Self-similar Markov processes
Part II: higher dimensions

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A more thorough set of lecture notes can be found here:
`https://arxiv.org/abs/1707.04343`
Other related material found here
`https://arxiv.org/abs/1511.06356`
`https://arxiv.org/abs/1706.09924`

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§7. Isotropic stable processes in dimension $d \geq 2$ seen as Lévy processes

Isotropic α-stable process in dimension $d \geq 2$

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- Associated Lévy measure satisfies, for $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned}\Pi(B) &= \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|} \int_B \frac{1}{|y|^{\alpha+d}} \mathrm{d}y \\ &= \frac{2^{\alpha-1}\Gamma((d+\alpha)/2)\Gamma(d/2)}{\pi^d \left|\Gamma(-\alpha/2)\right|} \int_{\mathbb{S}_{d-1}} r^{d-1} \sigma_1(\mathrm{d}\theta) \int_0^\infty \mathbf{1}_B(r\theta) \frac{1}{r^{\alpha+d}} \mathrm{d}r,\end{aligned}$$

  where $\sigma_1(\mathrm{d}\theta)$ is the surface measure on $\mathbb{S}_{d-1}$ normalised to have unit mass.
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- Stable processes are also self-similar. For $c > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$,

$$\text{under } \mathbb{P}_x, \text{ the law of } (cX_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P}_{cx}.$$

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- Isotropy means, for all orthogonal transformations (e.g. rotations) $U : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $x \in \mathbb{R}^d$,

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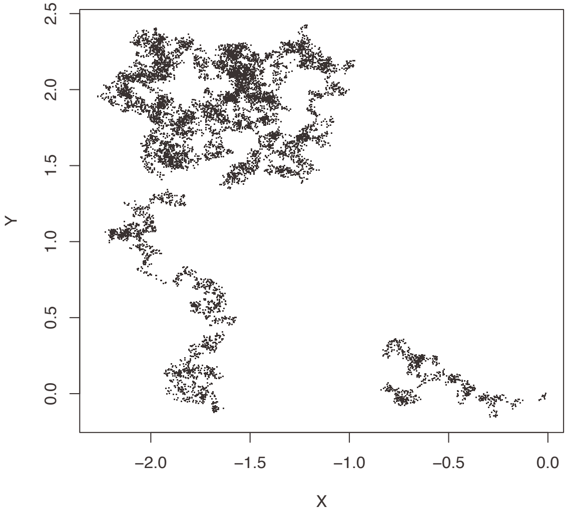
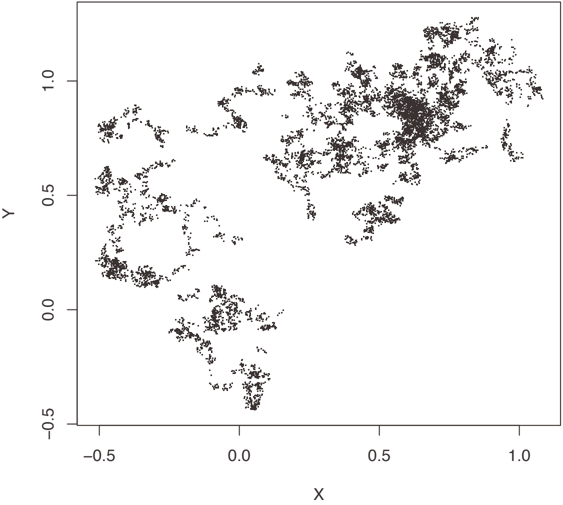
- If $(S_t, t \geq 0)$ is a stable subordinator with index $\alpha/2$ (a Lévy process with Laplace exponent $-t^{-1} \log \mathbb{E}[\mathrm{e}^{-\lambda S_t}] = \lambda^\alpha$) and $(B_t, t \geq 0)$ for a standard (isotropic) d-dimensional Brownian motion, then it is known that $X_t := \sqrt{2} B_{S_t}$, $t \geq 0$, is a stable process with index α.

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\theta X_t}] = \mathbb{E}\left[\mathrm{e}^{-\theta^2 S_t}\right] = \mathrm{e}^{-|\theta|^\alpha t}, \qquad \theta \in \mathbb{R}.$$

