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Perpetual Integrals for Lévy Processes

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Abstract Given a Lévy process ξ, we find necessary and sufficient conditions for almost sure finiteness of the perpetual integral $\int_0^\infty f(\xi_s)\mathrm{d}s$, where f is a positive locally integrable function. If $\mu = \mathbb{E}[\xi_1] \in (0, \infty)$ and ξ has local times we prove the 0–1 law

$$\mathbb{P}\Big(\int_0^\infty f(\xi_s)\,\mathrm{d}s < \infty \Big) \in \{0, 1\}$$

with the exact characterization

$$\mathbb{P}\Big(\int_0^\infty f(\xi_s)\,\mathrm{d}s < \infty \Big) = 0 \qquad \Longleftrightarrow \qquad \int^\infty f(x)\,\mathrm{d}x = \infty.$$

The proof uses spatially stationary Lévy processes, local time calculations, Jeulin's lemma and the Hewitt–Savage 0–1 law.

Keywords Lévy processes · Fluctuation theory · Perpetual integral

Mathematics Subject Classification 60J25 · 60J55 · 60J75

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Main Result

The study of perpetual integrals $\int f(X_s)\mathrm{d}s$ with finite or infinite horizon for diffusion processes X has a long history partially because of their use in the analysis of stochastic differential equations and insurance, financial mathematics as the present value of a continuous stream of perpetuities.

The main result of the present article is a characterization of finiteness for perpetual integrals of Lévy processes:

Theorem 1 *Suppose that ξ is a Lévy process that has strictly positive mean $\mu < \infty$, local times and is not a compound Poisson process. If f is a measurable locally integrable positive function, then the following 0–1 law holds:*

$$\mathbb{P}\Big(\int_0^\infty f(\xi_s)\,\mathrm{d}s < \infty\Big) = 1 \quad \Longleftrightarrow \quad \int^\infty f(x)\,\mathrm{d}x < \infty \tag{T1}$$

and

$$\mathbb{P}\Big(\int_0^\infty f(\xi_s)\,\mathrm{d}s < \infty\Big) = 0 \quad \Longleftrightarrow \quad \int^\infty f(x)\,\mathrm{d}x = \infty. \tag{T2}$$

Let us briefly compare the theorem with the existing literature:

(i) If ξ is a Brownian motion with positive drift, then results were obtained through the Ray-Knight theorem, Jeulin's lemma and Khashminkii's lemma by Salminen and Yor [10,11].

(ii) For spectrally negative ξ, i.e., ξ only jumps downwards, the equivalence was obtained in Khoshnevisan et al. [11], see also Example 3.9 of Schilling and Voncracek [8]. The spectrally negative case also turns out to be easier in our proof.

(iii) If f is (ultimately) decreasing, results for general Lévy processes have been proved in Erickson and Maller [5]. In this case, the result stated in Theorem 1 follows easily from the law of large numbers by estimating $2\mu t > \xi_t > \frac{1}{2}\mu t$ for t big enough. The very same argument also shows that for $\mu = +\infty$ the integral test (T) fails in general.

Remark 2 It is not clear whether or not the assumption that ξ has local time is necessary for (T) to hold. For (ultimately) decreasing f, the existence of local time is clearly not needed, whereas we have no conjecture for general f.

Proof of Theorem 1

Before going into the proof, let us fix some notation and facts that will be needed later on. For additional background on the theory of Lévy processes, we refer for instance to [1] or [7]. The law of ξ issued from $x \in \mathbb{R}$ will be denoted by $\mathbb{P}^x$, abbreviating $\mathbb{P} = \mathbb{P}^0$, and the characteristic exponent is defined as

$$\Psi(\lambda) := -\log \mathbb{E}\big[\exp(i\lambda\xi_1)\big], \quad \lambda \in \mathbb{R}.$$

We recall from Theorem V.1 of [1] that ξ has local times $\big(L_t(x)\big)_{t\geq 0, x\in\mathbb{R}}$ if and only if

$$\int_{-\infty}^{\infty} \mathcal{R}\left(\frac{1}{1+\Psi(r)}\right) \mathrm{d}r < \infty. \tag{1}$$

This means that, for any bounded measurable function $f : \mathbb{R} \to [0,\infty)$, the occupation time formula

$$\int_0^t f(\xi_s)\,\mathrm{d}s = \int_{\mathbb{R}} f(x)L_t(x)\,\mathrm{d}x, \quad t \geq 0,$$

holds almost surely. An additional consequence of (1) is that points are non-polar. More precisely, a Theorem of Kesten and Bretagnolle states that $\mathbb{P}(\tau_x < \infty) > 0$ for all $x > 0$, where $\tau_x = \inf\{t : \xi_t = x\}$. See for instance Theorem 7.12 of [7]. Our assumption of a finite and strictly positive mean implies that ξ is transient, hence, $\int_{-\varepsilon}^{\varepsilon} \mathcal{R}(\frac{1}{\psi(r)})\,\mathrm{d}r < \infty$ and, consequently, (1) implies $\int_{-\infty}^{\infty} \mathcal{R}\left(\frac{1}{\Psi(r)}\right) \mathrm{d}r < \infty$. In that case, Theorem II.16 of [1] implies that the potential measure

$$U(\mathrm{d}x) = \int_0^{\infty} \mathbb{P}\big(\xi_s \in \mathrm{d}x\big)\,\mathrm{d}s$$

has a bounded density $u(x)$ with respect to the Lebesgue measure.

