\Gamma(x,y)f(y)\frac{H(y)}{H(x)}\\
&=\int_\Gamma \mu(\mathrm{d}x)g(x)\int_\Gamma \mathrm{d}y\, p_t^{\triangleright}(x,y)f(y),\qquad t\geq 0,
\end{aligned}
$$

as required. □

4.12 Proof of Theorem 14

To prove the weak convergence on the Skorokhod space of $\mathbb{P}_x$, as $x\to 0$, to $\mathbb{P}_0$, we appeal to the following proposition, lifted from Dereich et al. [30] and written in the language of the present context.

Proposition 24. Define $\tau_\varepsilon^\ominus=\inf\{t:|X_t|\geq\varepsilon\}$, $\varepsilon>0$. Suppose that the following conditions hold:

(a) $\lim_{\varepsilon\to 0}\limsup_{\Gamma\ni z\to 0}\mathbb{E}_z^{\triangleleft}[\tau_\varepsilon^\ominus]=0$

(b) $\lim_{\Gamma\ni z\to 0}\mathbb{P}_z^{\triangleleft}(X_{\tau_\varepsilon^\ominus}\in\cdot)=:\mu_\varepsilon(\cdot)$ exists for all $\varepsilon>0$

(c) $\mathbb{P}_0^{\triangleleft}$-almost surely, $X_0=0$ and $X_t\neq 0$ for all $t>0$

(d) $\mathbb{P}_0^{\triangleleft}((X_{\tau_\varepsilon^\ominus+t})_{t\geq 0}\in\cdot) = \int_\Gamma \mu_\varepsilon(\mathrm{d}y)\mathbb{P}_y^{\triangleleft}(\cdot)$ for every $\varepsilon>0$

Then the mapping

$$\Gamma \ni z \mapsto \mathbb{P}_z^{\triangleleft}$$

is continuous in the weak topology on the Skorokhod space.

Verification of Condition (a). Define $G^{\triangleleft}(x,y)$ via the relation

$$\int_\Gamma f(y)G^{\triangleleft}(x,y)\mathrm{d}y = \mathbb{E}_x^{\triangleleft}\left[\int_0^\infty f(X_t)\mathrm{d}t\right],$$

and note that $G^{\triangleleft}(x,y) = M(y)G^{\Gamma}(x,y)/M(x)$, $x,y\in\Gamma$. Then, for f positive, bounded, measurable and compactly supported and $x\in\Gamma\cup\{0\}$, then (4.56) and Lemma 14 tells us that

$$\lim_{\Gamma\ni z\to 0}\int_\Gamma f(y)G^{\triangleleft}(z,y)\mathrm{d}y = \int_\Gamma f(y)G^{\triangleleft}(0,y)\mathrm{d}y.$$

Now note that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\limsup_{\Gamma\ni z\to 0}\mathbb{E}_z^{\triangleleft}[\tau_\varepsilon^\ominus] &= \lim_{\varepsilon\to 0}\limsup_{\Gamma\ni z\to 0}\mathbb{E}_z^{\triangleleft}\left[\int_0^{\tau_\varepsilon^\ominus}\mathbf{1}_{(|X_t|<\varepsilon)}\mathrm{d}t\right]\\
&\leq \lim_{\varepsilon\to 0}\limsup_{\Gamma\ni z\to 0}\mathbb{E}_z^{\triangleleft}\left[\int_0^{\infty}\mathbf{1}_{(|X_t|<\varepsilon)}\mathrm{d}t\right]\\
&\leq \lim_{\varepsilon\to 0}\limsup_{\Gamma\ni z\to 0}\int_{|y|<\varepsilon}G^{\triangleleft}(z,y)\mathrm{d}y\\
&\leq C\lim_{\varepsilon\to 0}\int_{|y|<\varepsilon}H(y)M(y)\mathrm{d}y\\
&\leq C\lim_{\varepsilon\to 0}\int_\Omega\sigma_1(\mathrm{d}\theta)M(\theta)^2\int_0^\varepsilon r^{\alpha-\beta-1}\mathrm{d}r\\
&\leq C\lim_{\varepsilon\to 0}\varepsilon^{\alpha-\beta}\\
&= 0,
\end{aligned}$$

where $C\in(0,\infty)$ is an unimportant constant which changes its value in each line and $\sigma_1(\mathrm{d}\theta)$ is the surface measure on $\mathbb{S}^{d-1}$ normalised to have unit mass.

