The Asymptotic Properties of Homogeneous Fragmentation Processes

submitted by

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Signature of Author ....

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For my parents.

SUMMARY

We prove three results concerning homogeneous fragmentations of the unit interval. First we study the asymptotic behaviour of the size of the largest fragment at large times. It is known that for some constant $v_{\max} > 0$ the size of the largest fragment is roughly equal to $\exp(-v_{\max}t)$ at the large time t. We refine this result by showing that the largest fragment's size is roughly equal to $t^{-\alpha}\exp(-v_{\max}t)$ whenever t is large, where $\alpha > 0$ is a constant we identify explicitly in terms of the characteristics of the fragmentation process.
Next we turn our attention to killed fragmentation processes, in which fragments smaller than $\exp(-vt)$ are removed from the system at time t. We show that when $v = v_{\max} + \epsilon$, the killed process survives with probability roughly equal to $\exp(-\beta\epsilon^{-1/2})$ provided ϵ is small, for a constant $\beta > 0$ which we calculate. Finally, when $v = v_{\max}$ we show that the killed process survives until the large time t with probability roughly equal to $\exp(-\gamma t^{1/3})$ for a constant $\gamma > 0$ which we calculate.

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CONTENTS

CHAPTER 1

INTRODUCTION

Fragmentation processes are mathematical models that describe the phenomenon of objects breaking apart. This fracturing process arises naturally in many contexts: "studies of stellar fragments in astrophysics, fractures and earthquakes in geophysics, breaking of crystals in crystallograpy, degredation of large polymer chains in chemistry, DNA fragmentation in biology, fission of atoms in nuclear physics, [and] fragmentation of a hard drive in computer science" [14], among others. A glance at the contents page of *Fragmentation Phenomena* [16] will convince the reader of the ubiquity of fragmentation phenomena in the physical sciences, with titles ranging from the prosaic *Experimental Results on Single Particle of Cement Clinker* to the playful *Surface and Coulomb Instabilities of Sheets, Bubbles and Donuts.*

In fact, the first serious probabilistic work concerning fragmentation processes, dating from 1941, and written by the father of modern probability himself, was motivated by just such an application. Kolmogorov begins his paper *On the logarithmic normal distribution of particle sizes under grinding* [30] (in a translation from the Russian by G. Lindquist) by remarking that "In a recent paper N.K. Razumovskii indicates many cases when the logarithms of particle sizes (gold grits in gold placers, rock particles under grinding, etc.) obey approximately the Gauss distribution law". In an article of less than four pages, he proceeds to explain this observation by proving a result that we would now describe as a "special case of the central limit theorem for branching random walks" [14].

Work on fragmentation processes continued sporadically throughout the rest of the twentieth century. In 1961, Aleksei Filippov, a student of Kolmogorov, proved a result on the typical particle size of conservative self-similar fragmentations [24], a result later rediscovered, for binary fragmentations, by Brennan and Durrett in 1987 [20]. Several contributions to the subject have also appeared in the physics literature, though the results obtained there are typically special cases of the more general results appearing soon after in the work to be discussed below; see, for instance,[6].

All the fragmentation processes considered in the work mentioned above have in common that the time taken for a given particle to fracture is assumed to be almost surely positive; moreover, the number of fragments produced by such an event, though in some cases random, is always taken to be finite. It wasn't until the year 2001 that a rigorous formulation of fragmentation processes in their full generality appeared in the literature. In his paper *Homogeneous fragmentation processes* [10], Jean Bertoin used a powerful discrete construction based on a Poisson point process to allow for instantaneous shattering of fragments into infinitely many fragments. This pioneering work was continued by Bertoin in [11], with further constructive groundwork laid by Julien Berestycki [7] and Anne-Laure Basdevant [5]. These papers also contain important first properties of fragmentation processes, and some further interesting results.

The study of fragmentation processes now forms an active area of research, reflecting the preponderance of natural questions one can ask about them. Using the theory of regular variation, Bertoin describes the small-time proliferation of very small particles in [13]. In [15], sharp large deviation results are obtained for fragment sizes by using the theory of probability tilting, discretization and the famous derivative martingale. In his paper *Multifractal spectra of fragmentation processes*, Julien Berestycki employs martingale theory to calculate the Hausdorff dimension of fragments with atypical exponential rates of decay. In [29], so-called "killed fragmentation processes" are defined, and a criterion for their survival is identified. These papers form only a drop in the ocean; since 2010, at least forty-five preprints with the word "fragmentation" in their title have appeared on the arXiv under the "probability" classification.

In this thesis we will study conservative homogeneous fragmentations of the unit interval. The word "conservative" means that, at all times, the total length of all the fragments of the unit interval is unity. The word "homogeneous" means that the rate at which particles break up does not depend on their size. Some of the work mentioned above generalises such processes in two directions. First, *dissipative* fragmentations allow for loss of mass through a deterministic process called *erosion*. Second, *self-similar* fragmentations allow fragments to break up at a rate dependent on their size. The latter generalization is of particular interest in physical applications, where, for instance, one might imagine that smaller fragments of an object are more fragile, and hence more prone to fracture. We content ourselves with the remark that these more general processes can be recovered from conservative homogeneous fragmentation processes via fairly simple transformations. Processes incorporating erosion are obtained simply by introducing a deterministic exponential decay to the particle sizes of a corresponding process with no erosion. Self-similar fragmentation processes are obtained from homogeneous fragmentation processes via the now-classical Lamperti time transform, first introduced in 1962 [37]. Therefore, although the subclass of fragmentation processes we consider may seem restrictive, results can, at least in theory, be translated to the more general setting using these ideas.

Our main results concern the large-time asymptotics of fragmentation processes. First we consider the size of the largest fragment at large times. The first work in this direction is contained in [11], where the rate of exponential decay of this fragment is calculated. We identify the polynomial correction to this rate of decay, which we express explicitly in terms of parameters intrinsic to the fragmentation process.
Our second and third results concern the survival probability of killed fragmentations, using the killing regime introduced in [29]. At time t, particles of size smaller than $\exp(-ct)$ are removed from the system. Depending on the value of the pa-

rameter c, this system may survive or die. In [29], this behaviour is classified into super- and sub-critical regimes, and the critical parameter value is identified. In the super-critical regime, the process survives with positive probability. Our second result estimates this survival probability for values of c slightly larger than the critical value. In the critical case, the process dies almost surely. With this in mind, our third result estimates the probability that a critically killed fragmentation survives until a given large time.

We will employ a variety of tools and techniques to prove these results. The overarching theme is the theory of spines and changes of measure, championed in the seminal paper by Lyons, Pemantle and Peres [38]. Roughly speaking, this approach seeks to simplify the study of branching phenomena by identifying a "privileged" embedded stochastic process called the spine, before relating its behaviour to the underlying process by using a change of measure. This technique is now a cornerstone of modern branching process theory, and has been used to elegantly tackle problems arising in a variety of contexts. Examples include work by Chauvin and Roualt on branching Brownian motion [21], Kyprianou on branching diffusions [34], Bertoin and Roualt on homogeneous fragmentations [15], Athreya on Markov chains [4], Biggins and Kyprianou [19] and Hu and Shi [27] on branching random walk, and Engländer and Kyprianou on superprocesses [23]; these references are borrowed from Matt Roberts' thesis *Spine Changes of Measure and Branching Diffusions* [42].

Our spine turns out to be a centered Lévy process with further special characteristics. An important part of our work will therefore consist in generalising several well known results from the theory of random walk to the Lévy setting. This includes a Lévy version of Mogulskii's Theorem for large deviation probabilities of random walk. With these tools in hand, the proofs will be completed using the second moment method.

The structure of this thesis is as follows. In the remainder of this chapter, we define conservative homogeneous fragmentation processes, and survey their basic properties; none of this work is original. In the next chapter, we will prove several preliminary lemmas on these processes, and transfer results from random walks to Lévy processes. Chapter 3 contains the proof of our first main result, and we prove our second and third main results in Chapters 4 and 5 respectively.

1.1 What is a conservative homogeneous fragmentation process?

1.1.1 Informal discussion

Before entering into technical formalities, we orientate ourselves with a dynamic description of a particular *finite activity* conservative homogeneous fragmentation process. We start with the unit interval $(0, 1)$, which serves as the "object" undergoing fragmentation. After a random period of time τ with an exponential distribution of rate 1, we generate some random number $1 < N < \infty$ of $(0, 1)$–valued uniform random variables, which we arrange in the usual order: $0 < a_1 < \cdots < a_N < 1$. At times in $[0, \tau)$, the "value" of the fragmentation process is the interval $(0, 1)$; the initial fragment has so far remained intact. At time τ, the fragmentation process jumps to the "value" $[(0, a_1), (a_1, a_2), \cdots, (a_N, 1)]$. The interval-valued entries in this vector are referred to as *particles*, *fragments* or *blocks*, and sometimes as *children* of the *parent* interval $(0, 1)$. The jump discontinuity producing them from the parent particle is variously referred to using the verbs *fracture*, *fragment*, *split*, *dislocate*, and *shatter*. The subsequent evolution (from time τ onwards) of each of these particles is identical to the evolution of the parent particle $(0, 1)$, provided we scale them to unit length at their birth times. Finally, distinct children evolve independently of their history and of one another. For instance, the interval $(0, a_1)$ shatters at time $\tau + \sigma$, where σ is an independent copy of τ; the children of $(0, a_1)$ born at this time are described by L uniform random variables concentrated on this interval, where L is an independent copy of N.

This example illustrates many of the salient features of conservative homogeneous fragmentation processes. First, the process is Markovian. Second, after scaling to unit length, all particles evolve in exactly the same way and independently of one another. This means, in particular, that the rate at which a particle fragments is independent of its size; this is what "homogeneous" means. Finally, no mass is lost: in the example above, the Lebesgue measure of the collection of fragments at any time is, by construction, unity. This property is signified by the word "conservative".

There are several phenomena consistent with these properties, however, that our example lacks. In general, we allow particles to shatter *instantaneously*. More precisely, we allow the infimum of times at which the process differs from $(0, 1)$ to be 0. Second, we allow particles to shatter into infinitely many pieces (though all fragments must have positive length). The word "crumbling" captures what happens to the unit interval in this most general setting.

Providing a formal description of fragmentation processes incorporating these additional features is no easy feat. Before setting out to do so, Jean Bertoin remarks in [10] that

> " ... the analysis of random processes in continuous times may be much more subtle than that of their counterparts in discrete times. For instance, the law of a random walk, say in $\mathbb{R}^d$, is completely characterized

> by the distribution of its generic step, which is an arbitrary probability measure on $\mathbb{R}^d$. The continuous time analogue of a random walk is a Lévy process... [and] its structure is only revealed by the combination of the celebrated Lévy-Khintchine formula and Lévy-Itô decomposition, which are two deep and difficult results of probability theory."

Bertoin achieves his explicit construction of homogeneous fragmentation processes by encoding them as stochastic processes taking values in the space of partitions on $\mathbb{N}$, an approach based on the important paper *The coalescent* by John Kingman, published in 1982 [28]. Briefly put, particle sizes are represented by the asymptotic frequencies of blocks in the partitions, which exist almost surely in certain circumstances.

The laws of Bertoin's partition-valued fragmentation processes are in one-to-one correspondence with the laws of so-called *ranked* fragmentations, as shown by Julien Berestycki in [7]; these are not quite what we want. In this thesis we will make use of *interval* fragmentations. The example we began this section with is an example of an interval fragmentation: in short, we retain information about the spatial position of our fragments. For instance, the state $[(0, \frac{1}{4}), (\frac{1}{4}, \frac{1}{2}), (\frac{1}{2}, 1)]$ is considered to be different to the state $[(0, \frac{1}{4}), (\frac{1}{4}, \frac{3}{4}), (\frac{3}{4}, 1)]$, despite the fact that the collection of block sizes equals $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ in both cases . In a ranked fragmentation process, particles are always arranged in order of their size. As a result, the only information captured by a ranked fragmentation process is the collection of particle sizes: clear genealogical structure is not available. Unsurprisingly, the class of interval fragmentations can be mapped surjectively to the class of ranked fragmentations, but the correspondence is not one-to-one.

Further work done by Anne-Laure Basdevant in 2006 [5], and based on the earlier work of Bertoin and Berestycki, completes the picture. She provides an explicit construction of interval fragmentations using fragmentations taking values in the space of *compositions* (ordered partitions) of $\mathbb{N}$. Her paper also provides the fundamental link between interval fragmentations and composition-valued fragmentations, and summarizes the links between the four kinds of fragmentations we have discussed. Briefly put, interval and composition-valued fragmentations are in one-to-one correspondence in law, as are ranked and partition-valued fragmentations; the former classes are mapped surjectively to the latter classes by discarding all information about the order of particles.