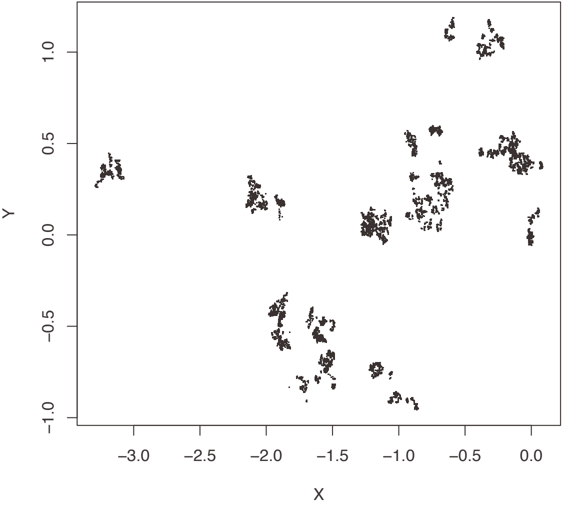
SAMPLE PATH, $\alpha = 1.9$



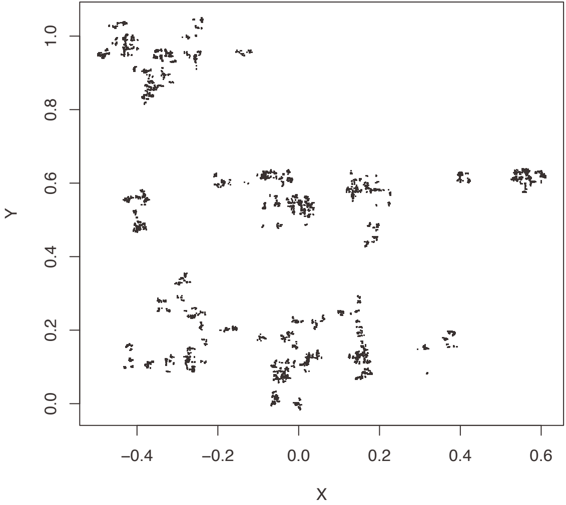
SAMPLE PATH, $\alpha = 1.7$



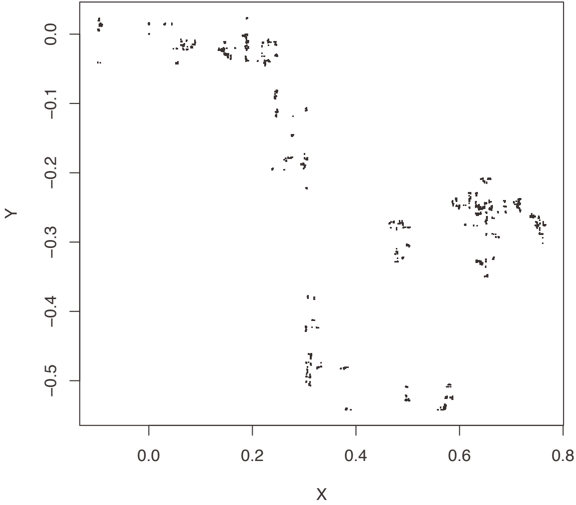
SAMPLE PATH, $\alpha = 1.5$



SAMPLE PATH, $\alpha = 1.2$



Sample path, $\alpha = 0.9$



Some classical properties: Transience

We are interested in the potential measure

$$U(x, \mathrm{d}y) = \int_0^\infty \mathbb{P}_x(X_t \in \mathrm{d}y)\mathrm{d}t = \left(\int_0^\infty \mathrm{p}_t(y-x)\mathrm{d}t\right)\mathrm{d}y, \qquad x, y \in \mathbb{R}.$$

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Theorem

The potential of X is absolutely continuous with respect to Lebesgue measure, in which case, its density in collaboration with spatial homogeneity satisfies $U(x, \mathrm{d}y) = u(y-x)\mathrm{d}y$, $x, y \in \mathbb{R}^d$, where

$$u(z) = 2^{-\alpha}\pi^{-d/2}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}|z|^{\alpha-d}, \qquad z \in \mathbb{R}^d.$$

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In this respect X is transient. It can be shown moreover that

$$\lim_{t\to\infty}|X_t| = \infty$$

almost surely

PROOF OF THEOREM

Now note that, for bounded and measurable $f : \mathbb{R}^d \mapsto \mathbb{R}^d$,

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty f(X_t)\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^\infty f(\sqrt{2}B_{S_t})\mathrm{d}t\right] \\
&= \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}t\,\mathbb{P}(S_t \in \mathrm{d}s) \int_{\mathbb{R}} \mathbb{P}(B_s \in \mathrm{d}x) f(\sqrt{2}x) \\
&= \frac{1}{\Gamma(\alpha/2)\pi^{d/2}2^d} \int_{\mathbb{R}} \mathrm{d}y \int_0^\infty \mathrm{d}s\,\mathrm{e}^{-|y|^2/4s} s^{-1+(\alpha-d)/2} f(y) \\
&= \frac{1}{2^\alpha\Gamma(\alpha/2)\pi^{d/2}} \int_{\mathbb{R}} \mathrm{d}y\,|y|^{(\alpha-d)} \int_0^\infty \mathrm{d}u\,e^{-u} u^{-1+(d-\alpha/2)} f(y) \\
&= \frac{\Gamma((d-\alpha)/2)}{2^\alpha\Gamma(\alpha/2)\pi^{d/2}} \int_{\mathbb{R}} \mathrm{d}y\,|y|^{(\alpha-d)} f(y).
\end{aligned}$$

Some classical properties: Polarity

- Kesten-Bretagnolle integral test, in dimension $d \geq 2$,

$$\int_{\mathbb{R}} \mathrm{Re}\left(\frac{1}{1+\Psi(z)}\right) \mathrm{d}z = \int_{\mathbb{R}} \frac{1}{1+|z|^{\alpha}} \mathrm{d}z \propto \int_{\mathbb{R}} \frac{1}{1+r^{\alpha}} r^{d-1} \mathrm{d}r \, \sigma_1(\mathrm{d}\theta) = \infty.$$

- $\mathbb{P}_x(\tau^{\{y\}} < \infty) = 0$, for $x, y \in \mathbb{R}^d$.
- i.e. the stable process cannot hit individual points almost surely.

§8. Isotropic stable processes in dimension $d \geq 2$ seen as a self-similar Markov process

Lamperti-transform of $|X|$

Theorem (Caballero-Pardo-Perez (2011))

For the pssMp constructed using the radial part of an isotropic d-dimensional stable process, the underlying Lévy process, ξ that appears through the Lamperti has characteristic exponent given by

$$\Psi(z) = 2^{\alpha}\frac{\Gamma(\frac{1}{2}(-\mathrm{i}z+\alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)}\frac{\Gamma(\frac{1}{2}(\mathrm{i}z+d))}{\Gamma(\frac{1}{2}(\mathrm{i}z+d-\alpha))}, \qquad z \in \mathbb{R}.$$

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Here are some facts that can be deduced from the above Theorem that are exercises in the tutorial:

- The fact that $\lim_{t\to\infty} |X_t| = \infty$
- The fact that

$$|X_t|^{\alpha-d}, \qquad t \geq 0,$$

  is a martingale.

Conditioned stable process

- We can define the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^\circ_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \qquad t \geq 0, x \neq 0$$

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- Suppose that f is a bounded measurable function then, for all $c > 0$,

$$\begin{aligned}\mathbb{E}_x^\circ[f(cX_{c^{-\alpha}s}, s \leq t)] &= \mathbb{E}_x\left[\frac{|cX_{c^{-\alpha}t}|^{\alpha-d}}{|cx|^{d-\alpha}} f(cX_{c^{-\alpha}s}, s \leq t)\right] \\ &= \mathbb{E}_{cx}\left[\frac{|X_t|^{\alpha-d}}{|cx|^{d-\alpha}} f(X_s, s \leq t)\right] = \mathbb{E}_{cx}^\circ[f(X_s, , s \leq t)]\end{aligned}$$

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- Similarly $(|X|, \mathbb{P}^\circ_x)$, $x\neq 0$ is a positive self-similar Markov process.

Conditioned stable process

- It turns out that $(X, \mathbb{P}^\circ_x)$, $x \neq 0$, corresponds to the stable process conditioned to be continuously absorbed at the origin.

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- More precisely, for $A \in \sigma(X_s, s \leq t)$, if we set $\{0\}$ to be 'cemetery' state and $\mathtt{k} = \inf\{t > 0 : X_t = 0\}$, then

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- In light of the associated Esscher transform on ξ, we note that the Lamperti transform of $(|X|, \mathbb{P}^\circ_x)$, $x \neq 0$, corresponds to the Lévy process with characteristic exponent

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- Given the pathwise interpretation of $(X, \mathbb{P}_x^\circ)$, $x \neq 0$, it follows immediately that $\lim_{t \to \infty} \xi_t = -\infty$, $\mathbb{P}_x^\circ$ almost surely, for any $x \neq 0$.