Verification of Condition (b). This condition is covered by Theorem 16. Note, moreover, that μ_ε does not depend on ε.

Verification of Condition (c). This condition is covered by Theorem 20.

Verification of Condition (d). We have that, for $|x|<\eta<\varepsilon$, from the Strong Markov Property,

$$\mathbb{E}_x^{\triangleleft}[f((X_{\tau_\varepsilon^\ominus+t}:t\geq 0)] = \mathbb{E}_x^{\triangleleft}\left[\mathbb{E}_{X_{\tau_\varepsilon^\ominus}}[f(X_t:t\geq 0)]\right] = \mathbb{E}_x^{\triangleleft}\left[g(X_{\tau_\eta^\ominus})\right] \tag{4.57}$$

for bounded, measurable f, where

$$g(y) = \mathbb{E}_y^{\triangleleft}\left[\mathbb{E}^{\triangleleft}_{X_{\tau_\varepsilon^\ominus}}[f(X_t : t \geq 0)].\right]$$

is bounded and measurable. From Theorem 16 and the Skorokhod continuity of $X, \mathbb{P}^{\triangleleft}$, which follows from the Lamperti–Kiu representation (4.13), we can take limits in (4.57) to get

$$\mathbb{E}_0^{\triangleleft}[f((X_{\tau_\varepsilon^\ominus + t} : t \geq 0)] = \mathbb{E}_0^{\triangleleft}\left[\mathbb{E}_{X_{\tau_\varepsilon^\ominus}}[f(X_t : t \geq 0)]\right] = \mathbb{E}_0^{\triangleleft}\left[g(X_{\tau_\eta^\ominus})\right] \tag{4.58}$$

Now appealing to Theorem 20 (ii), thanks to càdlàg paths, we know that $X_{\tau_\eta^\ominus} \to 0$ almost surely under $\mathbb{P}_0^{\triangleleft}$. As a consequence, we can appeal to the Dominated Convergence Theorem in (4.58), together with condition (a) and, again, the Skorokhod continuity of X under $\mathbb{P}_y^{\triangleleft}$, $y \in \Gamma$, and deduce the statement in condition (d).

4.13 Itô synthesis and proof of Theorem 15

The basis of Theorem 15 is the classical method of Itô synthesis of Markov processes and an extension of the main ideas in [52]. That is to say, the technique of piecing together excursions end to end under appropriate conditions, whilst ensuring that the strong Markov property holds. In our case, we are also charged with ensuring that self-similarity is preserved as well. We split the proof of Theorem 15 into the construction of the recurrent extension and the existence and characterisation of a stationary distribution.

4.13.1 Some general facts on self-similar recurrent extensions

As described before the statement of Theorem 15, according to Itô's synthesis theory a self-similar recurrent extension of $(X, \mathbb{P}^\Gamma)$ can be build from a self-similar excursion measure, i.e. a measure on $\mathbb{D}$ satisfying the conditions (i)-(iv) stated just before Theorem 15.

Suppose that $\mathbf{N}^\Gamma$ is a self-similar excursion measure compatible with the semigroup of $(X, \mathbb{P}^\Gamma)$. Define a Poisson point process $((s, \chi_s), s > 0)$ on $(0, \infty) \times \mathbb{D}$ with intensity $\mathrm{d}t \times \mathbf{N}^\Gamma(\mathrm{d}\chi)$ and let each excursion length be denoted by

$$\zeta_s := \inf\{t > 0 : \chi_s(t) = 0\} > 0.$$

Then, via the subordinator

$$\varsigma_t = \sum_{s \leq t} \zeta_s, \qquad t \geq 0,$$

we can define a local time process at 0 by

$$L_t = \inf\{r > 0 : \varsigma_r > t\}, \qquad t \geq 0$$

Note, for each $t \geq 0$, by considering the Laplace transform of ς_t, Campbell's formula and the assumption that $\mathbf{N}^\Gamma(1 - \mathrm{e}^{-\zeta}) < \infty$ ensures that $(\varsigma_t, t \geq 0)$ is well defined as a subordinator with jump measure given by $\nu(\mathrm{d}s) = \mathbf{N}^\Gamma(\zeta \in \mathrm{d}s)$, $s > 0$.