The questions we address in this thesis make sense and have the same answers regardless of which type of fragmentation we use, so we end this section with a few words on why we will make exclusive use (with one exception) of interval fragmentations. Ranked fragmentations are out of the question: the re-ordering of particles by size destroys *genealogical* information that will play a central role in our proofs. In contrast, given a fragment alive at some non-zero time in an interval fragmentation process, we can trace its lineage back to the unit interval. That is, if we look at a fragment $(a, b) \subset (0, 1)$ alive at time t, we can find nested intervals $(a_s, b_s) \subset (0, 1)$ (for $s \in [0, t]$) such that $(a_0, b_0) = (0, 1)$ and $(a_t, b_t) = (a, b)$. As it happens, both partition- and composition-valued fragmentation processes also contain genealogical information. Our final choice between the three appropriate candidates boils down to the fact that useful pictures can most easily be drawn of interval fragmentations; they are the most intuitive of the four models, and the easiest to get a handle on. The principle value of partition- and composition-valued fragmentations is their discrete nature, which we will not use (except on one important occasion).

1.1.2 The formal defintion

As we have seen, our fragments will be represented by disjoint open subintervals of $(0,1)$ whose union is the unit interval minus a countable set. In [11], Bertoin establishes an elegant way of encoding such collections, by metrizing the space $\mathcal{U}$ of *open subsets* of $(0,1)$. Given an open set $u \in \mathcal{U}$, he introduces the continuous function χ_u on $[0,1]$ defined by

$$\chi_u(x) \;:=\; \min\{|x-y| : \, y \in u^c\}.$$

The distance between two open sets $u, v \in \mathcal{U}$ is then given by

$$d(u,v) \;:=\; \|\chi_u - \chi_v\|_\infty.$$

As Bertoin remarks, this definition coincides with the Hausdorff distance between u^c and v^c; moreover, the space $(\mathcal{U}, d)$ is compact. We also endow $\mathcal{U}$ with the σ–algebra generated by the open sets corresponding to this distance, which we denote by $\mathcal{B}(\mathcal{U})$.

It is an elementary fact that that any open set in $\mathbb{R}$ has a unique *decomposition* into a countable collection of disjoint open intervals. We will frequently need to enumerate this countable collection, and do so by defining the following total order on the collection of finite subintervals of $\mathbb{R}$:

$$(a,b) < (c,d) \qquad \Longleftrightarrow \qquad b-a < d-c \quad \text{or} \quad (b-a = d-c \text{ and } a < c).$$

Given $u \in \mathcal{U}$ we will write $(u_i : i \in \mathbb{N})$ for the elements in the decomposition of u written in decreasing $<$–order, filling the tail of this sequence with empty sets in case u has only finitely many components. Technical details aside, the point is that, given a set $u \in \mathcal{U}$, the intervals in the collection $(u_i : i \in \mathbb{N})$ correctly capture our intuitive image of what "fragments" of $(0,1)$ are.

With our state-space in hand, we need to specify how fragments are to break apart. Our basic datum is a family $(q_t : t > 0)$ of probability measures defined on $(\mathcal{U}, \mathcal{B}(\mathcal{U}))$. We fix an interval $I := (a,b) \subseteq (0,1)$ and write $\mathcal{I}$ for the set of open subsets of I (with the distance inherited from $\mathcal{U}$ and the corresponding σ–algebra). We let g_I stand for the unique affine map sending $(0,1)$ to I, and retain the notation g_I for its natural extension to a map from $\mathcal{U}$ to $\mathcal{I}$. We write q_t^I for the image measure of q_t under the map g_I, so that q_t^I is a probability measure on $\mathcal{I}$. Given an open set $u \in \mathcal{U}$, we let q_t^u stand for the distribution of $\cup X_i$, where the X_i are independent random variables with laws $q_t^{u_i}$ respectively.

We are now ready to say exactly what a conservative homogeneous fragmentation is. The following definition is lifted from [5].

Definition 1.1. *A Markov process $U := (U(t) : t \geq 0)$ taking values in $\mathcal{U}$ is called a conservative homogeneous interval fragmentation if it has the following properties:*

1. *U is continuous in probability: for all $t \geq 0$ and $\epsilon > 0$,*

$$\mathbf{P}\big(d(U(t), U(s)) \geq \epsilon\big) \to 0 \qquad as \qquad s \to t.$$

2. *U is nested: $s > t \implies U(s) \subseteq U(t)$.*

3. *Fragmentation property: there exists some family $(q_t : t > 0)$ of probability measures on $\mathcal{U}$ such that*

$$\forall t \geq 0 \quad \forall s > t \quad \forall A \in \mathcal{B}(\mathcal{U}) \quad \mathbf{P}\big(U(s) \in A \,\big|\, U(t)\big) \;=\; q_{s-t}^{U(t)}(A).$$

4. *Conservative property: $|U(t)| = 1$ for all $t \geq 0$.*

Throughout this thesis, $|A|$ stands for the Lebesgue measure of a Borel set A. The filtration generated by U is denoted by $\mathcal{F} := (\mathcal{F}_t : t \geq 0)$ after the usual completion. Sometimes we will want to start our fragmentation process from some $u \in \mathcal{U}$ other than $(0,1)$. Accordingly, we write $\mathbf{P}_u$ for the law of the fragmentation process started from $u \in \mathcal{U}$, and $\mathbf{E}_u$ for the corresponding expectation operator. We also define $\mathbf{P} := \mathbf{P}_{(0,1)}$, with expectation operator $\mathbf{E}$.

All fragmentation processes have regular versions. Indeed, for a fragmentation U let us define a new process U^+ by setting

$$U^+(t) \;=\; \bigcup_{s \in (t,\infty)\cap\mathbb{Q}} U(s).$$

Using the continuity in probability of U, it is easily verified that U^+ is a version of U, and that the sample paths of U^+ are almost surely càdlàg (a map from $\mathbb{R}_+$ to a metric space is called càdlàg if it has left limits and is right continuous). In the remainder of this thesis, whenever we talk about a fragmentation U, we are implicitly working with U^+.

Since, with one exception, we will only consider conservative, homogeneous, interval fragmentation processes in this thesis, such processes will henceforth be referred to as "fragmentation processes", without qualification, in those contexts where no confusion can arise.

With Definition 1.1 in hand, we move on to an exposition of the key ideas that we will use in our proofs. This entails studying several martingales, changes of measure, spines and Lévy process theory.

1.2 Technical preliminaries

1.2.1 Dislocation measures

The kernel $(q_t \colon t > 0)$ will make no further appearance in the rest of this thesis. Instead, we work with the more useful class of *dislocation measures*, which capture the same information [5]. Under the total order introduced in the previous section, u_1 stands for the largest interval component in the decomposition of a given $u \in \mathcal{U}$. (In case there are several largest components, u_1 is the one with the largest left endpoint.)

Definition 1.2. *A measure ν on $\mathcal{U}$ is called a dislocation measure if*

$$\nu((0,1)) = 0 \qquad \textit{and} \qquad \int_{\mathcal{U}} (1 - |u_1|)\nu(du) < \infty\,.$$

In [5], Basdevant shows that the collection of laws of conservative homogeneous interval fragmentation processes is in one-to-one correspondence with the collection of dislocation measures. Her proof proceeds by establishing a bijection between the laws of interval fragmentation processes and the laws of fragmentation processes taking values in the space of ordered partitions. She then shows that the latter class is in bijective correspondence with the class of dislocation measures. This work is precisely analogous to the work contained in [8], where ranked fragmentations and partition fragmentations are shown to be in bijective correspondence. We remark that, for an interval fragmentation process with corresponding dislocation measure ν, the value $\nu(A)$ (which may be infinite) describes the rate at which a given particle breaks up into fragments described by a given collection of possible configurations $A \in \mathcal{B}(\mathcal{U})$. The proof of all these statements can be found in [5].

The first condition in Definition 1.2 is self-explanatory; it says that something actually happens at fragmentation events. Whenever $\nu(\mathcal{U}) = \infty$, fragments shatter instantaneously:

$$\inf\{t \geq 0 : \; U(t) \neq (0,1)\} \;=\; 0, \qquad \text{almost surely.}$$

The second condition is necessary in this case to prevent fragments from being immediately reduced to dust (particles of zero size).

In view of the display above, a fragmentation processes corresponding to a dislocation measure with infinite mass is called an *infinite activity* fragmentation process. When ν has finite mass, the first time when $U(t)$ differs from $(0,1)$ is exponentially distributed with rate $\nu(\mathcal{U})$. Fragmentation processes corresponding to finite dislocation measures are therefore given the qualifier *finite activity.*

We emphasize that phrase "finite activity" signifies the fact that a *given* fragment waits a positive time before shattering; it does not mean that fragmentation events are isolated in time. To illustrate this comment, consider the fragmentation process corresponding to the dislocation measure which assigns unit mass to an atom at $\bigcup(2^{-(n+1)}, 2^{-n})$ and no mass elsewhere. Then the *time of the first dislocation event*

τ is well-defined. Furthermore, the set of all splitting times is dense in $[\tau, \infty)$ almost surely. To see why, write σ_n for the time at which the interval $(2^{-(n+1)}, 2^{-n})$, necessarily alive at time τ, fractures. Then $(\sigma_n : n \in \mathbb{N})$ is collection of independent exponentially distributed random variables with rate 1. It remains to make the trivial observation that any infinite collection of independent continuous random variables whose laws have support equal to $[0, \infty)$ is dense there almost surely.

In a similar vein, it is possible to have an infinite activity process in which each dislocation event produces only finitely many new fragments. Indeed, define elements $v^{(n)} := (0, \frac{1}{n}) \cup (\frac{1}{n}, 1) \in \mathcal{U}$ for each $n \geq 2$. Let ν be a measure on $\mathcal{U}$ assigning mass $\frac{1}{n}$ to $v^{(n)}$, and no mass elsewhere. Then

$$\int_{\mathcal{U}} (1 - |u_1|)\nu(du) \;=\; \sum_{n=2}^{\infty} \frac{1}{n^2} \;<\; \infty,$$

so ν is a dislocation measure. On the other hand, $\nu(\mathcal{U}) = \infty$. Although jump discontinuities produce only finitely many new fragments, their infinite rate of arrival guarantees that infinitely many fragments exist at arbitrarily small times.

We conclude this section by defining *geometric* dislocation measures. The dislocation measure ν is called *geometric* if there exists a number $r > 1$ such that

$$\left\{u \in \mathcal{U} : \;\forall i \in \mathbb{N} \;\exists n \in \mathbb{N} \text{ such that } \; u_i = r^{-n}\right\}$$

is a ν–full set (that is, its complement is null with respect to ν). In view of the integrability condition in Definition 1.2, geometric dislocation measures are necessarily finite. This is because the value $1 - |u_1|$ is bounded below by $1 - r^{-1} > 0$, ν–almost everywhere, for some $r > 1$.

1.2.2 Some Lévy Process Theory

Before carrying on with our discussion of fragmentation processes, it is now necessary to say a bit about Lévy processes. Afterwards, we will discover important Lévy processes embedded in the class of fragmentation processes, which will be invaluable to us in the rest of this thesis.

A *Lévy process* is a Markov process issued from the origin with stationary and independent increments, and almost surely càdlàg paths. Their importance in the present context has already been hinted at, and will be elucidated in the following sections. Here we will discuss several special kinds of Lévy processes, and a few of their basic properties. There are several monographs on the subject; we will refer to [35] in this section. Other references include [9] and [44].

A *subordinator* is a Lévy process with increasing sample paths. A *pure jump subordinator* is a Lévy process that can be expressed as a Poisson random sum of positive jumps. To be precise, we say that a Lévy process S is a pure jump subordinator if there exists a Poisson random measure N on $[0, \infty) \times (0, \infty)$ such that for all $t \geq 0$,

$$S_t \;=\; \int_{[0,t]\times(0,\infty)} x \; N(ds \times dx).$$

The Poisson random measure corresponding to a pure jump subordinator is uniquely characterized by its Lévy measure Π, which is defined by the equation $\mathbf{E}N(ds \times dx) = ds\Pi(dx)$, and is concentrated on $(0, \infty)$. If Π is the Lévy measure corresponding to a pure jump subordinator, it is automatically true [35, Lemma 2.14] that

$$\int_{(0,\infty)} (1 \wedge x) \; \Pi(dx) \;\; < \;\; \infty.$$

For Lévy processes in general, this inequality is only guaranteed to hold when the integrand is replaced by $1 \wedge x^2$.