$\mathbb{R}^d$-SELF-SIMILAR MARKOV PROCESSES

Definition

A $\mathbb{R}^d$-valued regular Feller process $Z = (Z_t, t \geq 0)$ is called a $\mathbb{R}^d$*-valued self-similar Markov process* if there exists a constant $\alpha > 0$ such that, for any $x > 0$ and $c > 0$,

$$\text{the law of } (cZ_{c^{-\alpha}t}, t \geq 0) \text{ under } P_x \text{ is } P_{cx},$$

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- Same definition as before except process now lives on $\mathbb{R}^d$.
- Is there an analogue of the Lamperti representation?

Lamperti–Kiu transform

In order to introduce the analogue of the Lamperti transform in d-dimensions, we need to remind ourselves of what we mean by a Markov additive process in this context.

Definition

An $\mathbb{R} \times E$ valued regular Feller process $(\xi, \Theta) = ((\xi_t, \Theta_t) : t \geq 0)$ with probabilities $\mathbf{P}_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in E$, and cemetery state $(-\infty, \dagger)$ is called a *Markov additive process* (MAP) if Θ is a regular Feller process on E with cemetery state $\dagger$ such that, for every bounded measurable function $f : (\mathbb{R} \cup \{-\infty\}) \times (E \cup \{\dagger\}) \to \mathbb{R}$, $t, s \geq 0$ and $(x, \theta) \in \mathbb{R} \times E$, on $\{t < \varsigma\}$,

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where $\varsigma = \inf\{t > 0 : \Theta_t = \dagger\}$.

- Roughly speaking, one thinks of a MAP as a 'Markov modulated' Lévy process

Lamperti–Kiu transform

In order to introduce the analogue of the Lamperti transform in d-dimensions, we need to remind ourselves of what we mean by a Markov additive process in this context.

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- Roughly speaking, one thinks of a MAP as a 'Markov modulated' Lévy process
- It has 'conditional stationary and independent increments'
- Think of the E-valued Markov process Θ as modulating the characteristics of ξ (which would otherwise be a Lévy processes).

Lamperti–Kiu transform

Theorem

Fix $\alpha > 0$. The process Z is a ssMp with index α if and only if there exists a (killed) MAP, (ξ, Θ) on $\mathbb{R} \times \mathbb{S}_{d-1}$ such that

$$Z_t := \mathrm{e}^{\xi_{\varphi(t)}} \Theta_{\varphi(t)} \qquad , \qquad t \leq I_\varsigma,$$

where

$$\varphi(t) = \inf\left\{ s > 0 : \int_0^s \mathrm{e}^{\alpha \xi_u}\, \mathrm{d}u > t \right\}, \qquad t \leq I_\varsigma,$$

and $I_\varsigma = \int_0^\varsigma \mathrm{e}^{\alpha\xi_s} \mathrm{d}s$ is the lifetime of Z until absorption at the origin. Here, we interpret $\exp\{-\infty\} \times \dagger := 0$ and $\inf \emptyset := \infty$.

- In the above representation, the time to absorption in the origin,

$$\zeta = \inf\{t > 0 : Z_t = 0\},$$

  satisfies $\zeta = I_\varsigma$.
- Note $x \in \mathbb{R}^d$ if and only if

$$x = (|x|, \mathrm{Arg}(x)),$$

  where $\mathrm{Arg}(x) = x/|x| \in \mathbb{S}_{d-1}$. The Lamperti–Kiu decomposition therefore gives us a d-dimensional skew product decomposition of self-similar Markov processes.

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- **How do we characterise its underlying MAP** (ξ, Θ)?
- We already know that $|X|$ is a positive similar Markov process and hence ξ is a Lévy process, albeit corollated to Θ
- What properties does Θ and what properties to the pair (ξ, Θ) have?

MAP ISOTROPY

Theorem

Suppose (ξ, Θ) is the MAP underlying the stable process. Then $((\xi, U^{-1}\Theta), \mathbf{P}_{x,\theta})$ is equal in law to $((\xi, \Theta), \mathbf{P}_{x,U^{-1}\theta})$, for every orthogonal d-dimensional matrix U and $x \in \mathbb{R}^d$, $\theta \in \mathbb{S}_{d-1}$.

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Proof.

First note that $\varphi(t) = \int_0^t |X_u|^{-\alpha} \mathrm{d}u$. It follows that

$$(\xi_t, \Theta_t) = (\log |X_{A(t)}|, \operatorname{Arg}(X_{A(t)})), \qquad t \geq 0,$$

where the random times $A(t) = \inf\left\{s > 0 : \int_0^s |X_u|^{-\alpha} \mathrm{d}u > t\right\}$ are stopping times in the natural filtration of X.

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Now suppose that U is any orthogonal d-dimensional matrix and let $X' = U^{-1}X$. Since X is isotropic and since $|X'| = |X|$, and $\operatorname{Arg}(X') = U^{-1}\operatorname{Arg}(X)$, we see that, for $x \in \mathbb{R}$ and $\theta \in \mathbb{S}_{d-1}$

$$\begin{aligned}
((\xi, U^{-1}\Theta), \mathbf{P}_{\log|x|,\theta}) &= ((\log |X_{A(\cdot)}|, U^{-1}\operatorname{Arg}(X_{A(\cdot)})), \mathbb{P}_x) \\
&\stackrel{d}{=} ((\log |X_{A(\cdot)}|, \operatorname{Arg}(X_{A(\cdot)})), \mathbb{P}_{U^{-1}x}) \\
&= ((\xi, \Theta), \mathbf{P}_{\log|x|,U^{-1}\theta})
\end{aligned}$$

as required. $\square$

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Theorem (Bo Li, Victor Rivero, Bertoin-Werner (1996))

Suppose that f is a bounded measurable function on $[0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}_{d-1} \times \mathbb{S}_{d-1}$ such that $f(\cdot,\cdot,0,\cdot,\cdot) = 0$, then, for all $\theta \in \mathbb{S}_{d-1}$,

$$\mathbf{E}_{0,\theta}\left(\sum_{s>0} f(s,\xi_{s-},\Delta\xi_s,\Theta_{s-},\Theta_s)\right)$$
$$= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{R}} V_\theta(\mathrm{d}s,\mathrm{d}x,\mathrm{d}\vartheta)\sigma_1(\mathrm{d}\phi)\mathrm{d}y \frac{c(\alpha)\mathrm{e}^{yd}}{|\mathrm{e}^y\phi - \vartheta|^{\alpha+d}} f(s,x,y,\vartheta,\phi),$$

where

$$V_\theta(\mathrm{d}s, dx, \mathrm{d}\vartheta) = \mathbf{P}_{0,\theta}(\xi_s \in \mathrm{d}x, \Theta_s \in \mathrm{d}\vartheta)\mathrm{d}s, \qquad x \in \mathbb{R}, \vartheta \in \mathbb{S}_{d-1}, s \geq 0,$$

is the space-time potential of (ξ,Θ) under $\mathbf{P}_{0,\theta}$, $\sigma_1(\phi)$ is the surface measure on $\mathbb{S}_{d-1}$ normalised to have unit mass and

$$c(\alpha) = 2^{\alpha-1}\pi^{-d}\Gamma((d+\alpha)/2)\Gamma(d/2)/\left|\Gamma(-\alpha/2)\right|.$$