Now, we define $(\overset{\leftrightarrow}{X}_t, t \geq 0)$ with the following pathwise construction. For $t \geq 0$, let $L_t = s$, then $\sigma_{s-} \leq t \leq \sigma_{s-}$ and define

$$\overset{\leftrightarrow}{X}_t := \begin{cases} \Delta_s(t - \varsigma_{s-}), & \text{if } \varsigma_{s-} < \varsigma_s, \\ 0, & \text{if } \varsigma_{s-} = \varsigma_s \text{ or } s = 0. \end{cases}$$

Salisbury [54, 55] demonstrates how the process constructed above preserves the Markov property. In fact, one can easily adapt the arguments provided by Blumenthal [16], who considers only $[0, \infty)$ valued processes, to show that, under some regularity hypotheses on the semigroup of the minimal process $(X, \mathbb{P}^\Gamma)$, the process constructed above is a Feller process. This is due to the fact that, here we are considering an extension from Γ to $\Gamma \cup \{0\}$, for $(X, \mathbb{P}^\Gamma)$, which, by decomposing this process into polar coordinates, is equivalent to extend the radial part from $(0, \infty)$ to $[0, \infty)$.

To thus verify the Feller property, suppose that $C_0(\Gamma)$ is the space of continuous functions on Γ vanishing at 0 and ∞, and we write $(\mathcal{P}_t^\Gamma, t \geq 0)$ for the semigroup of $(X, \mathbb{P}^\Gamma)$. The aforesaid regularity hypothesis needed to adapt the argument given by Blumenthal [16] are:

(i) If $f \in C_0(\Gamma)$, then $\mathcal{P}_t^\Gamma f \in C_0(\Gamma)$ and $\mathcal{P}_t^\Gamma f \mapsto f$ uniformly as $t \to 0$;

(ii) For each $q > 0$, the mapping $x \mapsto \mathbb{E}_x^\Gamma[\mathrm{e}^{-q\zeta}]$ is continuous in Γ;

(iii) The following limits hold;

$$\lim_{\Gamma \ni x \to 0} \mathbb{E}_x^\Gamma[\mathrm{e}^{-\zeta}] = 1 \text{ and } \lim_{x \in \Gamma,\, |x| \to \infty} \mathbb{E}_x^\Gamma[\mathrm{e}^{-\zeta}] = 0.$$

All of these are easily verified using the Lamperti–Kiu representation of $(X, \mathbb{P}^\Gamma)$.

Now that we know that the process $(\overset{\leftrightarrow}{X}_t, t \geq 0)$ defined above is a strong Markov process, in fact a Feller process, we should verify that such a process has the scaling property. But this is a consequence of the condition (iv) above, as can be easily verified using the arguments in the proof of Lemma 2 in [52].

We will next describe all the excursion measures $\mathbf{N}^\Gamma$, compatible with $(X, \mathbb{P}^\Gamma)$. To that end, we recall that the entrance law $(\mathbf{N}_t^\Gamma(dy), t > 0)$ of an excursion measure $\mathbf{N}^\Gamma$, is defined by

$$\mathbf{N}_t^\Gamma(dy) := \mathbf{N}^\Gamma(X_t \in dy, t < \zeta), \qquad t > 0.$$

Lemma 15. Let $\mathbf{N}^\Gamma$ be a self-similar excursion measure compatible with $(X, \mathbb{P}^\Gamma)$, and γ the index appearing in (iv). Then, its entrance law admits the following representation: there is a constant $a \geq 0$, such that for all $t > 0$ and any $f : \Gamma \mapsto \mathbb{R}^+$ continuous and bounded

$$\begin{aligned} \mathbf{N}^\Gamma(f(X_t), t < \zeta) = a \lim_{|x|\to 0} \frac{\mathbb{E}_x[f(X_t), t < \kappa_\Gamma]}{M(x)} \\ + \int_{|y|>0} \mathbf{N}^\Gamma(X_{0+} \in \mathrm{d}y)\mathbb{E}_y[f(X_t), t < \kappa_\Gamma]. \end{aligned} \tag{4.59}$$

Furthermore, there is a measure π^Γ on Ω such that

$$\mathbf{N}^\Gamma(|X_{0+}| \in \mathrm{d}r, \arg(X_{0+}) \in \mathrm{d}\theta) = \frac{\mathrm{d}r}{r^{1+\alpha\gamma}} \pi^\Gamma(\mathrm{d}\theta), \tag{4.60}$$

and $\int_\Omega \pi^\Gamma(\mathrm{d}\theta)M(\theta) < \infty$. Finally, necessarily $\gamma \in (0, 1)$, and $\gamma \leq \beta/\alpha$; if $\gamma = \beta/\alpha$ then the measure $\pi^\Gamma \equiv 0$, whilst if $\gamma < \beta/\alpha$, then $a \equiv 0$.