Because subordinators only take positive values, they have Laplace exponents defined on the non-negative half-line. The Laplace exponent Φ of a subordinator X is defined for $\lambda \geq 0$ by the equation

$$\exp(-\Phi(\lambda)) \;\; = \;\; \mathbf{E} \exp(-\lambda X_1).$$

If S is a pure-jump subordinator, its Laplace exponent can be neatly expressed [35, pg. 116] in terms of its Lévy measure Π:

$$\Phi(\lambda) \;\; = \;\; \int_{(0,\infty)} \left(1 - e^{-\lambda x}\right) \; \Pi(dx). \tag{1.1}$$

This equation together with elementary measure theory tells us a lot about the Laplace exponent of a pure jump subordinator. For instance, Φ is infinitely differentiable on $(0, \infty)$. Differentiating once tells us that Φ increases strictly, and differentiating twice tells us that Φ is strictly concave. Clearly $\Phi(0) = 0$ for any Laplace exponent, and $\Phi(\infty) < \infty$ if and only if S is a compound Poisson process. Of course, Φ may exist for negative values too. If λ_0 is the infimum of those values for which Φ exists, then Φ is infinitely differentiable, strictly increasing, and strictly concave on (λ_0, ∞).

We will make frequent use of *spectrally one-sided Lévy processes of bounded variation.* Spectrally one-sided Lévy processes are those with jumps of only one sign and non-monotone paths. The class of spectrally one-sided processes of bounded variation coincides with the class of those Lévy processes that are the difference between a pure jump subordinator and a deterministic linear drift, or vice versa [35, pg. 58].

A *spectrally positive* (respectively *negative*) process has only positive (respectively negative) jumps and non-monotone paths. If X is a spectrally positive Lévy process of bounded variation, then 0 is irregular for $[0, \infty)$ relative to X [35, Theorem 6.5]. This means that the negative drift initially triumphs over the positive jumps, and the process almost surely starts its life with a dip in the negative half-line. That is,

$$\mathbf{P}\Big(\inf\{t > 0 : X_t \in [0, \infty)\} = 0\Big) \;\; = \;\; 0.$$

Similarly, if X is a spectrally negative process of bounded variation, then 0 is irregular for $(-\infty, 0]$ relative to X.

To state the next property of Lévy processes we need, we fix a Lévy process X and introduce random variables $\tau_a^- := \inf\{t > 0 : X_t \leq a\}$ and $\tau_b^+ := \inf\{t > 0 : X_t \geq b\}$ for $a, b \in \mathbb{R}$. If X is a spectrally positive process of bounded variation, we claim that $\mathbf{P}(\tau_{-a}^- < \tau_0^+) > 0$ for all $a > 0$. Indeed, by the irregularity of 0 for $[0, \infty)$, there exist some $t_0, r > 0$ such that

$$p \;:=\; \mathbf{P}\big(X_{t_0} < -r \;\text{ and }\; X_t < 0 \;\text{ on }\; (0, t_0]\big) \;>\; 0.$$

By the stationarity and independence of increments, we conclude that

$$\mathbf{P}(\tau_{-a}^- < \tau_0^+) \;\geq\; p^{\lceil a/r \rceil} \;>\; 0.$$

Finally, we introduce the *Esscher transform* of a subordinator X. Let Φ stand for the Laplace exponent of X, which we suppose exists on the set A. (The set A is an infinite interval, closed or open, with left end-point in $(-\infty, 0]$.) For $p \in A$ we introduce the process $(\mathcal{E}(t, p) : t \geq 0)$ defined by

$$\mathcal{E}(t, p) := \exp\big(\Phi(p)t - pX_t\big).$$

The process $\mathcal{E}(\cdot, p)$ is a unit mean $(\mathcal{F}_t)$–martingale for each $p \in A$ [35, pg. 82], allowing us to define the family of probability measures $\big(\mathbf{P}^p : p \in A)$ by

$$\left.\frac{d\mathbf{P}^p}{d\mathbf{P}}\right|_{\mathcal{F}_t} \;=\; \mathcal{E}(t, p) \qquad \text{for} \quad t \geq 0.$$

In the sequel, it will be important for us to understand the characteristics of spectrally positive processes of bounded variation under these changes of measure. Let Z be such a process, in which case it necessarily has the form $Z_t = X_t - ct$ for some pure jump subordinator X and some $c > 0$. Under $\mathbf{P}^p$ (defined relative to X, as above), the process Z is *still* a spectrally positive process of bounded variation, and has the same drift coefficient c. Write Π for the Lévy measure of X under $\mathbf{P}$. Then the Lévy measure of Z under $\mathbf{P}^p$ is $e^{-px}\Pi(dx)$ for $x \in (0, \infty)$. Finally, Z under $\mathbf{P}^p$ also has a Laplace exponent on A; it is given by the function $\lambda \mapsto \Phi(\lambda + p) - \Phi(p) - c\lambda$, where Φ denotes the Laplace exponent of X under $\mathbf{P}$. We refer to [35, Theorem 3.9] for a proof of these facts in a more general setting; see also [35, pg. 233] for further discussion.

This completes our brief survey of Lévy process theory. We are now ready to explore fragmentation processes in more detail.

1.2.3 Tagged fragments and the intrinsic subordinator

Given a fragmentation process U, it is natural to follow those intervals containing a particular value $x \in (0, 1)$ as time passes. Accordingly, for all $t \geq 0$ and $x \in (0, 1)$ we write

$$\mathcal{I}_t^x \;:=\; \begin{cases} (U(t))_i & \text{if } x \in (U(t))_i \\ \emptyset & \text{if } x \notin U(t), \end{cases}$$

for the component of $U(t)$ which contains $x \in (0, 1)$. (Recall that $(U(t))_i$ denotes the i'th component of $U(t)$ under the order introduced in §1.1.2.) We call $\mathcal{I}_t^x$ the

x–*tagged fragment at time* t, and define corresponding length processes by $I_t^x := |\mathcal{I}_t^x|$. We also use the x–tagged processes to define an important collection $(\xi^x : x \in (0,1))$ of stochastic processes by setting

$$\xi_t^x \quad := \quad -\log I_t^x \quad \text{for} \quad t \geq 0,$$

where we define $-\log 0 := \infty$. We call this collection the $-\log$ transform of U (see Figure 1.1, page 14); it inherits genealogical information from U, allowing us to talk about ξ–particles, ξ–children, and so on. In section §1.4 we show that the collection $(\xi^x : x \in (0,1))$ naturally corresponds to a branching random walk, whenever the dislocation measure of U is finite.

Different x–tagged fragments have different laws, and are intrinsically dependent on one another. For instance, it should be clear that

$$\mathcal{I}_t^{\frac{1}{4}} \neq \mathcal{I}_t^{\frac{1}{2}} \quad \Longrightarrow \quad \mathcal{I}_t^{\frac{1}{4}} \neq \mathcal{I}_t^{\frac{3}{4}}.$$

In order to overcome these difficulties, the key idea [10, 8] is to follow a *randomly tagged* fragment and relate its behaviour to the fragmentation process as a whole. To this end, we assume that the underlying probability space is rich enough to support a random variable χ which is distributed uniformly on $(0,1)$ and is *independent of the fragmentation process*. Using this uniform random variable, we define the *randomly tagged interval process* $\mathcal{I} := (\mathcal{I}_t : t \geq 0)$ by setting $\mathcal{I}_t := \mathcal{I}_t^\chi$. We also define the length process $I = (I_t : t \geq 0)$ by $I_t := |\mathcal{I}_t|$. We note that, since the sets $(U_t : t \geq 0)$ are nested, and each of them has Lebesgue measure 1 almost surely, the set

$$\bigcup \{(0,1) \setminus U(t) : \ t \geq 0\}$$

has Lebesgue measure 0 almost surely. It follows that

$$\mathbf{P}(\mathcal{I}_t \neq \emptyset \ \text{ for all } \ t \geq 0) \quad = \quad 1.$$

Definition 1.3. *We continue to write* $\mathbf{P}$ *for the joint law of the fragmentation process and the random tag* χ*, and* $\mathbf{E}$ *for the corresponding expectation operator. We write* $\mathcal{G} = (\mathcal{G}_t : t \geq 0)$ *for the enriched filtration defined by* $\mathcal{G}_t := \sigma(\mathcal{F}_t, \mathcal{I}_t)$*. In particular, note that* $\mathcal{I}$ *is* $\mathcal{G}$*-adapted.*

The following fundamental result was first proven in [10] in the context of partition-valued fragmentations. The following formulation, appropriate in the interval-valued fragmentation context, can be found in [11]; see also [5].

Theorem 1.4. *The* $[0,\infty)$*–valued stochastic process* ξ *defined by*

$$\xi_t \quad := \quad -\log I_t \quad \textit{for} \quad t \geq 0$$

is a pure-jump $\mathcal{G}$*–subordinator. Its Lévy measure is given by*

$$L(dx) \quad = \quad e^{-x} \sum_{n=1}^{\infty} \nu(-\log |u_n| \in dx), \quad x \in (0,\infty),$$

where ν *is the dislocation measure of the underlying fragmentation process.*

The process ξ is called the *intrinsic* subordinator corresponding to the fragmentation process U. Using equation (1.1), we can write its Laplace exponent Φ in terms of L:

$$\Phi(p) \;=\; \int_{(0,\infty)} \left(1 - e^{-px}\right) \, L(dx).$$

We can also express Φ in terms of the measure ν [10, pg. 16]:

$$\Phi(p) \;=\; \int_{\mathcal{U}} \left(1 - \sum_{n=1}^{\infty} |u_n|^{1+p}\right) \nu(du). \tag{1.2}$$

Note, in particular, that $\Phi(0) = 0$. This is because dislocation measures corresponding to conservative fragmentation processes only give mass to those configurations $u \in \mathcal{U}$ with $\sum_{n=1}^{\infty} |u_n| = 1$ (c.f. [5, pg. 408]).

We already know that $\Phi(p)$ exists for values $p \geq 0$, and obviously the sum $\sum u_n^{1+p}$ diverges whenever $p < -1$. It is possible, however, that (1.2) exists (that is, takes a finite, necessarily negative value) for $p \in [-1, 0)$. With this in mind, we define the parameter

$$\underline{p} \;:=\; \inf \left\{ p \in \mathbb{R} : \int \sum_{n=2}^{\infty} |u_n|^{1+p} \nu(du) < \infty \right\}.$$

Note that the sum starts at $n = 2$; the integrability condition in Definition 1.2 controls the size of $|u_1|$. Using this integrability condition, it's easy to check that $\underline{p}$ coincides with the infimum of those $p \in \mathbb{R}$ for which

$$\int_{\mathcal{U}} \left| 1 - \sum_{n=1}^{\infty} |u_n|^{1+p} \right| \nu(du) < \infty.$$

As discussed, we know that $\underline{p} \in [-1, 0]$. The values $\Phi(\underline{p})$ and $\Phi'(\underline{p}+)$ may or may not be finite.

Next, we introduce the parameter $\overline{p}$, which is of central importance in characterising the asymptotic properties of fragmentation processes. The following lemma is taken from [12, pg. 10].

Lemma 1.5. *The equation*

$$\Phi'(p) \;=\; \frac{\Phi(p)}{1+p}$$

has a unique solution $\overline{p} \in (\underline{p}, \infty)$. *This solution is necessarily positive. We have the equivalence*

$$(1+p)\Phi'(p) - \Phi(p) > 0 \quad \Longleftrightarrow \quad p \in (\,\underline{p}, \overline{p}\,).$$

The function

$$p \mapsto \frac{\Phi(p)}{1+p}$$

increases on $(\underline{p}, \overline{p})$ *and decreases on* $(\overline{p}, \infty)$.