MAP OF $(X, \mathbb{P}^\circ_\cdot)$

- Recall that $(|X_t|^{\alpha-d}, t \geq 0)$, is a martingale.
- Informally, we should expect $\mathcal{L}h = 0$, where $h(x) = |x|^{\alpha-d}$ and $\mathcal{L}$ is the infinitesimal generator of the stable process, which has action

$$\mathcal{L}f(x) = \mathtt{a} \cdot \nabla f(x) + \int_{\mathbb{R}^d} [f(x+y) - f(x) - \mathbf{1}_{(|y|\leq 1)} y \cdot \nabla f(x)]\Pi(\mathrm{d}y), \qquad |x| > 0,$$

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- Equivalently, the rate at which $(X, \mathbb{P}^\circ_x)$, $x \neq 0$ jumps given by

$$\Pi^\circ(x, B) := \frac{2^{\alpha-1}\Gamma((d+\alpha)/2)\Gamma(d/2)}{\pi^d |\Gamma(-\alpha/2)|} \int_{\mathbb{S}_{d-1}} \mathrm{d}\sigma_1(\phi) \int_{(0,\infty)} \mathbf{1}_B(r\phi) \frac{\mathrm{d}r}{r^{\alpha+1}} \frac{|x + r\phi|^{\alpha-d}}{|x|^{\alpha-d}},$$

for $|x| > 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$.

MAP OF $(X, \mathbb{P}^\circ_\cdot)$

Theorem

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where

$$V^\circ_\theta(\mathrm{d}s, dx, \mathrm{d}\vartheta) = \mathbf{P}^\circ_{0,\theta}(\xi_s \in \mathrm{d}x, \Theta_s \in \mathrm{d}\vartheta)\mathrm{d}s, \qquad x \in \mathbb{R}, \vartheta \in \mathbb{S}_{d-1}, s \geq 0,$$

is the space-time potential of (ξ, Θ) under $\mathbf{P}^\circ_{0,\theta}$.

Comparing the right-hand side above with that of the previous Theorem, it now becomes immediately clear that the the jump structure of (ξ, Θ) under $\mathbf{P}^\circ_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in \mathbb{S}_{d-1}$, is precisely that of $(-\xi, \Theta)$ under $\mathbf{P}_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in \mathbb{S}_{d-1}$.

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§9. Riesz–Bogdan–Żak transform

Riesz–Bogdan–Żak transform

- Define the transformation $K : \mathbb{R}^d \mapsto \mathbb{R}^d$, by

$$Kx = \frac{x}{|x|^2}, \qquad x \in \mathbb{R}^d \setminus \{0\}.$$

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- Write $x \in \mathbb{R}^d$ in skew product form $x = (|x|, \mathrm{Arg}(x))$, and note that

$$Kx = (|x|^{-1}, \mathrm{Arg}(x)), \qquad x \in \mathbb{R}^d \backslash \{0\},$$

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Theorem (d-dimensional Riesz–Bogdan–Żak Transform, $d \geq 2$)

Suppose that X is a d-dimensional isotropic stable process with $d \geq 2$. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0. \tag{1}$$

Then, for all $x \in \mathbb{R}^d \backslash \{0\}$, $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^\circ_{Kx})$.

Proof of Riesz–Bogdan–Żak transform

We give a proof, different to the original proof of Bogdan and Żak (2010).

- Recall that $X_t = \mathrm{e}^{\xi_{\varphi(t)}}\Theta_{\varphi(t)}$, where

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- Differentiating,

$$\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = \mathrm{e}^{-\alpha\xi_{\varphi(t)}} \text{ and } \frac{\mathrm{d}\eta(t)}{\mathrm{d}t} = \mathrm{e}^{2\alpha\xi_{\varphi\circ\eta(t)}}, \qquad \eta(t) < \tau^{\{0\}}.$$

  and chain rule now tells us that

$$\frac{\mathrm{d}(\varphi\circ\eta)(t)}{\mathrm{d}t} = \left.\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s}\right|_{s=\eta(t)} \frac{\mathrm{d}\eta(t)}{\mathrm{d}t} = \mathrm{e}^{\alpha\xi_{\varphi\circ\eta(t)}}.$$

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- Said another way,
$$\int_0^{\varphi\circ\eta(t)} \mathrm{e}^{-\alpha\xi_u}\mathrm{d}u = t, \qquad t \geq 0,$$
or
$$\varphi\circ\eta(t) = \inf\{s > 0 : \int_0^s \mathrm{e}^{-\alpha\xi_u}\mathrm{d}u > t\}$$

Proof of Riesz–Bogdan–Żak transform

- Next note that

$$KX_{\eta(t)} = \mathrm{e}^{-\xi_{\varphi\circ\eta(t)}}\Theta_{\varphi\circ\eta(t)}, \qquad t \geq 0,$$

and we have just shown that

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- We have also seen that $(X, \mathbb{P}^\circ_x)$, $x \neq 0$, is also a self-similar Markov process with underlying MAP given by $(-\xi, \Theta)$.

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- We have also seen that $(X, \mathbb{P}^\circ_x)$, $x \neq 0$, is also a self-similar Markov process with underlying MAP given by $(-\xi, \Theta)$.
- The statement of the theorem follows.

§10. Hitting spheres

Port's Sphere hitting probability

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Theorem (Port (1969))

If $\alpha \in (1,2)$*, then*

$$\mathbb{P}_x(\tau^{\odot} < \infty)$$
$$= \frac{\Gamma\left(\frac{\alpha+d}{2} - 1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha - 1)} \begin{cases} {}_2F_1((d-\alpha)/2, 1-\alpha/2, d/2; |x|^2) & 1 > |x| \\ |x|^{\alpha-d} {}_2F_1((d-\alpha)/2, 1-\alpha/2, d/2; 1/|x|^2) & 1 \leq |x|. \end{cases}$$

Otherwise, if $\alpha \in (0,1]$*, then* $\mathbb{P}_x(\tau^{\odot} = \infty) = 1$ *for all* $x \in \mathbb{R}^d$.