Proof. In order to prove the decomposition (4.59) we start by noticing that for all $s, t > 0$, we have

$$\mathbf{N}^\Gamma(f(X_t), t < \zeta) = \mathbf{N}^\Gamma(\lim_{s\to 0} f(X_{s+t}), s + t < \zeta),$$

which is a consequence of the dominated convergence theorem, since

$$\mathbf{N}^\Gamma(f(X_{s+t}), s + t < \zeta) \leq ||f||\mathbf{N}^\Gamma(s + t < \zeta) \leq ||f||\mathbf{N}^\Gamma(t < \zeta) < \infty,$$

since $\mathbf{N}^\Gamma(t < \zeta)$ is always finite for any $t > 0$ because

$$\infty > \mathbf{N}^\Gamma(1 - \mathrm{e}^{-\zeta}) > \mathbf{N}^\Gamma(1 - \mathrm{e}^{-\zeta}, t < \zeta) > (1 - \mathrm{e}^{-t})\mathbf{N}^\Gamma(t < \zeta).$$

The former, together with the Markov property under $\mathbf{N}^\Gamma$, implies that

$$\begin{aligned} \mathbf{N}^\Gamma(f(X_t), t < \zeta) &= \mathbf{N}^\Gamma(\lim_{s\to 0} f(X_{s+t}), s + t < \zeta) \\ &= \lim_{s\to 0} \mathbf{N}^\Gamma(f(X_{s+t}), s + t < \zeta) \\ &= \lim_{s\to 0} \mathbf{N}^\Gamma(\mathbb{E}_{X_s}[f(X_t), t < \kappa_\Gamma], s < \zeta) \\ &= \lim_{\epsilon\to 0} \lim_{s\to 0} \mathbf{N}^\Gamma(\mathbb{E}_{X_s}[f(X_t), t < \kappa_\Gamma], |X_s| < \epsilon, s < \zeta) \\ &\quad + \lim_{\epsilon\to 0} \lim_{s\to 0} \mathbf{N}^\Gamma(\mathbb{E}_{X_s}[f(X_t), t < \kappa_\Gamma], |X_s| > \epsilon, s < \zeta) \\ &= \lim_{\epsilon\to 0} \lim_{s\to 0} \mathbf{N}^\Gamma\left(M(X_s)\frac{\mathbb{E}_{X_s}[f(X_t), t < \kappa_\Gamma]}{M(X_s)}, |X_s| < \epsilon, s < \zeta\right) \\ &\quad + \int_{y\in\Gamma, |y|>0} \mathbf{N}^\Gamma(X_{0+} \in \mathrm{d}y)\mathbb{E}_y[f(X_t), t < \kappa_\Gamma]; \end{aligned}$$

where in the final equality we used the continuity of the mapping $y \mapsto \mathbb{E}_y[f(X_t), t < \kappa_\Gamma]$, $y \in \Gamma$.

The main result in [20] implies that the limit $\lim_{|x|\to 0}\mathbb{E}_x[f(X_t), t<\kappa_\Gamma]/M(x)$ exists. This implies from the right hand side above that

$$\mathbf{N}^\Gamma(f(X_t), t<\zeta) = a\lim_{|x|\to 0}\frac{\mathbb{E}_x[f(X_t), t<\kappa_\Gamma]}{M(x)} + \int_{y\in\Gamma,|y|>0}\mathbf{N}^\Gamma(X_{0+}\in \mathrm{d}y)\mathbb{E}_y[f(X_t), t<\kappa_\Gamma],$$

where

$$\begin{aligned}
a &= \lim_{\epsilon\to 0}\lim_{s\to 0}\mathbf{N}^\Gamma(M(X_s), |X_s|<\epsilon, s<\zeta)\\
&= \lim_{\epsilon\to 0}\lim_{s\to 0}\mathbf{N}^\Gamma\left(\frac{M(X_s)}{\mathbb{P}_{X_s}(\kappa_\Gamma>1)}\mathbb{P}_{X_s}(\kappa_\Gamma>1), |X_s|<\epsilon, s<\zeta\right)\\
&= \left(\lim_{|x|\to 0}\frac{M(x)}{\mathbb{P}_x(\kappa_\Gamma>1)}\right)\lim_{\epsilon\to 0}\lim_{s\to 0}\mathbf{N}^\Gamma(|X_s|<\epsilon, 1+s<\zeta)<\infty.
\end{aligned}$$