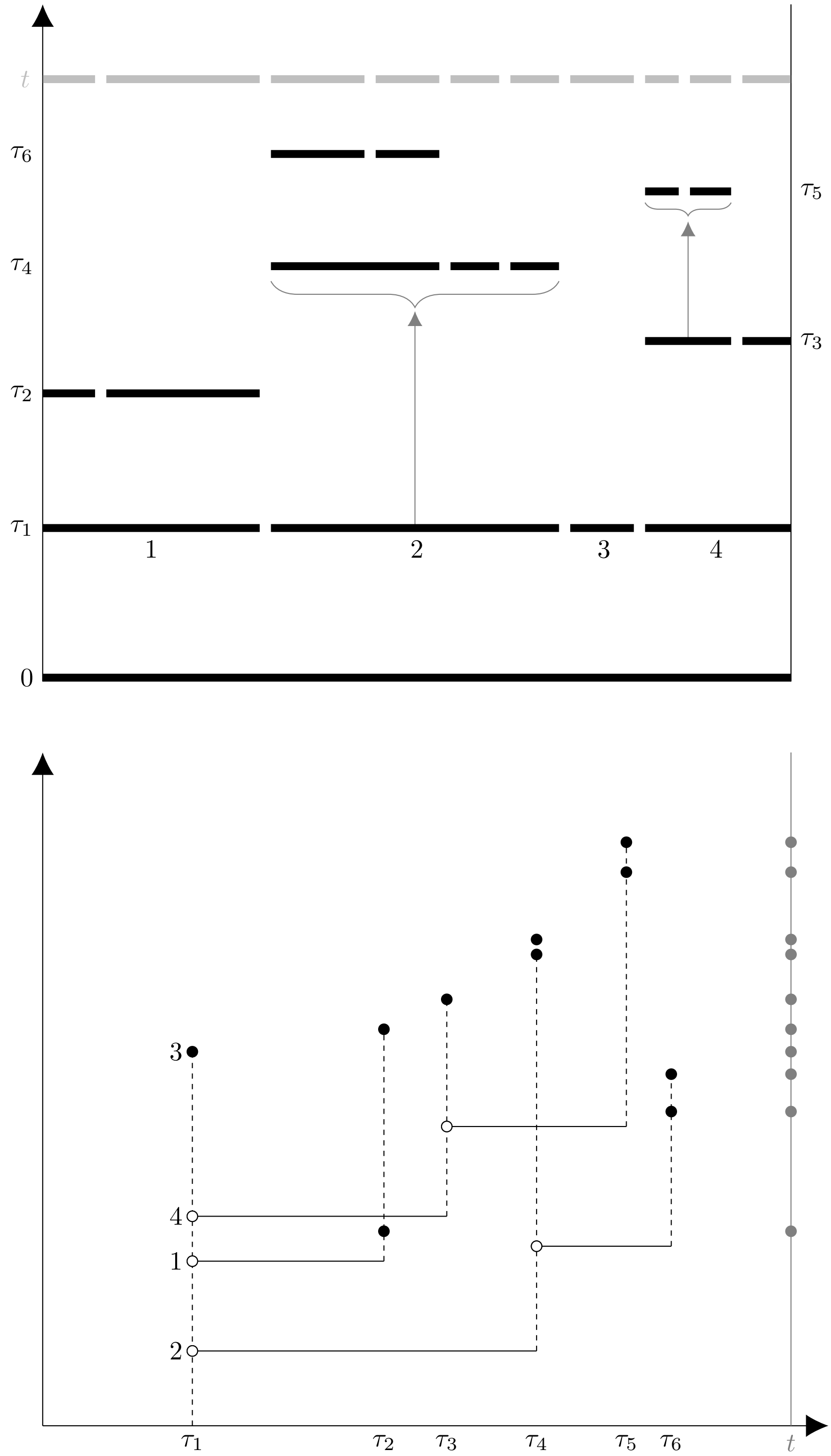

Figure 1.1. A sample path of a finite activity fragmentation process U (top), and the corresponding sample paths of the collection $((\xi_t^x)_{t\geq 0} : x \in (0,1))$ (bottom). The intervals with labels $1, 2, 3, 4 \in (0,1)$ correspond to the ξ–particle positions with respective labels. In particular, note that $-\log u$ is a decreasing map, so the largest particle is mapped to the smallest ξ–value. Particles that reproduce before time t are represented by hollow circles, and those that do not by solid circles. The union of the grey intervals equals $U(t)$, and the collection of grey circles are the corresponding realization of the point process generated by the collection $(\xi_t^x : x \in (0,1))$.

Many interesting properties of fragmentation processes can be expressed in terms of Φ and the parameters associated with it. In §1.3 we discuss some large-time asymptotic properties of fragmentation processes, which will hopefully give the reader some intuition about how fragmentation processes behave. For now, we continue with our technical survey.

1.2.4 Martingales, changes of measure, and spines

Before introducing our first class of martingales, we introduce the following useful notation. For a Borel set $A \subset (0,1)$, we use the notation $\sum_{[x]_t:A}$ to represent sums taken over the (countable) collection of *distinct* fragments alive at time t that are subsets of A. We also write $\sum_{[x]_t}$ for $\sum_{[x]_t:(0,1)}$, the sum taken over *all* distinct fragments at time t.

Let us briefly explain how such sums can be rigorously constructed as random variables. We denote by $\mathcal{D}([0,t])$ the space of càdlàg functions on $[0,t]$, and endow this space with the Skorokhod topology. We endow $\mathcal{D}([0,t])$ with the σ–algebra generated by the open sets of this topology, and write $m^+\mathcal{D}([0,t])$ for the collection of measurable functionals mapping $\mathcal{D}([0,t])$ to $[0,\infty)$. Finally, we let $m(\alpha)$ denote the midpoint of the finite interval α, and let $m_{t,i} := m((U_t)_i)$, where we recall that $(U_t)_i$ is the i'th element in the decreasing rearrangement of the components of U_t under the total order introduced in §1.1.2. Then, given $F \in m^+\mathcal{D}([0,t])$ and a Borel set $A \subseteq (0,1)$, we define

$$\sum_{[x]_t:A} F(|\mathcal{I}_s^x| : s \leq t) \ := \ \sum_{i\in\mathbb{N}} \mathbf{1}_{\{(U_t)_i \subseteq A\}} \cdot F(|\mathcal{I}_s^{m_{t,i}}| : s \leq t).$$

The first important class of martingales we will use are the *intrinsic additive martingales*, first defined in [12, pg. 10]. For $p > \underline{p}$, define

$$M(t,p) \ := \ \exp\left(\Phi(p)\, t\right) \sum_{[x]_t} (I_t^x)^{1+p}.$$

It is easy to show that $(M(t,p) : t \geq 0)$ is a non-negative, unit mean $(\mathcal{F}_t)$–martingale whenever $p > \underline{p}$. The martingale convergence theorem then tells us that $M(\cdot,p)$ converges almost surely to an almost surely finite random variable, $M(\infty,p)$. In fact [12, Theorem 2], whenever $p \in (\underline{p},\overline{p})$, the process $M(\cdot,p)$ is uniformly integrable, and $M(\infty,p) > 0$ almost surely. This result is the analogue of the work on branching random walk contained in the famous paper [17] by John Biggins.

The second class of martingales we will use are the exponential martingales introduced in §1.2.2, but now corresponding to the particular $\mathcal{G}$–subordinator ξ. In fact, we will only use $\mathcal{E}(\cdot,\overline{p})$. To see why, write $c_{\overline{p}} := \Phi(\overline{p})/(1+\overline{p}) = \Phi'(\overline{p})$, and introduce the process ζ defined by $\zeta_t := \xi_t - c_{\overline{p}}t$. Following the general theory in §1.2.2, we also define a measure $\mathbf{Q}$ on $\mathcal{G}_\infty := \sigma(\bigcup\{\mathcal{G}_t : t \geq 0\})$ by setting

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{G}_t} \ = \ \mathcal{E}(t,\overline{p}) \qquad \text{for} \quad t \geq 0.$$

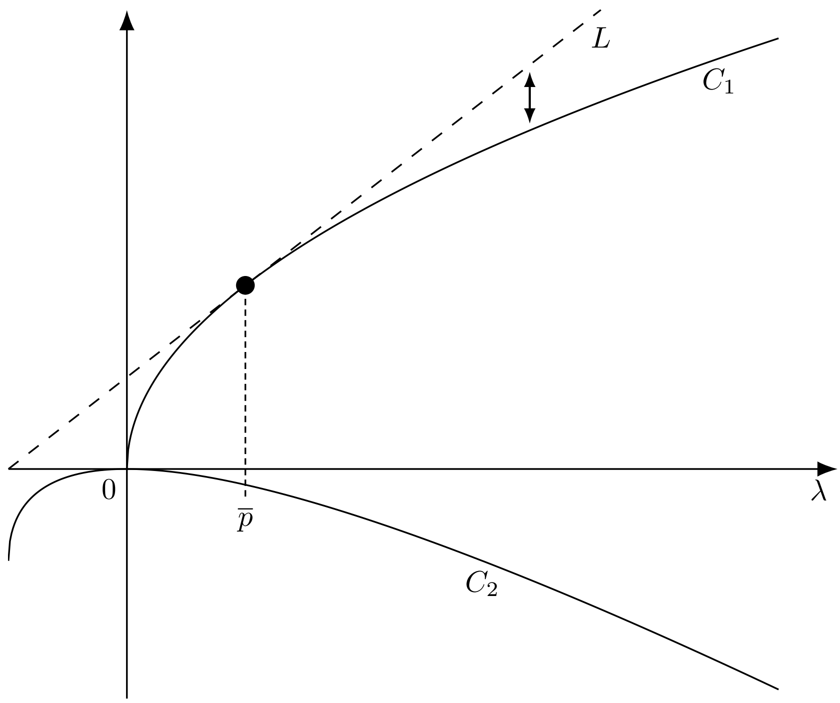

Figure 1.2. Illustration of the power of the transformation $(\xi, \mathbf{P}) \to (\zeta, \mathbf{Q})$ in the worst-case scenario, in which $\underline{p} = 0$, with $\Phi'(0+) = \infty$. C_1 maps the Laplace exponent Φ of $(\xi, \mathbf{P})$, and C_2 maps the Laplace exponent $\lambda \mapsto \Phi(\lambda + \overline{p}) - \Phi(\overline{p}) - c_{\overline{p}}\lambda$ of $(\zeta, \mathbf{Q})$. The line L is the tangent to C_1 at $\lambda = \overline{p}$, and therefore has gradient $\Phi'(\overline{p}) = c_{\overline{p}}$. Clearly, C_1 is sent to C_2 by placing a new origin at the solid circle, and taking differences between L and C_1 as illustrated by the double-headed arrow. The diagram makes it clear that the Lévy process $(\zeta, \mathbf{Q})$ is centred and has finite moments of all orders.

We have the following:

Lemma 1.6. *The process* $(\zeta, \mathbf{Q})$

1. *is a centered spectrally positive Lévy process of bounded variation;*
2. *has drift coefficient* $c_{\overline{p}}$*, and Lévy measure* $e^{-\overline{p}x}L(dx)$*; and*
3. *has finite exponential moments:*

$$\forall \epsilon \in [0, \overline{p} - \underline{p}) \;\; \forall t \geq 0 \quad \mathbf{Q} \exp(\epsilon |\zeta_t|) < \infty.$$

In other words, $(\zeta, \mathbf{Q})$ is just about the nicest kind of Lévy process there is, after the compound Poisson process. Facts 1 and 2 (except the centeredness property) are particular instances of the general theory described in §1.2.2. Fact 3 can be found in [36, §4.1]. So we just need to show that the process is centered. From §1.2.2, we know that the Laplace exponent of $(\zeta, \mathbf{Q})$ is $\lambda \mapsto \Phi(\lambda + \overline{p}) - \Phi(\overline{p}) - c_{\overline{p}}\lambda$. The derivative of this function at 0, which coincides with the value $\mathbf{Q}\,\zeta_1$, equals $\Phi'(\overline{p}) - c_{\overline{p}}$. By the definitions of $\overline{p}$ and $c_{\overline{p}}$ we know that $\Phi'(\overline{p}) - c_{\overline{p}} = 0$. An illustration of the transformation $(\xi, \mathbf{P}) \to (\zeta, \mathbf{Q})$ in terms of Laplace exponents is given in Figure 1.2.

The importance of the process $(\zeta, \mathbf{Q})$ is reflected by the following *Many-to-One Lemma*, which will be a fundamental tool in our proofs. To state it, we define the family of processes $(\zeta^x : x \in (0,1))$ by $\zeta^x_t := \xi^x_t - c_{\overline{p}}t$ for $t \geq 0$.

Lemma 1.7. (MT1) *For any map $F \in m^+\mathcal{D}([0,t])$ and any starting configuration $u \in \mathcal{U}$ we have*

$$\mathbf{E}_u \sum_{[x]_t} F(\zeta_s^x : s \le t) \;=\; \sum_{i=1}^{\infty} \mathbf{Q}\big(e^{(1+\overline{p})\zeta_t} F(\zeta_s - \log|u_i| : s \le t)\big).$$

In particular,

$$\mathbf{E} \sum_{[x]_t} F(\zeta_s^x : s \le t) \;=\; \mathbf{Q}\big(e^{(1+\overline{p})\zeta_t} F(\zeta_s : s \le t)\big).$$

Proof. First we remind the reader of the notation $\sum_{[x]_t:A}$ introduced earlier in this section. In particular, $\sum_{[x]_t:u_i}$ stands for the sum taken over distinct particles at time t which result from the fragmentation of the interval u_i, which is $\mathbf{E}_u$–almost surely a component of $U(0)$. Now we make the following simple calculation:

$$\begin{aligned}
\mathbf{E}_u \sum_{[x]_t} F(\zeta_s^x : s \le t) &= \sum_{i=1}^{\infty} \mathbf{E}_u \sum_{[x]_t:u_i} F(\zeta_s^x : s \le t) \\
&= \sum_{i=1}^{\infty} \mathbf{E} \sum_{[x]_t:(0,1)} F(\zeta_s^x - \log|u_i| : s \le t),
\end{aligned}$$

where the sums in i should be regarded as finite in case u consists of finitely many blocks. In the second equality we have used the fact that, fixing $x \in u_i$, the law of I_t^x under $\mathbf{E}_u$ is the same as the law of $|u_i| \cdot I_t^{g(x)}$ under $\mathbf{E}$, were g is the affine map sending u_i to $(0,1)$. This means that the law of ζ_t^x under $\mathbf{E}_u$ equals the law of $\zeta_t^{g(x)} - \log|u_i|$ under $\mathbf{E}$.