We start with the easy direction of Theorem 1:

Proof of Theorem 1, Sufficiency of Integral Test Suppose that $\int_{\mathbb{R}} f(x)\mathrm{d}x < \infty$. It follows from the assumption $\mu \in (0,\infty)$ that ξ is transient. Since ξ is furthermore assumed to have a local time we can use the existence and boundedness of the potential density to obtain

$$\begin{aligned}\mathbb{E}\Big[\int_0^{\infty} f(\xi_s)\,\mathrm{d}s\Big] &= \int_{\mathbb{R}} f(x) \int_0^{\infty} \mathbb{P}(\xi_s \in \mathrm{d}x)\,\mathrm{d}s \\ &= \int_{\mathbb{R}} f(x)u(x)\,\mathrm{d}x \leq \sup_{x\in\mathbb{R}} u(x) \int_{\mathbb{R}} f(x)\,\mathrm{d}x < \infty.\end{aligned}$$

Finiteness of the expectation implies almost sure finiteness of $\int_0^{\infty} f(\xi_s)\,\mathrm{d}s$, thus, the sufficiency of the integral test for almost sure finiteness of the perpetual integral is proved. □

For the reverse direction, we use Jeulin's lemma, here is a simple version:

Lemma 3 *Suppose $(X_x)_{x\in\mathbb{R}}$ are non-negative, non-trivial and identically distributed random variables on some probability space $(\Omega, \mathcal{F}, P)$. Then*

$$P\Big(\int_{\mathbb{R}} f(x)X_x\,\mathrm{d}x < \infty\Big) = 1 \quad \Longrightarrow \quad \int_{\mathbb{R}} f(x)\,\mathrm{d}x < \infty.$$

Proof Since X_x are identically distributed, we may choose $\varepsilon > 0$ so that $P(X_x > \varepsilon) = \delta > 0$, for all $x \in \mathbb{R}$. Since $\int_{\mathbb{R}} f(x) X_x \, \mathrm{d}x$ is almost surely finite, there is some $r > 0$ so that $P(A_r) > 1 - \delta/2$ with $A_r = \{\int_{\mathbb{R}} f(x) X_x \, \mathrm{d}x < r\}$. Hence, we have

$$E\big[X_x 1_{A_r}\big] \geq \varepsilon P(\{X_x > \varepsilon\} \cap A_r) > \varepsilon\delta/2 > 0.$$

But then this implies that

$$r \geq r P(A_r) \geq E\Big[\int_{\mathbb{R}} f(x) X_x \, \mathrm{d}x \, 1_{A_r}\Big] = \int_{\mathbb{R}} f(x) E[X_x 1_{A_r}] \, \mathrm{d}x \geq \varepsilon\delta/2 \int_{\mathbb{R}} f(x) \, \mathrm{d}x.$$

The proof is now complete. □

Note that there are different versions of Jeulin's lemma (see for instance [9]). Most commonly, Jeulin's lemma is stated with the assumption that $P(\int_{\mathbb{R}} f(x) X_x \, \mathrm{d}x < \infty) > 0$ with some additional assumptions on the variables $(X_x)_{x \in \mathbb{R}}$. Those extra assumptions on $(X_x)_{x \in \mathbb{R}}$ are not satisfied in our setting, but we will employ a 0–1 law that allows us to work with $P(\int_{\mathbb{R}} f(x) X_x \, \mathrm{d}x < \infty) = 1$.

We would like to apply Jeulin's lemma via the occupation time formula

$$\int_0^\infty f(\xi_s) \, \mathrm{d}s = \lim_{t \uparrow \infty} \int_0^t f(\xi_s) \, \mathrm{d}s = \lim_{t \uparrow \infty} \int_0^\infty f(x) L_t(x) \, \mathrm{d}x = \int_0^\infty f(x) L_\infty(x) \, \mathrm{d}x$$