This finishes the proof of the identity (4.59). We will next prove the identity (4.60). The latter decomposition together with the convergence (4.10), applied to $f(x) = M(x)g(\arg(x))\mathbf{1}_{(|x|\in(0,1))}$ with g any continuous and bounded function on Γ, implies that

$$\begin{aligned}
&\mathbf{N}^\Gamma\left(M(X_t)g(\arg(X_t))\mathbf{1}_{(|X_t|\in(0,1)}, t<\zeta\right)\\
&= aC\int_{|y|>1} g(\arg(y))n_t(y)dy\\
&\qquad + \int_{y\in\Gamma,|y|>0}\mathbf{N}^\Gamma(X_{0+}\in \mathrm{d}y)\mathbb{E}_y[M(X_t)g(\arg(X_t))\mathbf{1}_{(|X_t|\in(1,\infty))}, t<\kappa_\Gamma]
\end{aligned}$$

By the scaling property (iv') applied to $f(x) = M(x)g(\arg(x))\mathbf{1}_{(|x|\in(0,1))}$.

$$\begin{aligned}
\mathbf{N}^\Gamma(M(X_{0+})g(\arg(X_{0+})), |X_{0+}|\in(0,1))) &= c^{\alpha\gamma}\mathbf{N}^\Gamma(M(c^{-1}X_{0+})g(X_{0+}), |X_{0+}|\in(0,c))\\
&= c^{\alpha\gamma-\beta}\mathbf{N}^\Gamma(M(X_{0+})g(X_{0+}), |X_{0+}|\in(0,c)).
\end{aligned}$$

Notice that this is always finite because for $|x|<1$, $M(x)<K\mathbb{P}_x(\kappa_\Gamma>1)$, for some $K>0$. So, by the Markov property, the latter is bounded by $c^{\alpha\gamma-\beta}K||g||\mathbf{N}^\Gamma(\zeta>1)$. Differentiating in $c>0$, one gets

$$\begin{aligned}
&(\beta-\alpha\gamma)r^{\beta-\alpha\gamma-1}\mathrm{d}r\,\mathbf{N}^\Gamma(M(X_{0+})g(\arg(X_{0+})), |X_{0+}|\in(0,1))\\
&\qquad = \mathbf{N}^\Gamma(M(X_{0+})g(\arg(X_{0+})), |X_{0+}|\in \mathrm{d}r).
\end{aligned}$$

Observe that since the right hand side is positive as soon as g is positive, we get as a side consequence that $\beta-\alpha\gamma\geq 0$. Using again that $M(x)=|x|^\beta M(\arg(x))$, one gets

$$\begin{aligned}
&(\beta-\alpha\gamma)r^{-\alpha\gamma-1}\mathrm{d}r\,\mathbf{N}^\Gamma(|X_{0+}|^\beta M(\arg(X_{0+}))g(\arg(X_{0+})), |X_{0+}|\in(0,1))\\
&\qquad = \mathbf{N}^\Gamma(M(\arg(X_{0+}))g(\arg(X_{0+})), |X_{0+}|\in \mathrm{d}r).
\end{aligned}$$

Since this identity holds for any g continuous and bounded, we derive that, when $\beta > \alpha\gamma$, the equality of measures

$$\mathbf{N}^{\Gamma}(|X_{0+}| \in \mathrm{d}r,\ \arg(X_{0+}) \in \mathrm{d}\theta)) = \frac{\mathrm{d}r}{r^{\alpha\gamma+1}}\pi^{\Gamma}(\mathrm{d}\theta),$$

holds, where,

$$\pi^{\Gamma}(\mathrm{d}\theta) = \frac{\mathbf{N}^{\Gamma}(|X_{0+}|^{\beta}, |X_{0+}| \in (0,1), \arg(X_{0+}) \in \mathrm{d}\theta))}{\beta - \alpha\gamma}.$$