Proof of Port's hitting probability

- If (ξ, Θ) is the underlying MAP then

$$\mathbb{P}_x(\tau^{\odot} < \infty) = \mathbf{P}_{\log|x|}(\tau^{\{0\}} < \infty) = \mathbf{P}_0(\tau^{\{\log(1/|x|)\}} < \infty),$$

where $\tau^{\{z\}} = \inf\{t > 0 : \xi_t = z\}$, $z \in \mathbb{R}$. (Note, the time change in the Lamperti–Kiu representation does not level out.)

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- Using Sterling's formula, we have, $|\Gamma(x + iy)| = \sqrt{2\pi} e^{-\frac{\pi}{2}|y|}|y|^{x-\frac{1}{2}}(1 + o(1))$, for $x, y \in \mathbb{R}$, as $y \to \infty$, uniformly in any finite interval $-\infty < a \leq x \leq b < \infty$. Hence,

$$\frac{1}{\Psi(z)} = \frac{\Gamma(-\frac{1}{2}\mathrm{i}z)}{\Gamma(\frac{1}{2}(-\mathrm{i}z + \alpha))} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z + d - \alpha))}{\Gamma(\frac{1}{2}(\mathrm{i}z + d))} \sim |z|^{-\alpha}$$

uniformly on $\mathbb{R}$ as $|z| \to \infty$.

PROOF OF PORT'S HITTING PROBABILITY

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uniformly on $\mathbb{R}$ as $|z| \to \infty$.

- From Kesten-Brestagnolle integral test we conclude that $(1 + \Psi(z))^{-1}$ is integrable and each sphere $\mathbb{S}_{d-1}$ can be reached with positive probability from any x with $|x| \neq 1$ if and only if $\alpha \in (1, 2)$.

Proof of Port's hitting probability

- Note that
$$\frac{\Gamma(\frac{1}{2}(-\mathrm{i}z+\alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)}\,\frac{\Gamma(\frac{1}{2}(\mathrm{i}z+d))}{\Gamma(\frac{1}{2}(\mathrm{i}z+d-\alpha))}$$
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- We can use the identity
$$\mathbb{P}_x(\tau^{\odot}<\infty)=\frac{u_\xi(\log(1/|x|))}{u_\xi(0)},$$
providing
$$u_\xi(x)=\frac{1}{2\pi\mathrm{i}}\int_{c+\mathrm{i}\mathbb{R}}\frac{\mathrm{e}^{-zx}}{\Psi(-\mathrm{i}z)}\mathrm{d}z,\qquad x\in\mathbb{R},$$
for $c\in(\alpha-d,0)$.

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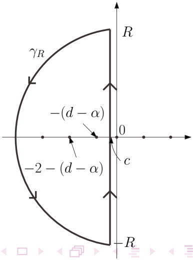
for $c\in(\alpha-d,0)$.
- Build the contour integral around simple poles at $\{-2n-(d-\alpha):n\geq 0\}$.

$$\begin{aligned}&\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}R}^{c+\mathrm{i}R}\frac{\mathrm{e}^{-zx}}{\Psi(-\mathrm{i}z)}\mathrm{d}z\\&=-\frac{1}{2\pi\mathrm{i}}\int_{c+R\mathrm{e}^{\mathrm{i}\theta}:\theta\in(\pi/2,3\pi/2)}\frac{\mathrm{e}^{-zx}}{\Psi(-\mathrm{i}z)}\mathrm{d}z\\&+\sum_{1\leq n\leq\lfloor R\rfloor}\mathrm{Res}\left(\frac{\mathrm{e}^{-zx}}{\Psi(-\mathrm{i}z)};z=-2n-(d-\alpha)\right).\end{aligned}$$



Proof of Port's hitting probability

- Now fix $x \leq 0$ and recall estimate $|1/\Psi(-\mathrm{i}z)| \lessapprox |z|^{-\alpha}$. The assumption $x \leq 0$ and the fact that the arc length of $\{c + R\mathrm{e}^{\mathrm{i}\theta} : \theta \in (\pi/2, 3\pi/2)\}$ is πR, gives us

$$\left| \int_{c+R\mathrm{e}^{\mathrm{i}\theta}:\theta\in(\pi/2,3\pi/2)} \frac{\mathrm{e}^{-xz}}{\Psi(-\mathrm{i}z)} \mathrm{d}z \right| \leq CR^{-(\alpha-1)} \to 0$$

as $R \to \infty$ for some constant $C > 0$.

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- Moreover,

$$\begin{aligned} u_\xi(x) &= \sum_{n\ge1} \mathrm{Res}\left(\frac{\mathrm{e}^{-zx}}{\Psi(-\mathrm{i}z)}; z = -2n - (d-\alpha) \right) \\ &= \sum_0^\infty (-1)^{n+1} \frac{\Gamma(n + (d-\alpha)/2)}{\Gamma(-n+\alpha/2)\Gamma(n+d/2)} \frac{\mathrm{e}^{2nx}}{n!} \\ &= \mathrm{e}^{x(d-\alpha)} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)\Gamma(d/2)} {}_2F_1((d-\alpha)/2, 1-\alpha/2, d/2; \mathrm{e}^{2x}), \end{aligned}$$

  Which also gives a value for $u_\xi(0)$.

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  Which also gives a value for $u_\xi(0)$.
- Hence, for $1 \leq |x|$,

$$\begin{aligned} \mathbb{P}_x(\tau^{\odot} < \infty) &= \frac{u_\xi(\log(1/|x|))}{u_\xi(0)} \\ &= \frac{\Gamma\left(\frac{\alpha+d}{2} - 1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)} |x|^{\alpha-d} {}_2F_1((d-\alpha)/2, 1-\alpha/2, d/2; |x|^{-2}). \end{aligned}$$

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- To this end we note that, for $|x| < 1$, $|Kx| > 1$

$$\mathbb{P}_{Kx}(\tau^{\odot} < \infty) = \mathbb{P}^{\circ}_{x}(\tau^{\odot} < \infty) = \mathbb{E}_{x}\left[\frac{|X_{\tau^{\odot}}|^{\alpha-d}}{|x|^{\alpha-d}}\mathbf{1}_{(\tau^{\odot}<\infty)}\right] = \frac{1}{|x|^{\alpha-d}}\mathbb{P}_{x}(\tau^{\odot} < \infty)$$

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- Hence plugging in the expression for $|x| < 1$,

$$\mathbb{P}_x(\tau^{\odot} < \infty) = \frac{\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)}\,{}_2F_1((d-\alpha)/2, 1-\alpha/2, d/2; |x|^2),$$

  thus completing the proof.
- To deal with the case $x = 0$, take limits in the established identity as $|x| \to 0$.