with $L_\infty(x) := \lim_{t \uparrow \infty} L_t(x)$ playing the role of X_x, $x \in \mathbb{R}$. However, this is not quite in the right format as the distribution of $L_\infty(x)$ depends on x. An exception to this is when ξ is spectrally negative, in which case the laws $L_\infty(x)$ are independent of x by the strong Markov property. To make this idea work in the general framework, we work with a randomized point of issue for ξ instead. We chose a particularly convenient initial distribution motivated by a result from fluctuation theory (see Lemma 3 of [2]). Our assumption $\mathbb{E}[\xi_1] < \infty$ implies that

$$\mathbb{P}\big(\xi_{T_z} - z \in \mathrm{d}y\big) \overset{z \to \infty}{\Longrightarrow} \rho(\mathrm{d}y), \tag{2}$$

where $T_z = \inf\{t \geq 0 : \xi_t \geq z\}$ and the existence of the non-degenerate weak limit ρ, called the stationary overshoot distribution, is a classical result coming from renewal theory. As a stationary overshoot, ρ has the property that

$$\mathbb{P}^\rho\big(\xi_{T_a} - a \in \mathrm{d}y\big) := \int \mathbb{P}^x\big(\xi_{T_a} - a \in \mathrm{d}y\big)\rho(\mathrm{d}x) = \rho(\mathrm{d}y), \quad \forall a > 0,$$

and hence spatial stationarity holds due to the strong Markov property; i.e., under $\mathbb{P}^\rho$,

$$(\xi_t^{(a)})_{t \geq 0} := (\xi_{T_a + t} - a)_{t \geq 0} \tag{3}$$

has law $\mathbb{P}^\rho$ for all $a > 0$. The stationarity property (3) will be the key to applying Jeulin's lemma.

Lemma 4 *For any $x > 0$ we have*

$$\mathbb{P}^{\rho}(L_{\infty}(x) \in dy) = \mathbb{P}^{\rho}(L_{\infty}(1) \in dy).$$

Proof First note that $\mathbb{P}^{\rho}(T_x < \infty) = 1$ for all $x > 0$ and $L_{\cdot}(x)$ only starts to increase at some time at or after T_x. We therefore have that

$$\begin{aligned}
\mathbb{P}^{\rho}(L_{\infty}(x) \in \mathrm{d}y) &= \int_0^{\infty} \mathbb{P}^{z}(L_{\infty}(x) \in \mathrm{d}y)\mathbb{P}^{\rho}(\xi_{T_x} \in \mathrm{d}z) \\
&= \int_0^{\infty} \mathbb{P}^{z+x}(L_{\infty}(x) \in \mathrm{d}y)\mathbb{P}^{\rho}(\xi_{T_x} - x \in \mathrm{d}z) \\
&= \int_0^{\infty} \mathbb{P}^{z+x}(L_{\infty}(x) \in \mathrm{d}y)\rho(\mathrm{d}z) \\
&= \int_0^{\infty} \mathbb{P}^{z}(L_{\infty}(0) \in \mathrm{d}y)\rho(\mathrm{d}z),
\end{aligned}$$

where we have used the strong Markov property, the spatial stationarity of $\mathbb{P}^{\rho}$ and spatial homogeneity of ξ. Since the righthand side is independent of x the proof is complete. □

Next, we use the Hewitt–Savage 0–1 law (see [4]) in order to implement our weaker version of Jeulin's lemma. If $X^0, X^1, \ldots$ denotes a sequence of random variables taking values in some measurable space, then an event $A \in \sigma(X^0, X^1, \ldots)$ is called exchangeable if it is invariant under finite permutations (i.e., only finitely many indices are changed) of the sequence $X^0, X^1, \ldots$. The Hewitt–Savage 0–1 law states that any exchangeable event of an iid sequence has probability 0 or 1.

Lemma 5 *If f is a measurable locally integrable positive function, then $\mathbb{P}\big(\int_0^{\infty} f(\xi_s)\, ds < \infty\big) \in \{0, 1\}$.*

Proof The idea of the proof is to write $\Lambda := \{\int_0^{\infty} f(\xi_s)\mathrm{d}s < \infty\}$ as an exchangeable event with respect to the iid increments of ξ on intervals $[n, n+1]$ so that $\mathbb{P}(\Lambda) \in \{0, 1\}$. Let $\mathcal{D}$ denote the RCLL functions $w : [0, 1] \to \mathbb{R}$. If ξ is the given Lévy process, then define the increment processes as

$$(\xi^n_t)_{t\in[0,1]} = (\xi_{n+t} - \xi_n)_{t\in[0,1]}.$$

The Lévy property implies that the sequence $\xi^0, \xi^1, \ldots$ is iid on $\mathcal{D}$. Furthermore, note that ξ can be reconstructed from the ξ^n through

$$\xi_r = \xi^n_{r-n} + \sum_{i=0}^{n-1} \xi^i_1 \qquad \forall r \in [n, n+1).$$