Whilst if $\beta = \alpha\gamma$, $\mathbf{N}^{\Gamma}(|X_{0+}| > 0) \equiv 0$, and thus $\pi^{\Gamma} \equiv 0$. We are just left to prove that when $\beta > \alpha\gamma$, then $\pi^{\Gamma}M < \infty$ and $a \equiv 0$. Indeed, that $\pi^{\Gamma}M < \infty$ follows from the following estimates

$$\begin{aligned}
\infty &> n(1 - \mathrm{e}^{\zeta}, X_{0+} \neq 0) \\
&= \int_0^{\infty} \mathrm{d}s\, n(s < \zeta, X_{0+} \neq 0)\mathrm{e}^{-s} \\
&= \int_0^{\infty} \mathrm{d}s\, \mathrm{e}^{-s} \int_{\Omega} \pi^{\Gamma}(\mathrm{d}\theta) \int_0^{\infty} \frac{\mathrm{d}r}{r^{1+\alpha\gamma}} \mathbb{P}_{r\theta}(\kappa_{\Gamma} > s) \\
&\geq \int_{\Omega} \pi^{\Gamma}(\mathrm{d}\theta) \int_0^1 \frac{\mathrm{d}r}{r^{1+\alpha\gamma}} \int_{r^{\alpha}}^{\infty} \mathrm{d}s\mathrm{e}^{-s} M(r\theta) s^{-\beta/\alpha} \\
&\geq \int_{\Omega} \pi^{\Gamma}(\mathrm{d}\theta) M(\theta) \int_0^1 \frac{\mathrm{d}r}{r^{1+\alpha\gamma-\beta}} \int_1^{\infty} \mathrm{d}s\mathrm{e}^{-s} s^{-\beta/\alpha},
\end{aligned}$$

where we used the estimate in Proposition 22. As claimed we derive that $\int_{\Omega} \pi^{\Gamma}(\mathrm{d}\theta)M(\theta) < \infty$. To finish, we observe that the identity (4.59) together with the scaling property (iv') implies that for any $t > 0$

$$t^{-\gamma}\mathbf{N}^{\Gamma}(\zeta > 1) = \mathbf{N}^{\Gamma}(\zeta > t) \geq a \lim_{|x| \to 0} \frac{\mathbb{P}_x(t < \kappa_{\Gamma})}{M(x)} = at^{-\beta/\alpha}C,$$

with $C > 0$ the constant appearing in Proposition 22. Since by assumption $\beta > \alpha\gamma$, we obtain by making $t \downarrow 0$, that $a \equiv 0$. We have thus finished the proof of Lemma 15. □

To finish the proof of Theorem 15 one should notice that, from the equation (4.10), for every $f : \Gamma \to \mathbb{R}^+$ continuous with compact support, one has the convergence

$$\lim_{|x| \to 0} \frac{\mathbb{E}_x[f(X_t), t < \kappa_{\Gamma}]}{M(x)} = C \int_{\Gamma} f(y) n_t(y) dy, \qquad t > 0.$$

This together with the decomposition in Lemma 15 implies that we necessarily have the following representation for the entrance law of any self-similiar excursion measure $\mathbf{N}^{\Gamma}$. There is a measure π^{Γ} on Ω, such that $\int_{\Omega} \pi^{\Gamma}(d\theta)M(\theta) < \infty$, and a constant $a \geq 0$ such that

$$\mathbf{N}^{\Gamma}(X_t \in \mathrm{d}y, t < \zeta) = an_t(y)dy + \int_0^{\infty} \frac{dr}{r^{1+\alpha\gamma}} \int_{\Omega} \pi^{\Gamma}(d\theta)\mathbb{E}_{r\theta}[X_t \in \mathrm{d}y,\ t < \kappa_{\Gamma}], \tag{4.61}$$

$a\pi^\Gamma \equiv 0$, if $a > 0$ then $\gamma = \beta/\alpha$, and if $\pi^\Gamma \neq 0$ then $\gamma < \beta/\alpha$, and $\pi^\Gamma M < \infty$.