Now we make a size-biased pick: for $\alpha \in \mathbb{R}$, we can write

$$\begin{aligned}
\mathbf{E} \sum_{[x]_t} F(\zeta_s^x + \alpha : s \le t) &= \mathbf{E} \sum_{[x]_t} I_t^x \cdot (I_t^x)^{-1} \cdot F(\zeta_s^x + \alpha : s \le t) \\
&= \mathbf{E} \sum_{[x]_t} \mathbf{P}(\chi \in \mathcal{I}_t^x | \mathcal{F}_t) \cdot (I_t^x)^{-1} \cdot F(\zeta_s^x + \alpha : s \le t) \\
&= \mathbf{E} \sum_{[x]_t} \mathbf{1}_{(\chi \in \mathcal{I}_t^x)} \cdot (I_t^x)^{-1} \cdot F(\zeta_s^x + \alpha : s \le t) \\
&= \mathbf{E}\,(I_t)^{-1} \cdot F(\zeta_s + \alpha : s \le t).
\end{aligned}$$

The first and final lines are trivial. In the second we use the fact that the uniform random variable χ is independent of the fragmentation process. In the third line we use the $\mathcal{F}_t$–measurability of everything outside the conditional probability in the line before.

To obtain the required result it remains to use the definitions of the measure $\mathbf{Q}$, the process ζ_t, and the special value $c_{\overline{p}} = \Phi(\overline{p})(1+\overline{p})^{-1}$:

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{G}_t} \;=\; \exp\big(\Phi(\overline{p})t - \overline{p}\xi_t\big) \;=\; I_t^{-1} \exp\big(-(1+\overline{p})\zeta_t\big).$$

Substituting this simple rearrangement into the previous display yields the required result. □

The Many-to-One Lemma relates functionals of paths of fragmentation processes to functionals of the paths of ζ. In view of this property, the process $(\zeta, \mathbf{Q})$ is referred to as the *spine* of the fragmentation process.

1.2.5 Frozen fragmentation processes

Usually the simple Many-to-One Lemma above will be sufficient for our purposes. On one occasion, however, we will need a version that we can apply to *frozen fragmentation processes*. The following definition was introduced by Bertoin [11]. His picturesquely named *frosts* bear the same relation to fragmentation processes as stopping times do to Markov processes.

Definition 1.8. *Let $\mathcal{F}^x := (\mathcal{F}^x_t : t \geq 0)$ denote the filtration (completed by null sets) generated by the process $(\mathcal{I}^x_t : t \geq 0)$. We call a random function $T : (0,1) \to [0,\infty]$ a frost for the fragmentation process U if*

1. *for all $x \in (0,1)$, T_x is an $\mathcal{F}^x$–stopping time; and*
2. *$T_x = T_y$ whenever $x \in (0,1)$ and $y \in \mathcal{I}^x_{T_x}$.*

A fragmentation process U together with a frost T and a time $t \geq 0$ naturally correspond to the element of $\mathcal{U}$ whose decomposition is given by

$$\left\{\mathcal{I}^x_{T_x} : x \in (0,1), T_x \leq t\right\} \cup \left\{\mathcal{I}^x_t : x \in (0,1), T_x > t\right\} ;$$

see Figure 1.3, page 19. We will use the notation $\sum_{(T,t)}$ to refer to sums taken over the distinct interval components of this decomposition. Such sums can be constructed rigorously as random variables, but this time we don't labour the point.

Given a frost T, we introduce the $\mathcal{G}$–stopping time $\tau(T) := T_\chi$, where χ is the uniformly distributed random tag used to define $\mathcal{I}$. We now state a Many-to-One Lemma for frosts. It is proven in the same way as the standard Many-to-One Lemma.

Lemma 1.9. *For each $s \in [0,t]$ let F_s be a map in $m^+\mathcal{D}([0,s])$. For any frost T and any starting configuration $u \in \mathcal{U}$ we have*

$$\mathbf{E}_u \sum_{(T,t)} F_{T_x \wedge t}(\zeta^x_s : s \leq T_x \wedge t) \quad = \quad \sum_{i=1}^{\infty} \mathbf{Q}\big(e^{\zeta_{\tau\wedge t}(\overline{p}+1)} F_{\tau\wedge t}(\zeta_s - \log|u_i| : s \leq \tau \wedge t)\big),$$

where $\tau := \tau(T)$. In particular,

$$\mathbf{E} \sum_{(T,t)} F_{T_x \wedge t}(\zeta^x_s : s \leq T_x \wedge t) \quad = \quad \mathbf{Q}\big(e^{\zeta_{\tau\wedge t}(\overline{p}+1)} F_{\tau\wedge t}(\zeta_s : s \leq \tau \wedge t)\big).$$

Our second and third main results concern the survival probability of *killed fragmentations*, which we proceed to define in the next section.

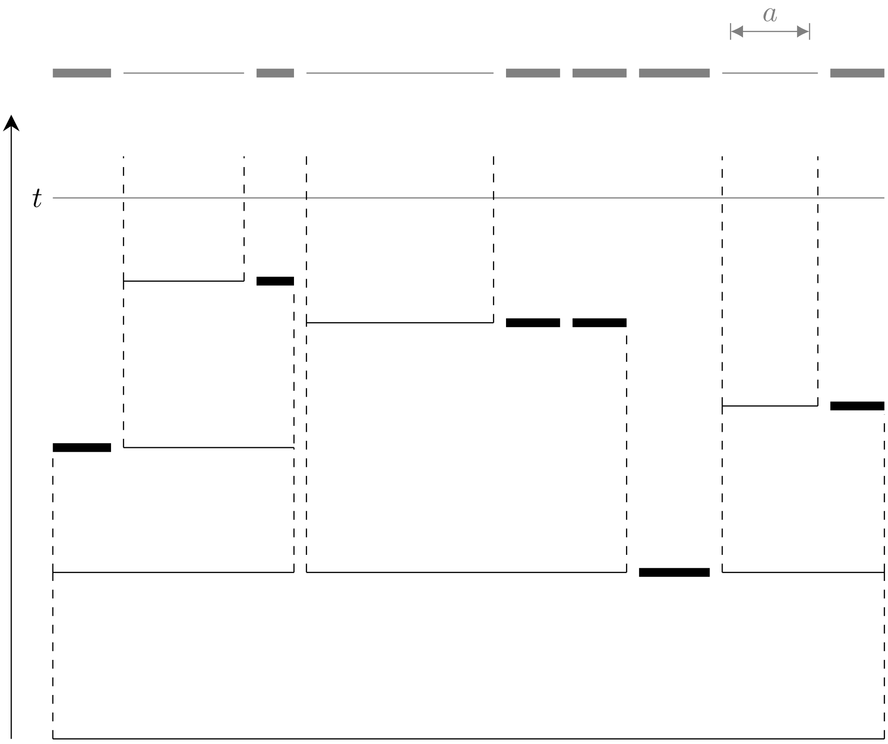

Figure 1.3. An illustration of the frost $T := x \mapsto \inf\{t \geq 0 : I_t^x < a\}$ for fixed $a \in (0,1)$. Time runs vertically. Whenever a fragment is produced whose size is smaller than a, it is "frozen", ceasing to break apart any further. These fragments, in their frozen state, are represented at the time of their birth as thick horizontal lines. The remaining fragments continue to evolve as usual; this is signified by the vertical dashed lines. At the bottom of the figure, we have illustrated the element of $\mathcal{U}$ corresponding to the frost T and the time t.

1.2.6 Fragmentation processes with killing

Given a fragmentation process, Knobloch and Kyprianou [29] fix parameters $a, c \in \mathbb{R}$, and introduce an embedded process in which particles born at time t are removed from the system if their size is less than $\exp(-a - ct)$. This corresponds to removing the particle tagged by $x \in (0,1)$ from the system at time t if and only if $\xi_t^x \geq ct + a$. We will call the resulting process the (a,c)–*killed fragmentation process*. When $a < 0$, the (a,c)–killed fragmentation process obviously dies immediately, so we can assume that $a \geq 0$. In fact, little loss of generality is incurred by making the assumption $a = 0$, and, for simplicity, we will do so in the statements and proofs of our second and third main results. We will refer to the $(0,c)$–killed fragmentation process as the c-killed process; see Figure 1.4, page 20.

Let us now fix a fragmentation process, and consider what happens to the c-killed process as c varies. It turns out that the case $c = c_{\overline{p}}$ is critical in the following sense. Whenever $c \leq c_{\overline{p}}$, the c–killed process almost surely dies within a finite period of time. When $c > c_{\overline{p}}$, the probability of survival lies in $(0,1)$. With these facts in mind, we say that a c–killed fragmentation process is *supercritically killed* when $c > c_{\overline{p}}$, *critically killed* when $c = c_{\overline{p}}$, and *subcritically killed* when $c < c_{\overline{p}}$.

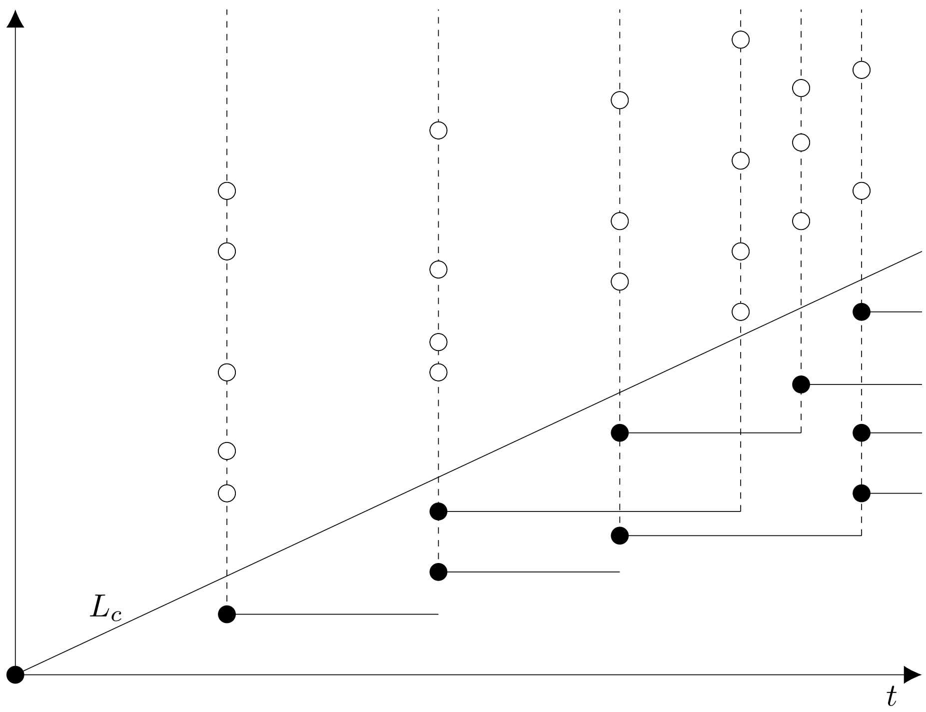

Figure 1.4. The $-\log$ transform of a c-killed fragmentation process in case $\nu(\mathcal{U}) < \infty$. L_c is the graph of the map $t \mapsto ct$. Particles whose ξ–values at birth fall above this line, are killed instantly; they are represented by hollow circles. The remaining particles (whose $-\log$ values correspond to the solid circles) reproduce as normal.

Finally, we introduce some more notation that will be used later. For $\epsilon > 0$, we let $\rho(\epsilon)$ stand for the survival probability of the $(c_{\overline{p}}+\epsilon)$–killed fragmentation process. We also let $\kappa(t)$ stand for the probability that the critically killed fragmentation survives until time $t \geq 0$. Later on, we will consider the asymptotics of $\rho(\epsilon)$ as $\epsilon \downarrow 0$, and the asymptotics of $\kappa(t)$ as $t \to \infty$.