Using that $g_1 : (w_t)_{t\in[0,1)} \mapsto (w_1)_{t\in[0,1)}$, $g_2 : (w, w')_{t\in[0,1)} \mapsto (w_t + w'_t)_{t\in[0,1)}$ and $g_3 : (w_t)_{t\in[0,1)} \mapsto \int_0^1 f(w_s)\mathrm{d}s$ are measurable mappings, we have that there are measurable mappings $g^n : \mathcal{D}^n \to \mathbb{R}$ satisfying

$$\int_0^1 f\Big(\xi_r^n + \sum_{i=0}^{n-1} \xi_1^i\Big)\,\mathrm{d}r = g^n(\xi^0, ..., \xi^n).$$

As a consequence, we find that

$$\begin{aligned}\Big\{\int_0^\infty f(\xi_s)\,\mathrm{d}s < \infty\Big\} &= \Big\{\sum_{n=0}^\infty \int_n^{n+1} f(\xi_s)\,\mathrm{d}s < \infty\Big\}\\ &= \Big\{\sum_{n=0}^\infty \int_0^1 f\Big(\xi_r^n + \sum_{i=0}^{n-1}\xi_1^i\Big)\,\mathrm{d}r < \infty\Big\}\\ &= \Big\{\sum_{n=0}^\infty g^n(\xi^0, ..., \xi^n) < \infty\Big\}\\ &\in \sigma(\xi^0, \xi^1, ...).\end{aligned}$$

Since clearly Λ is exchangeable for $\xi^0, \xi^1, ...$ the Hewitt–Savage 0–1 law implies the claim. □

Lemma 6 *Suppose* $\mathbb{P}(\int_0^\infty f(\xi_s)ds < \infty) = 1$, *then* $\mathbb{P}^\rho(\int_0^\infty f(\xi_s)ds < \infty) = 1$.

Proof The statement is obvious if ξ is a subordinator, so we assume this is not the case.

Next we show that $\mathbb{P}^x(\int_0^\infty f(\xi_s)\mathrm{d}s < \infty) = 1$ for any $x > 0$. To see this, we use the strong Markov property at $\tau_0 = \inf\{t : \xi_t = 0\}$ which is finite with positive probability since points in $\mathbb{R}$ are non-polar:

$$\begin{aligned}\mathbb{P}^x\Big(\int_0^\infty f(\xi_s)\,\mathrm{d}s < \infty\Big) &\geq \mathbb{P}^x\Big(\int_{\tau_0}^\infty f(\xi_s)\,\mathrm{d}s < \infty, \tau_0 < \infty\Big)\\ &= \mathbb{P}^x\Big(\int_0^\infty f(\xi_{s+\tau_0} - \xi_{\tau_0})\,\mathrm{d}s < \infty, \tau_0 < \infty\Big)\\ &= \mathbb{P}^0\Big(\int_0^\infty f(\xi_s)\,\mathrm{d}s < \infty\Big)\mathbb{P}^x(\tau_0 < \infty)\\ &> 0.\end{aligned}$$

But then the 0–1 law of Lemma 5 implies that $\mathbb{P}^x\Big(\int_0^\infty f(\xi_s)\,\mathrm{d}s < \infty\Big) = 1$. Finally, we obtain

$$\mathbb{P}^\rho\Big(\int_0^\infty f(\xi_s)\,\mathrm{d}s < \infty\Big) = \int_{\mathbb{R}} \mathbb{P}^x\Big(\int_0^\infty f(\xi_s)\,\mathrm{d}s < \infty\Big)\rho(\mathrm{d}x) = \int_{\mathbb{R}} \rho(\mathrm{d}x) = 1$$

and the proof is complete. □

Now we are ready to prove the more delicate part of Theorem 1.

Proof of Theorem 1, Necessity of Integral Test Suppose $\mathbb{P}(\int_0^\infty f(\xi_s)\,\mathrm{d}s < \infty) > 0$, which then implies $\mathbb{P}(\int_0^\infty f(\xi_s)\,\mathrm{d}s < \infty) = 1$ by Lemma 5 and hence, $\mathbb{P}^\rho(\int_0^\infty f(\xi_s)\,\mathrm{d}s < \infty) = 1$ by Lemma 6. Using the occupation time formula, we get

$$\begin{aligned}\int_0^\infty f(\xi_s)\,\mathrm{d}s &= \lim_{t\to\infty}\int_0^t f(\xi_s)\,\mathrm{d}s\\ &= \lim_{t\to\infty}\int_{\mathbb{R}} f(x)L_t(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)L_\infty(x)\,\mathrm{d}x \qquad \mathbb{P}^\rho\text{-a.s.}\end{aligned}$$

In Lemma 4, we proved that $L_\infty(x)$ is independent of x under $\mathbb{P}^\rho$ so that Jeulin's Lemma implies $\int_{\mathbb{R}} f(x)\mathrm{d}x < \infty$. $\square$

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