Furthermore, via cylinder sets, one can check that for $t > 0$ and $A \in \mathcal{F}_t$

$$\mathbf{N}^\Gamma(A\,,\,t<\zeta) = a\mathbb{E}_0^\triangleleft\left[\frac{1}{M(X_t)}\mathbf{1}_A\right] + \int_0^\infty \frac{dr}{r^{1+\alpha\gamma}}\int_\Omega \pi^\Gamma(d\theta)\mathbb{E}_{r\theta}[A\,,\,t<\kappa_\Gamma]. \tag{4.62}$$

with a and π^Γ as above. As a side consequence, we have that the measure $\tilde{\mathbf{N}}^\Gamma$ on $\mathbb{D}$, defined by the relation

$$\tilde{\mathbf{N}}^\Gamma(A\,,\,t<\zeta) := \mathbb{E}_0^\triangleleft\left[\frac{1}{M(X_t)}\mathbf{1}_A\right], \quad \text{for } A \in \mathcal{F}_t,\ t>0, \tag{4.63}$$

is a self-similar excursion measure, whose entrance law is $(n_t, t > 0)$, and such that $\tilde{\mathbf{N}}^\Gamma(X_{0+} \neq 0) = 0$. Finally this is the unique self-similar excursion measure bearing this property, and hence the self-similar recurrent extension associated to it leaves zero continuously, and it is the unique self-similar recurrent extension having this property.

4.13.2 Invariant measure

We start by computing the invariant measure according to Chapter XIX.46 of Dellacherie and Meyer [29]. There it is shown that the invariant measure $\overset{\leftrightarrow}{\pi}{}^\Gamma(\mathrm{d}y)$, $y \in \Gamma$, defined up to a multiplicative constant, is given by the excursion occupation measure so that

$$\int_\Gamma f(y)\,\overset{\leftrightarrow}{\pi}{}^\Gamma(\mathrm{d}y) = \mathbf{N}^\Gamma\left(\int_0^\zeta f(\chi_t)\mathrm{d}t\right),$$

for all bounded measurable f on Γ. Note, however, the computations in Lemma 14 can be used to show that

$$\tilde{\mathbf{N}}^\Gamma\left(\int_0^\zeta f(\chi_t)\mathrm{d}t\right) = \int_0^\infty\int_\Gamma f(z)n_t(z)\mathrm{d}z\,\mathrm{d}t = \int_\Gamma \frac{f(z)}{M(z)}G^\triangleleft(0,\mathrm{d}z) = \int_\Gamma f(z)H(z)\mathrm{d}z.$$

It is then straight forward to prove the final identity in the statement of Theorem 15.

Finally, to see that $\overset{\leftrightarrow}{\pi}$ is not a finite measure, we can compute its total mass, after converting to generalised polar coordinates (see e.g. Blumenson [15]), by

$$\int_\Gamma H(x)\mathrm{d}x = C\int_\Omega \sigma_1(\mathrm{d}\theta)M(\theta)\int_0^\infty r^{\alpha-\beta-1}\mathrm{d}r = \infty, \tag{4.64}$$

where $C > 0$ is an unimportant constant attached to the Jacobian in the change of variables to generalised polar coordinates, and $\sigma_1(\mathrm{d}\theta)$ is the surface measure on $\mathbb{S}^{d-1}$ normalised to have unit total mass. Moreover, we also have that if π^Γ is not trivial then

$$\int_0^\infty \frac{dr}{r^{1+\alpha\gamma}}\int_\Omega \pi^\Gamma(d\theta)\mathbb{E}_{r\theta}\left[\kappa_\Gamma\right] = \infty,$$

because by Proposition 22, $\mathbb{E}_{r\theta}\left[\kappa_\Gamma\right] = \infty$, for any $r > 0$, and Ω.

The failure of this measure to normalise to have unit mass means that a stationary distribution cannot exist, cf. Chapter XIX.46 of Dellacherie and Meyer [29] and hence $\overleftarrow{X}$ is a null-recurrent process. $\square$

4.14 Proof of Theorem 21

The proof of this result follows verbatim that of Theorem 15, albeit for some of the estimates that are used. Indeed, to establish Theorem 15, we used that for $(X, \mathbb{P}^{\Gamma})$ we have

(a) for $t < |x|^{\alpha}$,

$$\mathbb{P}_x(\kappa_\Gamma > t) \approx M(x) t^{-\beta/\alpha};$$

(b) for any $t > 0$ and $f : \Gamma \to \mathbb{R}^+$ continuous and bounded

$$\lim_{|x|\to 0} \frac{\mathbb{P}_x(f(X_t), t < \kappa_\Gamma)}{M(x)} \text{ exists.}$$

These conditions are replaced by the following conditions on $(X, \mathbb{P}^{\triangleright})$

(a') for $t < |x|^{\alpha}$,

$$\mathbb{P}^{\triangleright}_x(\kappa_\Gamma > t) \approx H(x) t^{(\alpha-d-2\beta)/\alpha},$$