Riesz representation of Port's hitting probability

Theorem

Suppose $\alpha \in (1,2)$. For all $x \in \mathbb{R}^d$,

$$\mathbb{P}_x(\tau^{\odot} < \infty) = \frac{\Gamma\left(\frac{\alpha+d}{2} - 1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha - 1)} \int_{\mathbb{S}_{d-1}} |z - x|^{\alpha - d} \sigma_1(\mathrm{d}z),$$

where $\sigma_1(\mathrm{d}z)$ is the uniform measure on $\mathbb{S}_{d-1}$, normalised to have unit mass. In particular, for $y \in \mathbb{S}_{d-1}$,

$$\int_{\mathbb{S}_{d-1}} |z - y|^{\alpha - d} \sigma_1(\mathrm{d}z) = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha - 1)}{\Gamma\left(\frac{\alpha+d}{2} - 1\right)\Gamma\left(\frac{\alpha}{2}\right)}.$$

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- We know that $|X_t - z|^{\alpha-d}$, $t \geq 0$ is a martingale.

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$$M_t := \int_{\mathbb{S}_{d-1}} |z - X_{t\wedge\tau^\odot}|^{\alpha-d}\sigma_1(\mathrm{d}z), \qquad t \geq 0,$$

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  is a martingale.
- Recall that $\lim_{t\to\infty} |X_t| = 0$ and $\alpha < d$ and hence

$$M_\infty := \lim_{t\to\infty} M_t = \int_{\mathbb{S}_{d-1}} |z - X_{\tau^{\odot}}|^{\alpha-d}\sigma_1(\mathrm{d}z)\mathbf{1}_{(\tau^{\odot}<\infty)} \stackrel{d}{=} C\mathbf{1}_{(\tau^{\odot}<\infty)}.$$

  where, despite the randomness in $X_{\tau^{\odot}}$, by rotational symmetry,

$$C = \int_{\mathbb{S}_{d-1}} |z - 1|^{\alpha-d}\sigma_1(\mathrm{d}z),$$

  and $1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$ is the 'North Pole' on $\mathbb{S}_{d-1}$.

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- Since M is a UI martingale, taking expectations of M_∞

$$\int_{\mathbb{S}_{d-1}} |z - x|^{\alpha-d}\sigma_1(\mathrm{d}z) = \mathbb{E}_x[M_0] = \mathbb{E}_x[M_\infty] = C\mathbb{P}_x(\tau^\odot < \infty)$$

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- Taking limits as $|x| \to 0$,
  $C = 1/\mathbb{P}(\tau^\odot < \infty) = \Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)/\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)$.

Sphere inversions

Sphere inversions

- Fix a point $b \in \mathbb{R}^d$ and a value $r > 0$.
- The spatial transformation $x^* : \mathbb{R}^d \backslash \{b\} \mapsto \mathbb{R}^d \backslash \{b\}$

$$x^* = b + \frac{r^2}{|x-b|^2}(x-b),$$

is called an *inversion through the sphere* $\mathbb{S}_{d-1}(b,r) := \{x \in \mathbb{R}^d : |x-b| = r\}$.

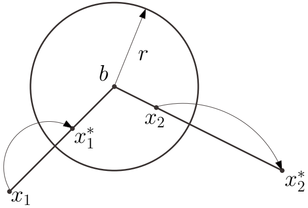


Figure: Inversion relative to the sphere $\mathbb{S}_{d-1}(b,r)$.

INVERSION THROUGH $\mathbb{S}_{d-1}(b, r)$: KEY PROPERTIES

Inversion through $\mathbb{S}_{d-1}(b, r)$

$$x^* = b + \frac{r^2}{|x-b|^2}(x-b),$$

The following can be deduced by straightforward algebra

- Self inverse

$$x = b + r^2 \frac{(x^* - b)}{|x^* - b|^2}$$

- Symmetry

$$r^2 = |x^* - b||x - b|$$

- Difference

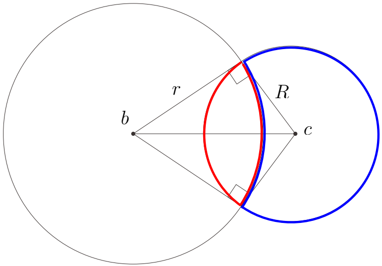
$$|x^* - y^*| = \frac{r^2 |x-y|}{|x-b||y-b|}$$

- Differential

$$\mathrm{d}x^* = \frac{r^{2d}}{|x-b|^{2d}} \mathrm{d}x$$

Inversion through $\mathbb{S}_{d-1}(b, r)$: key properties

- The sphere $\mathbb{S}_{d-1}(c, R)$ maps to itself under inversion through $\mathbb{S}_{d-1}(b, r)$ provided the former is orthogonal to the latter, which is equivalent to $r^2 + R^2 = |c - b|^2$.



- In particular, the area contained in the blue segment is mapped to the area in the red segment and vice versa.

Sphere inversion with reflection

A variant of the sphere inversion transform takes the form

$$x^\diamond = b - \frac{r^2}{|x-b|^2}(x-b),$$

and has properties

- Self inverse

$$x = b - \frac{r^2}{|x^\diamond - b|^2}(x^\diamond - b),$$

- Symmetry

$$r^2 = |x^\diamond - b||x-b|,$$

- Difference

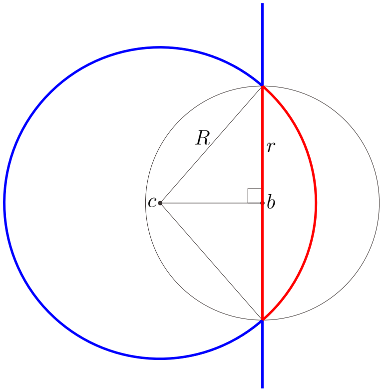
$$|x^\diamond - y^\diamond| = \frac{r^2|x-y|}{|x-b||y-b|}.$$

- Differential

$$\mathrm{d}x^\diamond = \frac{r^{2d}}{|x-b|^{2d}}\mathrm{d}x$$

Sphere inversion with reflection

- Fix $b \in \mathbb{R}^d$ and $r > 0$. The sphere $\mathbb{S}_{d-1}(c, R)$ maps to itself through $\mathbb{S}_{d-1}(b, r)$ providing $|c - b|^2 + r^2 = R^2$.



- However, this time, the exterior of the sphere $\mathbb{S}_{d-1}(c, R)$ maps to the interior of the sphere $\mathbb{S}_{d-1}(c, R)$ and vice versa. For example, the region in the exterior of $\mathbb{S}_{d-1}(c, R)$ contained by blue boundary maps to the portion of the interior of $\mathbb{S}_{d-1}(c, R)$ contained by the red boundary.