1.2.7 Partition-valued fragmentation processes

In Chapter 2, we will show how to calculate the second moments of random variables of the form

$$Z \ := \ \sum_{[x]_t} F(I_s^x \,:\, 0 \leq s \leq t),$$

for elements $F \in m^+\mathcal{D}([0,t])$. One way of tackling this problem would be to formulate a *Many-to-Two Lemma*, expressing second moments in terms of *two* randomly tagged particles. Harris and Roberts [26] develop this approach for a large class of branching processes, and address the more general problem of calculating k'th moments for any integer $k \geq 1$. Our approach is slightly different—we will express second moments in terms of a single randomly tagged particle—but the Many-to-Two approach would work just as well.

The first step is to write $\mathbf{E}Z^2 = \mathbf{E}Z + \mathbf{E}\Lambda$, with

$$\Lambda \;:=\; \sum_{[x]_t \neq [y]_t} F(I_s^x : 0 \leq s \leq t) F(I_s^y : 0 \leq s \leq t),$$

where the sum is over distinct components of the fragmentation process which are alive at time t. We would then like to proceed by using an *ancestral decomposition* of the fragmentation process. As will become clear, giving such a decomposition rigorous sense relies on the existence of a suitable Poissonian construction of fragmentation processes at the path-wise level. Such a construction is not available (in the literature) for $\mathcal{U}$–valued processes, beyond Basdevant's tantalizing remark that "A Poissonian construction of an [interval] fragmentation with no erosion is also possible... For more details, we refer to Berestycki [7] who has already proved this result for [ranked] fragmentation and the same approach works in our case." Rather than pursue this approach, we will show how to relate a given $\mathcal{U}$–valued fragmentation to a particular partition-valued process whose Poissonian construction has been considered extensively in the literature (see, for instance, [10, 15]). We attempt to keep the exposition as concise as possible. No material in this section is original; the constructive work is lifted from [10], and the *coup de grâce* is delivered by [5].

We start with a few definitions:

1. A *partition* of $A \subset \mathbb{N}$ is collection of pairwise disjoint, non-empty subsets of $\mathbb{N}$ whose union equals A. For $A \subset \mathbb{N}$, the symbol $\mathcal{P}(A)$ stands for the set of partitions of A. We write $\mathcal{P}$ for $\mathcal{P}(\mathbb{N})$.

2. Given a set $A \subset \mathbb{N}$ and a partition $\Gamma \in \mathcal{P}(A)$, we write $N_\Gamma \in \mathbb{N} \cup \{\infty\}$ for the number of blocks (i.e. subsets of $\mathbb{N}$) in Γ. We then write the blocks $(B_i^\Gamma : 1 \leq i \leq N_\Gamma)$ of Γ in order of increasing least element. Normally we will just write $\Gamma = (B_1, B_2, ...)$ with the understanding that this sequence may be finite.

3. For sets $A \subset B \subset \mathbb{N}$ and $\Gamma = (B_1, B_2, ...) \in \mathcal{P}(B)$, we write Γ_A for the element of $\mathcal{P}(A)$ whose blocks are the non-empty entries in the sequence $(B_1 \cap A, B_2 \cap A, ...)$.

4. For $A \subset \mathbb{N}$ we define $\mathcal{P}^*(A) := \{\Gamma \in \mathcal{P} : \Gamma_A \neq \{A\}\}$. We write $\mathcal{P}^*$ for $\mathcal{P}^*(\mathbb{N})$, which of course is just $\mathcal{P} - \{\mathbb{N}\}$.

5. We write $[n]$ for $\{1, ..., n\}$, $\mathcal{P}_n$ for $\mathcal{P}([n])$, and $\mathcal{P}_n^*$ for $\mathcal{P}^*([n])$. In particular, note that elements of $\mathcal{P}_n^*$ are partitions of $\mathbb{N}$—not of $[n]$.

6. Given a partition $\Gamma \in \mathcal{P}$, we write Γ^n for the block of Γ that contains n.

For elements Γ and Δ of $\mathcal{P}$, we write $n(\Gamma, \Delta)$ for the supremum of those $k \in \mathbb{N}$ witnessing $\Gamma_{[k]} = \Delta_{[k]}$. Note that $n(\Gamma, \Delta) = \infty$ if and only if $\Gamma = \Delta$. We then define $d(\Gamma, \Delta) := \exp(-n(\Gamma, \Delta))$. The pair $(\mathcal{P}, d)$ is a compact metric space. We endow $\mathcal{P}$ with the σ–algebra generated by the collection of d–open subsets of $\mathcal{P}$.

We also need to introduce a mechanism by which one partition can be used to dislocate another. To this end, fix a set $A \subset \mathbb{N}$, and a partition $\Gamma \in \mathcal{P}$. Let $(a_1, a_2, ...)$ be the increasing enumeration of the elements of A (this sequence may be finite). We define an equivalence relation on A by saying that $(a_i \sim a_j)$ if and only if i and j lie in the same block of Γ. The resulting equivalence classes form a partition of A which we call $\Gamma \circ A$.

Now we fix $A \subset \mathbb{N}$, a partition $\Delta = (\Delta_1, \Delta_2, ...) \in \mathcal{P}(A)$ and a partition $\Gamma \in \mathcal{P}$. Given $k \in \mathbb{N}$ we define a new partition $\Gamma \overset{k}{\circ} \Delta \in \mathcal{P}(A)$ as follows. If $k > N_\Delta$, then the operator $\Gamma \overset{k}{\circ} (\cdot)$ acts as the identity: $\Gamma \overset{k}{\circ} \Delta = \Delta$. Otherwise, we replace the block Δ_k with $\Gamma \circ \Delta_k$ (as defined in the previous paragraph) and leave the blocks $(\Delta_i : i \neq k)$ in tact.

We emphasize that $\Gamma \overset{k}{\circ} (\cdot)$ may act as the identity on Δ, even when $N_\Delta \geq k$. This happens precisely when the set $\{1, 2, ..., \mathrm{Card}(\Delta_k)\}$ is a subset of some block of Γ. (This condition reduces to $\Gamma = \{\mathbb{N}\}$ in case $\mathrm{Card}(\Delta_k) = \infty$, but otherwise has non-trivial import.)

Next we introduce the auxiliary space $\mathcal{S}^\downarrow \subset [0,1]^{\mathbb{N}}$ defined by

$$\mathcal{S}^\downarrow \;\; := \;\; \left\{ s_1 \geq s_2 \geq ... \geq 0 : \; \sum s_i = 1 \right\},$$

which we endow with the topology of point-wise convergence. A measure λ on $\mathcal{S}^\downarrow$ is called a *Lévy measure* if it assigns no mass to the singleton $\{(1,0,0,...)\}$, and satisfies the integrability condition

$$\int_{\mathcal{S}^\downarrow} (1 - s_1)\, \lambda(ds) \;\; < \;\; \infty. \tag{1.3}$$

Given an element $s \in \mathcal{S}^\downarrow$, we follow Bertoin following Kingman by defining a probability measure P^s on $\mathcal{P}$ using a "paint-box" construction. Let Y be a random variable specified by setting $\mathbb{P}(Y = n) = s_n$, and let $(Y_i : i \in \mathbb{N})$ be a sequence of independent copies of Y. We define an equivalence relation on $\mathbb{N}$ by writing

$$(i \sim j, s) \quad \Longleftrightarrow \quad Y_i = Y_j.$$

This relation generates a random partition of $\mathbb{N}$ whose law we denote by P^s. Given a Lévy measure λ, we define the mixture

$$\mu_\lambda(\cdot) \;\; := \;\; \int_{\mathcal{S}^\downarrow} \lambda(ds) \cdot P^s(\cdot)$$

which is a measure on $\mathcal{P}$. We note that μ_λ is sigma-finite for all Lévy measures λ. Indeed, for any $s \neq (1, 0, 0...)$, we have

$$P^s(\mathcal{P}^*_n) \;\; = \;\; 1 - \sum_{k=1}^{\infty} s_k^n \;\; \leq \;\; 1 - s_1^n \;\; \leq \;\; n(1 - s_1)$$

and also $P^s(\{\mathbb{N}\}) = 0$. It remains to note that $\mathcal{P} = \bigcup \mathcal{P}^*_n \cup \{\mathbb{N}\}$, that Lévy measures assign no mass to $\{(1, 0, 0, ...\}$, and to apply the integrability condition (1.3).

Now we fix a Lévy measure, and show how to associate it with a $\mathcal{P}$–valued stochastic process. We have seen that $(\mathcal{P}, d)$ is a compact metric space—in particular, Polish—and that $\mu = \mu_\lambda$ is a sigma finite measure on this space. We endow $\mathbb{N}$ with the discrete metric, and write $\#$ for the counting measure. Then $[0, \infty) \times \mathbb{N} \times \mathcal{P}$ is also Polish, and Leb $\times \# \times \mu$ is a σ–finite measure on this space. The machinery of Poisson point processes can therefore grind into action. We let M be the Poisson random measure on $[0, \infty) \times \mathbb{N} \times \mathcal{P}$ with intensity Leb $\times \# \times \mu$. We let M^n denote the projection of M to the space $[0, \infty) \times [n] \times \mathcal{P}_n^*$. The atoms of M^n arrive at the finite rate $n \cdot \mu(\mathcal{P}_n^*)$. Fixing $n \in \mathbb{N}$, this observation allows us to construct a process $(\pi_s^{(n)} : s \geq 0)$ according to the following rules:

1. $\pi_0^{(n)} := [n]$.
2. $\pi^{(n)}$ is a pure jump process which jumps at time t *only if* M^n has an atom on the fibre $\{t\} \times [n] \times \mathcal{P}_n^*$.
3. If M^n has an atom at (t, k, π) we define $\pi_t^{(n)} := \pi \overset{k}{\circ} \pi_{t-}^{(n)}$.

An atom (t, k, π) of M^n may act trivially in the construction above; see the previous paragraph beginning "We emphasize that...".

The processes $\pi^{(n)}$ are piecewise constant and, by construction, right-continuous (hence càdlàg). Moreover, it is quite easy to see that they are compatible with restriction in the sense that $(\pi_t^{(n+1)})_{[n]} = \pi_t^{(n)}$ for all $n \in \mathbb{N}$ at $t \geq 0$. This allows us to define a $\mathcal{P}$-valued process Π by insisting that for all $n \in \mathbb{N}$ and $t \geq 0$, the restriction of Π_t to $[n]$ equals $\pi_t^{(n)}$. Moreover, we can assert that Π is a pure jump càdlàg process.

To summarize, we have seen the following. There exists a pure jump càdlàg process that jumps only when M has atoms. When M has an atom at (t, k, π), the partition $\Pi(t-)$ is replaced by the partition $\pi \overset{k}{\circ} \Pi_{t-}$. There is clearly only one $\mathcal{P}$-valued process with these properties, and we call $\Pi = \Pi(\lambda)$ the Poissonian $\mathcal{P}$–fragmentation with Lévy measure λ. We make the following definition:

Definition 1.10. *A $\mathcal{P}$–valued process is called a conservative homogeneous $\mathcal{P}$–fragmentation process if it is equal in law to $\Pi(\lambda)$ for some Lévy measure λ.*

Now we turn our attention to the subject of *asymptotic frequencies.* A set $A \subset \mathbb{N}$ is said to have an asymptotic frequency if the limit

$$\lim_{n \to \infty} \frac{1}{n} \#(A \cap [n])$$

exists, and then we write $|A|$ for value of this limit. For an index set $I \subset \mathbb{N}$ and a sequence $(a_i : i \in I)$ of positive numbers with $\sum a_i \leq 1$ we write $(a_i : i \in I)^\downarrow$ for the decreasing rearrangement of $(a_i : i \in I)$ (in case some of the a_i are equal, we preserve their original ordering in the rearrangement).

Lemma 1.11. *Fix a Poissonian $\mathcal{P}$–valued fragmentation Π, write $(B_1(t), B_2(t), ...)$ for the blocks of $\Pi(t)$, and N_t for N_{Π_t}. The event*

$$|B_i(t)| \text{ exists and is postive for all } t \geq 0 \text{ and } 1 \leq i \leq N_t$$

occurs with probability 1. *On this event it is almost surely the case that*

$$\sum_{i=1}^{N_t} |B_i(t)| = 1 \text{ for all } t \geq 0.$$

Suppose M *has an atom at* (t, k, π), *and that* $N(t) \geq k$. *The sequence* $(|B_i(t)| : 1 \leq i \leq N_t)^{\downarrow}$ *is equal to the sequence obtained by interpolating the values* $(|\pi_j| \cdot |B_k(t-)| : 1 \leq j \leq N_\pi)^{\downarrow}$ *between the elements of* $(|B_i(t-)| : i \neq k,\ 1 \leq i \leq N(t-))^{\downarrow}$.

With regards to the final statement, we note that μ_λ–almost everywhere, $|\pi_j| \neq 0$ for all $1 \leq j \leq N_\pi$.