(b') for any $t > 0$ and $f : \Gamma \to \mathbb{R}^+$ continuous and bounded

$$\lim_{|x|\to 0} \frac{\mathbb{P}^{\triangleright}_x(f(X_t), t < \kappa_\Gamma)}{H(x)} \text{ exists.}$$

Moreover, in proving Theorem 21, where one reads β in the proof of Theorem 15, one should use $\beta' := 2\beta + d - \alpha$. From here we have the restriction $0 < \beta' = d + 2\beta - \alpha < \alpha$, which restricts β to the interval $((\alpha - d)/2) \vee 0 < \beta < (2\alpha - d)/2$.

Let us finish by noticing that the finiteness of $\mathbf{N}^{\triangleright}(1 - \mathrm{e}^{-\zeta})$ is equivalent to $d < 2(\alpha - \beta)$. To this end, we can appeal to Lemma 4.3 of [6] to see that, there exists a constant $C > 0$, such that, for $x \in \Gamma$,

$$C^{-1} s^{(\alpha-2\beta-d)/\alpha} < \lim_{x\to 0} |x|^{\alpha-2\beta-d} \mathbb{P}^{\triangleright}_x(s < \mathtt{k}^{\{0\}}) < C s^{(\alpha-2\beta-d)/\alpha}, \tag{4.65}$$

where $\mathtt{k}^{\{0\}} = \inf\{t > 0 : |X|_t = 0\}$. Moreover, if we set $J(x) = H(x)/M(x) = |x|^{\alpha-2\beta-d}$, note that, for $x \in \Gamma$, and $A \in \mathcal{F}_t$, $t \geq 0$,

$$\mathbb{E}^{\triangleleft}_x \left[\frac{J(X_t)}{J(x)} \mathbf{1}_A \right] = \mathbb{E}_x \left[\frac{H(X_t)}{H(x)} \mathbf{1}_{(A \cap \{t < \kappa_\Gamma\})} \right] = \mathbb{P}^{\triangleright}_x(A,\, t < \mathtt{k}^{\{0\}}). \tag{4.66}$$

Hence, using (4.66), we have

$$\begin{aligned}\mathbf{N}^{\triangleright}(1-\mathrm{e}^{-\zeta}) &= \int_0^\infty \mathrm{e}^{-s}\mathbf{N}^{\triangleright}(\zeta > s)\mathrm{d}s \\ &= \int_0^\infty \mathrm{d}s\mathrm{e}^{-s}\mathbb{E}_0^{\triangleleft}[J(X_s)] \\ &= \int_0^\infty \mathrm{d}s\mathrm{e}^{-s}\lim_{x\to 0}|x|^{\alpha-2\beta-d}\mathbb{P}_x^{\triangleright}(s < \mathtt{k}^{\{0\}}),\end{aligned}$$

and, thanks to (4.65), the right hand side either converges or explodes depending on whether $d < 2(\alpha-\beta)$. So, remember that by Campbell's theorem, the sum of the lengths $\sum_{s\le t}\zeta_s$ is finite a.s. for any $t>0$, if and only if $\mathbf{N}^{\triangleright}(1-\mathrm{e}^{-\zeta}) < \infty$, which is equivalent to $d < 2(\alpha-\beta)$. This justifies our comment following the statement of Theorem 21.

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Chapter 5

Conclusions

We have considered several problems for self-similar Markov processes in $\mathbb{R}^d$, conditioning a self-similar Markov process to hit/avoid the origin, computing some fluctuation identities for isotropic stable processes, and constructing a recurrent extension of a self-similar Markov process after hitting zero. We have used Lamperti-Kiu representation to translate a problem for self-similar Markov processes into a similar problem for Markov additive processes. This gives us access to many of our principal tools, for example Maisonneuve's exit formula and Markov additive renewal theorem. In high dimensions, we have given special attention to isotropic stable Lévy processes. Our analysis, in this case, benefits from Blumenthal–Getoor–Ray's first passage into a ball formula and the Boundary Harnack Principle for α-harmonic functions.

The methods used in Chapter 3 and Chapter 4 are designed for proving results in the case of isotropic stable processes. However, heuristically, the methods do not seem to be very specific to isotropic stable processes. As a possible future direction, it would be interesting to see if the methods used in this thesis can be adapted to the setting of $\mathbb{R}^d$ self-similar Markov process in general.