§11. Spherical hitting distribution

Port's Sphere hitting distribution

A richer version of the previous theorem:

Theorem (Port (1969))

Define the function

$$h^{\odot}(x,y) = \frac{\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)} \frac{||x|^2-1|^{\alpha-1}}{|x-y|^{\alpha+d-2}}$$

for $|x| \neq 1$, $|y| = 1$. *Then, if* $\alpha \in (1,2)$,

$$\mathbb{P}_x(X_{\tau^{\odot}} \in \mathrm{d}y) = h^{\odot}(x,y)\sigma_1(\mathrm{d}y)\mathbf{1}_{(|x|\neq 1)} + \delta_x(\mathrm{d}y)\mathbf{1}_{(|x|=1)}, \qquad |y| = 1,$$

where $\sigma_1(\mathrm{d}y)$ *is the surface measure on* $\mathbb{S}_{d-1}$, *normalised to have unit total mass.*

Otherwise, if $\alpha \in (0,1]$, $\mathbb{P}_x(\tau^{\odot} = \infty) = 1$, *for all* $|x| \neq 1$.

Proof of Port's sphere hitting distribution

- Write $\mu_x^{\odot}(\mathrm{d}z) = \mathbb{P}_x(X_{\tau^{\odot}} \in \mathrm{d}z)$ on $\mathbb{S}_{d-1}$ where $x \in \mathbb{R}^d \backslash \mathbb{S}_{d-1}$.

PROOF OF PORT'S SPHERE HITTING DISTRIBUTION

- Write $\mu_x^{\odot}(\mathrm{d}z) = \mathbb{P}_x(X_{\tau^{\odot}} \in \mathrm{d}z)$ on $\mathbb{S}_{d-1}$ where $x \in \mathbb{R}^d \backslash \mathbb{S}_{d-1}$.
- Recall the expression for the resolvent of the stable process in Theorem 1 which states that, due to transience,

$$\int_0^\infty \mathbb{P}_x(X_t \in \mathrm{d}y)\mathrm{d}t = C(\alpha)|x-y|^{\alpha-d}\mathrm{d}y, \qquad x, y \in \mathbb{R}^d,$$

where $C(\alpha)$ is an unimportant constant in the following discussion.

Proof of Port's sphere hitting distribution

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  where $C(\alpha)$ is an unimportant constant in the following discussion.
- The measure $\mu_x^{\odot}$ is the solution to the 'functional fixed point equation'

$$|x-y|^{\alpha-d} = \int_{\mathbb{S}_{d-1}} |z-y|^{\alpha-d}\mu(\mathrm{d}z), \qquad y \in \mathbb{S}_{d-1}.$$

  Note that $y \in \mathbb{S}_{d-1}$, so the occupation of y from x, will at least see the the process pass through the sphere $\mathbb{S}_{d-1}$ somewhere first (if not y).
- With a little work, we can show it is the unique solution in the class of probability measures.

Proof of Port's sphere hitting distribution

Recall, for $y^* \in \mathbb{S}_{d-1}$, from the Riesz representation of the sphere hitting probability,

$$\frac{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)}{\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)} = \int_{\mathbb{S}_{d-1}} |z^* - y^*|^{\alpha-d}\sigma_1(\mathrm{d}z^*).$$

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we are going to manipulate this identity using sphere inversion to solve the fixed point equation **first assuming that** $|x|>1$

- Apply the sphere inversion with respect to the sphere $\mathbb{S}_{d-1}(x, (|x|^2-1)^{1/2})$ remembering that this transformation maps $\mathbb{S}_{d-1}$ to itself and using

$$\frac{1}{|z^*-x|^{d-1}}\sigma_1(\mathrm{d}z^*) = \frac{1}{|z-x|^{d-1}}\sigma_1(\mathrm{d}z)$$

$$(|x|^2-1) = |z^*-x||z-x| \quad \text{and} \quad |z^*-y^*| = \frac{(|x|^2-1)|z-y|}{|z-x||y-x|}$$

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- We have

$$\frac{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)}{\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)} = \int_{\mathbb{S}_{d-1}} |z^*-x|^{d-1}|z^*-y^*|^{\alpha-d}\frac{\sigma_1(\mathrm{d}z^*)}{|z^*-x|^{d-1}}$$

$$= \frac{(|x|^2-1)^{\alpha-1}}{|y-x|^{\alpha-d}}\int_{\mathbb{S}_{d-1}} \frac{|z-y|^{\alpha-d}}{|z-x|^{\alpha+d-2}}\sigma_1(\mathrm{d}z).$$

- For the case $|x|<1$, use Riesz–Bogdan–Żak theorem again! (See exercises).

§12. Spherical entrance/exit distribution

Blumenthal–Getoor–Ray exit / entrance distribution

Theorem

Define the function

$$g(x,y) = \pi^{-(d/2+1)}\,\Gamma(d/2)\,\sin(\pi\alpha/2)\frac{|1-|x|^2|^{\alpha/2}}{|1-|y|^2|^{\alpha/2}}|x-y|^{-d}$$

for $x,y \in \mathbb{R}^d \backslash \mathbb{S}_{d-1}$. *Let*

$$\tau^{\oplus} := \inf\{t>0 : |X_t|<1\} \text{ and } \tau_a^{\ominus} := \inf\{t>0 : |X_t|>1\}.$$

(i) *Suppose that* $|x|<1$, *then*

$$\mathbb{P}_x(X_{\tau^{\ominus}} \in \mathrm{d}y) = g(x,y)\mathrm{d}y, \qquad |y| \geq 1.$$

(ii) *Suppose that* $|x|>1$, *then*

$$\mathbb{P}_x(X_{\tau^{\oplus}} \in \mathrm{d}y,\ \tau^{\oplus}<\infty) = g(x,y)\mathrm{d}y, \qquad |y| \leq 1.$$

Proof of B–G–R entrance/exit distribution (i)

- Appealing again to the potential density and the strong Markov property, it suffices to find a solution to

$$|x-y|^{\alpha-d} = \int_{|z|\geq 1} |z-y|^{\alpha-d}\mu(\mathrm{d}z), \qquad |y|>1>|x|,$$

with a straightforward argument providing uniqueness.