Let us summarise some other useful properties of conservative homogeneous $\mathcal{P}$–valued fragmentation processes. We recall that a permutation on $\mathbb{N}$ is called "finite" if its restriction to $\mathbb{N} - [n]$ acts as the identity for some $n \in \mathbb{N}$, and note that any permutation naturally induces a map from $\mathcal{P}$ to itself.

Theorem 1.12. *Let* Π *be a conservative homogeneous* $\mathcal{P}$*–valued fragmentation processes and assume that* Π *is càdlàg. Then*

1. Π *has the Feller property.*
2. Π *is exchangeable: for all finite permutations* σ *on* $\mathbb{N}$, $\sigma\Pi_t$ *is equal in law to* Π_t.
3. Π *has the fragmentation property: given* $\Pi_t = (B_1, B_2, ...)$, *the process* $(\Pi(t+s) : s \geq 0)$ *is equal in law to the process* $(\Gamma_s^{(1)} \circ B_1, \Gamma_s^{(2)} \circ B_2, ... : s \geq 0)$, *where the* $\Gamma^{(i)}$ *are independent copies of* Π.
4. *The event* $\{|\Pi_t^1|$ *exists for all* $t \geq 0\}$ *has probability* 1, *and* $(-\log |\Pi_t^1| : t \geq 0)$ *is a subordinator.*

The second two properties are usually used to *define* $\mathcal{P}$–valued fragmentations, before showing that every such process has a Poissonian version.

Now we explain the fundamental relationship between $\mathcal{U}$–valued and $\mathcal{P}$–valued fragmentation processes. Let us fix a $\mathcal{U}$–valued fragmentation process U. We assume that the underlying probability space is rich enough to support a sequence $(X_n : n \in \mathbb{N})$ of independent random variables each with the uniform distribution on $(0, 1)$, which are independent of U. In an abuse of notation, we will write $\mathbf{E}$ for the joint law of the fragmentation process U, this sequence of random variables, and the uniform random tag χ we used before. Recalling that the set

$$\bigcup \{(0,1) \setminus U(t) : \quad t \geq 0\}$$

almost surely has Lebesgue measure 0, it is $\mathbf{E}$–almost surely the case than for all t "simultaneously", we can well-define an equivalence relation on $\mathbb{N}$ by

$$(i \sim j,\, U_t) \iff X_i \text{ and } X_j \text{ lie in the same block of } U_t.$$

We let Π^U_t stand for the partition of $\mathbb{N}$ generated by this equivalence relation, and then let $\Pi(U) := (\Pi^U_t : t \geq 0)$. Finally, let us write $\varrho : \mathcal{U} \to \mathcal{S}^{\downarrow}$ for the map $u \mapsto (|u_1|, |u_2|, ...)$. We note that if ν is a dislocation measure, then $\varrho\nu$ is a Lévy measure. The following result is obtained from [5] by first mapping U to a fragmentation taking values in the space of ordered partitions of $\mathbb{N}$, and then projecting this fragmentation onto $\mathcal{P}$.

Theorem 1.13. *Fix a $\mathcal{U}$-valued fragmentation process U with dislocation measure ν. Then the process $\Pi(U)$ is equal in law to $\Pi(\varrho\nu)$.*

1.3 What do fragmentation processes look like?

In this section we will discuss some existing results that describe the asymptotic behaviour of fragmentation processes, focusing first on results concerning the *speed* of fragments, before turning our attention to killed fragmentation processes. We will see that different qualitative behaviours arise depending on the underlying dislocation measure ν. These different cases can be separated conveniently using the Laplace exponent Φ of the intrinsic subordinator ξ.

We say the particle labelled by $x \in (0,1)$ has *speed* $v \in [0,\infty]$ if we have the almost sure convergence

$$\frac{\xi^x_t}{t} \;=\; \frac{-\log I^x_t}{t} \;\to\; v \qquad \text{as } t \to \infty.$$

This means, of course, that at the large time t, the fragment containing the tag x has size approximately equal to $\exp(-vt)$. The interesting case where $v = \infty$ corresponds to particles exhibiting *superexponential* decay.

Before proceeding with our discussion, let's introduce a few parameters that will appear frequently. We define $v_{\text{typ}} := \Phi'(0+) \in (0,\infty]$, $v_{\min} := \Phi'(\overline{p}) = c_{\overline{p}} \in (0,\infty)$, and $v_{\max} := \Phi'(\underline{p}+) \in (0,\infty]$. As we will see, these values, when finite, are the typical, minimal and maximal particle speeds, respectively.

Jean Bertoin [12] carried out the first work on the asymptotic properties of fragmentation processes in their most general form. (Although [12] was only published in 2003, preprints existed as early as 2001.) In this paper, the paths of fragmentation processes are described using a family of random measures. For $t \geq 0$, introduce the measures ρ_t on the Borel sets of $[0,\infty)$, where

$$\rho_t \;:=\; \sum_{[x]_t} I^x_t \, \delta_{\xi^x_t/t} \, .$$

As usual, the measure δ_a attributes unit mass to the value $a \in [0,\infty)$. Bertoin shows that a law of large numbers and a central limit theorem hold for the measures $(\rho_t : t \geq 0)$ under certain hypotheses. To be precise, we introduce the value $\sigma^2 := -\Phi''(0+) \in (0,\infty]$, and let $\widehat{\rho_t}$ stand for the image of ρ under the map $x \mapsto \sqrt{t}(x - v_{\text{typ}})/\sigma$.

Then [12, Theorem 1(a)], as $t \to \infty$,

$$\rho_t \xrightarrow{\mathbf{P}} \delta_{v_{\text{typ}}} \qquad \text{whenever} \qquad v_{\text{typ}} < \infty, \qquad \text{and}$$
$$\widehat{\rho_t} \xrightarrow{\mathbf{P}} \mathcal{N} \qquad \text{whenever} \qquad \sigma^2 < \infty,$$

where $\mathcal{N}$ denotes the standard normal distribution.

One very simple consequence of the first convergence result in the previous display is that $\xi_t/t \to v_{\text{typ}}$ in probability, whenever $v_{\text{typ}} < \infty$. In fact, the strong law of large numbers for the subordinator ξ tells us that this convergence holds almost surely. This simple result has an interesting implication. As Bertoin notes [12, pg. 7], the measure ρ_t coincides with conditional distribution of ξ_t/t given the underlying fragmentation process (after this conditioning, the "only randomness" comes from the uniform random tag used in the definition of ξ). As a result, whenever $v_{\text{typ}} < \infty$,

$$\left|\{x \in (0,1) : I^x \text{ has speed } v_{\text{typ}}\}\right| \;=\; 1.$$

This makes precise the statement that v_{typ} is the *typical* fragment speed, whenever this value is finite.

In the case where $v_{\text{typ}} = \infty$, Bertoin [12, Proposition 1] extends the two convergence statements above, under the additional hypothesis that Φ varies regularly at 0. The statement of this theorem is rather complicated, so we omit it here.

Bertoin then proceeds to study large deviations of the measures $(\rho_t : t \geq 0)$ by applying the Gärtner-Ellis theorem. Rather than quoting these results in full, let us mention that they are used [12, pg. 15] to explicitly calculate the value of the function C on $(-\infty, 0)$ defined by

$$C(a) \;:=\; \lim_{\epsilon \downarrow 0} \lim_{t \to \infty} \log \# \left\{ \mathcal{I}_t^x : x \in (0,1),\;\; e^{(a-\epsilon)t} \leq I_t^x \leq e^{(a+\epsilon)t} \right\}$$

in terms of the Legendre transform of a function associated with Φ. Roughly speaking, $\exp(C(a)t)$ is the number of particles at time t of size $\exp(at)$, whenever t is sufficiently large. Bertoin then uses the properties of the function C to make the following remark: almost surely there exist particles of size roughly $\exp(-v_{\min}t)$ at time t, though the number of such particles is always less than $\exp(\eta t)$ for all $\eta > 0$. Moreover, there are *no* particles of larger size at time t.

The previous paragraph suggests that the size of the *largest* particle has size approximately equal to $\exp(-v_{\min}t)$ at time t. Indeed, we refer to [14, Corollary 1.4] for a proof of the following important fact: the convergence

$$\frac{1}{t} \min \{\xi_t^x : x \in (0,1)\} \;\to\; c_{\overline{p}}$$

holds almost surely as $t \to \infty$. We remark that this holds true for *all* (conservative, homogeneous) fragmentation processes; no special hypothesis on Φ (or equivalently, on the dislocation measure ν) is required.

We also make the important remark that this result does *not* imply the existence of an $x \in (0,1)$ such that $\xi_t^x/t \to v_{\min}$ as $t \to \infty$. In general, this is not the case; the largest particle at time $t > s$ is not necessarily a descendant of the largest particle at time s.

So far, we've seen that the largest particle has speed $v_{\min}$ and that the typical particle speed is v_{typ} whenever this value is finite. Since Φ is strictly concave and $\overline{p} > 0$, we know that $v_{\min} < v_{\text{typ}}$, as we'd expect. Pursuing these ideas further, it's natural to ask whether particles exist with other speeds. This question is addressed in Julien Berestycki's paper *Multifractal spectra of fragmentation processes*. He calculates the Hausdorff dimension of the set $\mathcal{G}_v$ of particles of speed v,

$$\mathcal{G}_v \quad := \quad \left\{ x \in (0,1) : \lim_{t\to\infty} \frac{\xi_t^x}{t} = v \text{ almost surely} \right\}$$

in terms of the Legendre transform of Φ. Let us summarize his results, which are obtained under the hypothesis that $\underline{p} < 0$, forcing $v_{\text{typ}} < \infty$. We use the notation $D(v)$ to stand for the Hausdorff dimension of $\mathcal{G}_v$. Berestycki proves the following statements: D is a continuous function on $(v_{\min}, v_{\max})$, $D(v_{\text{typ}}) = 1$, and $D(v)$ decreases as $|v - v_{\text{typ}}|$ increases. Interestingly, it is not necessarily the case that $D(v) \to 0$ as $v \to v_{\max}$. This reflects the existence of particles exhibiting superexponential decay, which correspond to the set $\mathcal{G}_\infty$. Whenever $\underline{p} > -1$ and $v_{\max} = \infty$, Berestycki shows that $D(\infty) = 1 + \underline{p}$.

Another interesting question one can ask about fragmentations concerns the cardinalities of the sets

$$\mathcal{H}_{v,a,b}(t) \quad := \quad \left\{ \mathcal{I}_t^x : \ x \in (0,1), \ ae^{-vt} \le I_t^x \le be^{-vt} \right\},$$

where $0 < a < b$. Using powerful discretization methods, Bertoin and Rouault [15, Corollary 2] are able to show that

$$\lim_{t\to\infty} \frac{1}{t} \log \#\mathcal{H}_{v,a,b}(t)$$

exists almost surely and takes positive values whenever $v \in (v_{\min}, v_{\max})$ and $0 < a < b$. This result makes precise the statement that the number of particles of size approximately equal to $\exp(-vt)$ grows exponentially for any possible speed v.

Now let us turn our attention to the theory of killed fragmentation processes, which we introduced in §1.2.6. The killing scheme defined there was first implemented explicitly in 2014 by Knobloch and Kyprianou in their paper *Survival of homogeneous fragmentation processes with killing* [29]. (In fact, as far as we know, killed fragmentation processes haven't been considered in the literature since.) Their paper contains three main results, which we now summarize, writing $\varsigma(a,c)$ for the probability that the (a,c)–killed fragmentation process survives.

First they show that $\varsigma(a,c) = 0$ whenever $c \le c_{\overline{p}}$, and that for fixed $c > c_{\overline{p}}$ the function $a \mapsto \varsigma(a,c)$ is an increasing $(0,1)$-valued function on $[0,\infty)$. Letting $N^{a,c}(t)$ stand for the number of particles alive in the (a,c)–killed fragmentation process at time t, they then show that whenever the process survives, $\limsup_{t\to\infty} N^{a,c}(t) = \infty$ almost surely. Finally, they show that the speed of the largest particle in a (a, c)–killed process is still $c_{\overline{p}}$ almost surely, on the event that the process survives.

Although [29] is the first paper to *explicitly* deal with killed fragmentation processes, we mention that several results in Natalie Krell's paper [31], published six years earlier, can be interpreted as concerning fragmentation processes with *two-sided* killing. In this context, she establishes more precise versions of some of the results mentioned in the previous paragraph, under the hypothesis that the absolutely continuous part of the dislocation measure assigns infinite mass to all intervals of the form $[0, \epsilon)$ with $\epsilon > 0$. Indeed, let $N^{a,b,c}(t)$ denote the number of particles alive at time t which have remained in the intervals $(\exp(a - cs), \exp(b - cs))$ for all $s \in [0, t]$. Krell shows that with positive probability $N^{a,b,c}(\infty) \neq 0$ whenever $a < 0 < b$ and $c > c_{\overline{p}}$. Whenever this obtains, she shows that $N^{a,b,c}(t)$ almost surely grows at an exponential rate, which she explicitly calculates in terms of the characteristics of the Lévy process ξ.