PROOF OF B–G–R ENTRANCE/EXIT DISTRIBUTION (I)

- Appealing again to the potential density and the strong Markov property, it suffices to find a solution to

$$|x-y|^{\alpha-d} = \int_{|z|\geq 1} |z-y|^{\alpha-d}\mu(\mathrm{d}z), \qquad |y|>1>|x|,$$

with a straightforward argument providing uniqueness.
- The proof is complete as soon as we can verify that

$$|x-y|^{\alpha-d} = c_{\alpha,d}\int_{|z|\geq 1} |z-y|^{\alpha-d}\frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}}|x-z|^{-d}\mathrm{d}z$$

for $|y|>1>|x|$, where

$$c_{\alpha,d} = \pi^{-(1+d/2)}\,\Gamma(d/2)\,\sin(\pi\alpha/2).$$

Proof of B–G–R entrance/exit distribution (I)

- Transform $z \mapsto z^\diamond$ (sphere inversion with reflection) through the sphere $\mathbb{S}_{d-1}(x, (1-|x|^2)^{1/2})$, noting in particular that

$$|z^\diamond - y^\diamond| = (1-|x|^2)\frac{|z-y|}{|z-x||y-x|} \quad \text{and} \quad |z|^2 - 1 = \frac{|z-x|^2}{1-|x|^2}(1-|z^\diamond|^2)$$

and

$$\mathrm{d}z^\diamond = (1-|x|^2)^d |z-x|^{-2d}\mathrm{d}z, \qquad z \in \mathbb{R}^d.$$

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  and
$$\mathrm{d}z^\diamond = (1-|x|^2)^d |z-x|^{-2d}\mathrm{d}z, \qquad z \in \mathbb{R}^d.$$

- For $|x| < 1 < |y|$,

$$\int_{|z|\geq 1} |z-y|^{\alpha-d}\frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}}|x-z|^{-d}\mathrm{d}z = |y-x|^{\alpha-d}\int_{|z^\diamond|\leq 1}\frac{|z^\diamond - y^\diamond|^{\alpha-d}}{|1-|z^\diamond|^2|^{\alpha/2}}\mathrm{d}z^\diamond.$$

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- Now perform similar transformation $z^\diamond \mapsto w$ (inversion with reflection), albeit through the sphere $\mathbb{S}_{d-1}(y^\diamond, (1-|y^\diamond|^2)^{1/2})$.

$$|y-x|^{\alpha-d}\int_{|z^\diamond|\leq 1}\frac{|z^\diamond - y^\diamond|^{\alpha-d}}{|1-|z^\diamond|^2|^{\alpha/2}}\mathrm{d}z^\diamond = |y-x|^{\alpha-d}\int_{|w|\geq 1}\frac{|1-|y^\diamond|^2|^{\alpha/2}}{|1-|w|^2|^{\alpha/2}}|w-y^\diamond|^{-d}\mathrm{d}w.$$

Proof of B–G–R entrance/exit distribution (I)

Thus far:

$$\int_{|z|\geq 1} |z-y|^{\alpha-d}\frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}}|x-z|^{-d}\mathrm{d}z = |y-x|^{\alpha-d}\int_{|w|\geq 1}\frac{|1-|y^\diamond|^2|^{\alpha/2}}{|1-|w|^2|^{\alpha/2}}|w-y^\diamond|^{-d}\mathrm{d}w.$$

- Taking the integral in red and decomposition into generalised spherical polar coordinates

$$\int_{|v|\geq 1}\frac{1}{|1-|w|^2|^{\alpha/2}}|w-y^\diamond|^{-d}\mathrm{d}w = \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_1^\infty \frac{r^{d-1}\mathrm{d}r}{|1-r^2|^{\alpha/2}}\int_{\mathbb{S}_{d-1}(0,r)}|z-y^\diamond|^{-d}\sigma_r(\mathrm{d}z)$$

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- Poisson's formula (the probability that a Brownian motion hits a sphere of radius $r>0$) states that

$$\int_{\mathbb{S}_{d-1}(0,r)}\frac{r^{d-2}(r^2-|y^\diamond|^2)}{|z-y^\diamond|^d}\sigma_r(\mathrm{d}z)=1, \qquad |y^\diamond|<1<r.$$

  gives us

$$\int_{|v|\geq 1}\frac{1}{|1-|w|^2|^{\alpha/2}}|w-y^\diamond|^{-d}\mathrm{d}w = \frac{\pi^{d/2}}{\Gamma(d/2)}\int_1^\infty\frac{2r}{(r^2-1)^{\alpha/2}(r^2-|y^\diamond|^2)}\mathrm{d}r$$
$$= \frac{\pi}{\sin(\alpha\pi/2)}\frac{1}{(1-|y^\diamond|^2)^{\alpha/2}}$$

- Plugging everything back in gives the result for $|x|<1$.

Exercises Set 2

EXERCISES

1. Use the fact that the radial part of a d-dimensional ($d \geq 2$) isotropic stable process has MAP (ξ, Θ), for which the first component is a Lévy process with characteristic exponent given by

$$\Psi(z) = 2^{\alpha} \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z + \alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z + d))}{\Gamma(\frac{1}{2}(\mathrm{i}z + d - \alpha))}, \qquad z \in \mathbb{R}.$$

to deduce the following facts:

- Irrespective of its point of issue, we have $\lim_{t\to\infty} |X_t| = \infty$ almost surely.

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   to deduce the following facts:
   - Irrespective of its point of issue, we have $\lim_{t\to\infty} |X_t| = \infty$ almost surely.
   - By considering the roots of Ψ show that

$$\exp((\alpha - d)\xi_t), \qquad t \geq 0,$$

     is a martingale.
   - Deduce that

$$|X_t|^{\alpha - d}, \qquad t \geq 0,$$

     is a martingale.

2. Remaining in d-dimensions ($d \geq 2$), recalling that

$$\left.\frac{\mathrm{d}\mathbb{P}^{\circ}_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \qquad t \geq 0, x \neq 0,$$

   show that under $\mathbb{P}^{\circ}$, X is absorbed continuously at the origin in an almost surely finite time.

Exercises

3. Recall the following theorem

Theorem

Define the function

$$g(x,y) = \pi^{-(d/2+1)}\,\Gamma(d/2)\,\sin(\pi\alpha/2)\frac{\left|1-|x|^2\right|^{\alpha/2}}{\left|1-|y|^2\right|^{\alpha/2}}|x-y|^{-d}$$

for $x, y \in \mathbb{R}^d \backslash \mathbb{S}_{d-1}$*. Let*

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(i) *Suppose that* $|x| < 1$*, then*

$$\mathbb{P}_x(X_{\tau^{\ominus}} \in \mathrm{d}y) = g(x,y)\mathrm{d}y, \qquad |y| \geq 1.$$

(ii) *Suppose that* $|x| > 1$*, then*

$$\mathbb{P}_x(X_{\tau^{\oplus}} \in \mathrm{d}y,\, \tau^{\oplus} < \infty) = g(x,y)\mathrm{d}y, \qquad |y| \leq 1.$$

Prove (ii) (i.e. $|x| > 1$) from the identity in (i) (i.e. $|x| < 1$).

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