Having given a flavour of the qualitative properties of fragmentation processes, we proceed in the next section to a discussion of our main results. This will include an explanation of the intimate connection between fragmentation processes and branching random walk, which provides the basis for our proofs. As we will explain, however, the infinite activity exhibited by fragmentation processes in their full generality leads to complications that must be treated with care.

1.4 Main results

Our first result concerns the size of the largest fragment of a conservative homogeneous fragmentation process. We show that the largest particle has roughly the size $t^{-\alpha} \exp(-c_{\overline{p}} t)$ at time t, where $\alpha > 0$ is a constant that we identify explicitly in terms of the characteristics of the underlying process.

Theorem 1.14. *Starting from any initial configuration in $\mathcal{U}$, the following convergence holds in probability, as $t \to \infty$:*

$$\frac{\min_{x\in(0,1)} \xi^x_t - c_{\overline{p}} t}{\log t} \quad \longrightarrow \quad \frac{3}{2}(1 + \overline{p})^{-1} \,.$$

Next we turn our attention to the class of killed fragmentation processes introduced in §1.2.6. First we show that the survival probability of a $(c_{\overline{p}}+\epsilon)$–killed fragmentation process is roughly $\exp\left(-\frac{\beta}{\epsilon^{1/2}}\right)$ whenever ϵ is small, where $\beta > 0$ is a constant that we identify.

Theorem 1.15. *The survival probability $\rho(\epsilon)$ of the $(c_{\overline{p}} + \epsilon)$–killed fragmentation process satisfies the following asymptotic identity:*

$$\lim_{\epsilon \downarrow 0} \epsilon^{1/2} \log \rho(\epsilon) \quad = \quad -\sqrt{\frac{\pi^2 (1 + \overline{p}) |\Phi''(\overline{p})|}{2}}.$$

Our third and final result concerns the long-term survival probability of a critically killed fragmentation. We will show that the probability that such a process survives until the large time t is roughly $\exp(-\gamma t^{1/3})$, where $\gamma > 0$ is a constant that we identify.

Theorem 1.16. *The probability $\kappa(t)$ that the critically killed fragmentation process survives until time t satisfies the following asymptotic identity:*

$$\lim_{t\to\infty} \frac{1}{t^{1/3}} \log \kappa(t) \quad = \quad -\left(\frac{3\pi^2(1+\overline{p})^2|\Phi''(\overline{p})|}{2}\right)^{1/3}.$$

Having stated our main results, we conclude this chapter by discussing the key idea that kick-starts our proofs: the connection between fragmentation processes and branching random walk. In particular, we highlight the usefulness and limitations of this relationship.

Given a fragmentation process U, we defined (in §1.2.3) an associated collection of stochastic processes $(\xi^x : x \in (0,1))$ by setting $\xi^x_t := -\log|\mathcal{I}^x_t|$. For fixed $t \geq 0$, we can encode the values ξ^x_t (as x ranges over representatives of interval components of $U(t)$) using the point process

$$\mathcal{W}_t \quad := \quad \sum_{[x]_t} \delta_{\xi^x_t}.$$

We write $-\log U$ for the stochastic process whose value at time t is $\mathcal{W}_t$, and we endow $-\log U$ with the genealogical information it naturally inherits from U.

Suppose now that U is a finite activity fragmentation process, and write τ for the time of the first dislocation event. Then the fragmentation property immediately implies that $-\log U$ is equal in law to the branching random walk in continuous time (with branching rate $\nu(\mathcal{U})$) generated by the point process $\mathcal{W}_\tau$. To see why, fix $x \in (0,1)$ and write $\sigma + \tau$ for the time when $\mathcal{I}^x_\tau$ fractures. By the fragmentation property, the following statement holds: τ and σ are independent exponential random variables with parameter $\nu(\mathcal{U})$ and we have the following equality in law of point processes:

$$(I^y_{\tau+\sigma}/I^x_\tau : \; y \in \mathcal{I}^x_\tau) \quad \overset{L}{=} \quad (I^x_\tau : \; x \in (0,1)).$$

Moreover, these point processes are independent. Taking minus logarithms, we deduce that

$$(\xi^y_{\tau+\sigma} : \; y \in \mathcal{I}^x_{\tau+\sigma}) - \xi^x_\tau \quad \overset{L}{=} \quad (\xi^x_\tau : x \in (0,1)),$$

and that these point processes are independent. It follows that $(\mathcal{W}_t : t \geq \tau)$ is equal in law to the point process obtained by rooting independent copies of $(\mathcal{W}_t : t \geq 0)$ at each ξ–particle alive at time τ, and taking their sum.

Branching random walk analogues of Theorems 1.14, 1.15 and 1.16 exist in the literature under hypotheses of varying severity; we refer to [3], [25] and [2] respectively. In light of the connection between fragmentation processes and branching random walks just discussed, it is natural to expect that we might be able to adapt the proofs contained in these papers to the present context; this is exactly what we do in the following chapters. As Jean Bertoin's comment (quoted on page 4) suggests, this process of proof adaptation is not a trivial technical exercise.

In contrast to our method of proof adaptation, it is sometimes possible to use a discretization method introduced by Bertoin and Rouault in [15] to directly transfer results from branching random walks to fragmentation processes. We conclude this chapter by briefly describing this method and explaining why it will not work for us (at least in the case of our second and third main results).
Bertoin and Rouault start by observing that the process $\mathcal{W}^h := (\mathcal{W}_{nh} : n \in \mathbb{N})$ is a branching random walk in *discrete* time, even in the infinite activity case. Speaking very roughly, they then send $h \downarrow 0$, and use the Croft-Kingman lemma, to extend a famous result of John Biggins concerning discrete time branching random walk to the fragmentation context. *This is only possible because Biggins proved his result in sufficient generality.* In particular, Biggins does not exclude the possibility of infinite offspring in his branching random walk. Nor does he assume that the point process describing offspring positions can be constructed using a sequence of independent and identically distributed random variables. In short, the branching random walk $\mathcal{W}^h$ satisfies the hypotheses of Biggins' paper.

In contrast, our three references, [3], [25] and [2], assume at the very least that the size of the first generation has a finite moment of order $1 + \delta$, for some $\delta > 0$. This already restricts the scope of their results to branching random walks in which birth events produce a finite number of offspring. Starker still, the third of these references assumes that the number of offspring of an individual is a fixed (finite) number, and also assumes that particle positions (relative to birth position) are given by independent identically distributed random variables. The branching random walk $\mathcal{W}^h$ does not satisfy these hypotheses: offspring numbers are random, and may be infinite; and particle positions are intrinsically dependent (the image of their values under the map $x \mapsto \exp(-x)$ sum to unity, almost surely).

In this direction, we make a few further remarks concerning our first main result, Theorem 1.14. In his paper *Convergence in law of the minimum of a branching random walk* [1], Elie Aïdékon shows that for some random variable D (which is almost surely positive on the event that the branching random walk survives) and some deterministic $c \in (0, \infty)$, the following convergence obtains:

$$\lim_{n\to\infty} \mathbf{P}\left(M_n \;\geq\; x + \frac{3}{2}\log n\right) \;=\; \mathbf{E}\left(e^{-c\,e^{x}\,D}\right),$$

where M_n denotes the minimal particle position. As the author remarks, "We can see our theorem as the analogue of the result of Lalley and Sellke in the case of the branching Brownian motion: the minimum converges to a random shift of the Gumbel distribution."

In the published version of this paper, various integrability conditions are assumed; one of them is that the first generation size is almost surely finite (though need not have a finite first moment). In the latest arXiv version (version 6, November 2013), however, even this mild hypothesis is not assumed, resulting in remarkably general theorem. As the author notes, though, the non-lattice hypothesis is a necessary condition for the result above to hold.

In fact, provided that the underlying dislocation measure is non-geometric, Dadoun [22] has recently shown that the branching random walk $\mathcal{W}^h$ satisfies *all* the hypotheses of arXiv version 6 of [1]. He then applies the Croft-Kingman lemma to deduce that the appropriate analogue of Aïdékon's result holds for fragmentation processes, thereby providing finer information about the size of the largest fragment than our Theorem 1.14 does.

We emphasize, though, that a branching random walk result with sufficient power to yield a fragmentation result via the discretization method is *only* available in the case of Theorem 1.14. Random walk results under sufficiently mild hypotheses do not exist in the literature to grant similar proofs of Theorems 1.15 and 1.16.

In any case, we aim to present proofs that give some insight into why the *intrinsic structure* of fragmentation processes negates the need for complicated collections of moment conditions in the first place. It is natural to suspect that the questions we ask (for instance, how large is the largest particle?) are not influenced by the presence of (even infinitely many) very small particles. Our proofs, we hope, make clear how this intuition plays out mathematically.

CHAPTER 2

PRELIMINARY RESULTS

In this chapter we will develop the technical machinery required to adapt proofs of branching random walk results to the fragmentation setting. The proofs of our three main results proceed by the second moment method. Accordingly, the most important result in this Chapter is Corollary 2.7, which provides a formula expressing the second moment of certain functionals of fragmentation paths in terms of the spine $(\zeta, \mathbf{Q})$ and the dislocation measure of the underlying fragmentation process. This result is contained in the first section of this chapter, where we also prove a collection of simple statements needed later.

Since we will express second moments in terms of the spine, we need to be able to estimate the probabilities that $(\zeta, \mathbf{Q})$ behaves in certain ways. The results we need are contained in the second section of this chapter, and concern the fluctuation theory of spectrally positive Lévy processes of bounded variation. We prove these results by making reference to the theory centred random walks with finite variance. In particular, we will give a "Lévy version" of Mogulskii's Theorem concerning the small deviations of paths of random walks, which is the main technical tool used in Chapters 4 and 5.

2.1 First results on fragmentation processes

We warm up with a simple application of the Many-to-One Lemma for frozen fragmentations, bounding from above the probability that at least one fragment remains "large" as time passes. This result will enable us to painlessly extract the easy halves of Theorem 1.15 and Theorem 1.16 from their branching random walk analogues.

Lemma 2.1. *Let f and g be non-negative continuous functions on $[0, \infty)$, and suppose that $g - f$ increases. Then, for any $t \geq 0$,*

$$\mathbf{P}\big(\exists x \in (0,1) : \ \zeta_s^x \leq g_s \ \ \forall s \leq t\big) \ \leq \ e^{(1+\overline{p})g_t} I_t + \sum_{i=0}^{\lceil t \rceil - 1} e^{(1+\overline{p})(g_{i+1} - f_{i+1})} I_i,$$

where $\ I_r := \mathbf{Q}\big(g_s - f_s < \zeta_s \leq g_s \quad \forall s \leq r\big)$ *for* $r \geq 0$.

Proof. We denote the probability we aim to bound above by $p(t)$, and define the frost T by

$$T_x \quad := \quad \inf\big\{t \geq 0: \ \ \zeta_t^x \leq g_t - f_t\big\},$$

with $\inf \emptyset := \infty$. We note that T_x is an $\mathcal{F}^x$–stopping time by the Début Theorem (see [43, II.76]), since the stochastic process $t \mapsto \zeta_t^x - (g_t - f_t)$ has càdlàg paths almost surely, and the set $[0, \infty)$ is closed. Writing indicators in the form $\mathbf{1}\{\cdot\}$ (to avoid subscripts), we make the trivial observation that

$$\begin{aligned} p(t) \ &\leq \ \mathbf{P}\big(\exists x \in (0,1): \quad \zeta_s^x \leq g_s \quad \forall s \leq T_x \wedge t\big) \\ &= \ \mathbf{P}\Big(\sum_{(T,t)} \mathbf{1}\big\{\zeta_s^x \leq g_s \ \forall s \leq T_x \wedge t\big\} \ \geq \ 1\Big). \end{aligned}$$

Now we apply Markov's inequality to bound $p(t)$ above by

$$\mathbf{E} \sum_{(T,t)} \mathbf{1}\big\{\zeta_s^x \leq g_s \ \forall s \leq T_x \wedge t\big\}.$$

Next we apply the Many-to-One Lemma for frosts to arrive at the bound

$$p(t) \ \leq \ \mathbf{Q}\big(e^{(1+\overline{p})\zeta_{\tau \wedge t}} \mathbf{1}\big\{\zeta_s \leq g_s \ \forall s \leq \tau \wedge t\big\}\big),$$

where τ is the $\mathcal{G}$–stopping time $\inf\big\{t \geq 0: \ \ \zeta_t \leq g_t - f_t\big\}$.

We continue by partitioning the sample space according to whether or not the inequality $\tau \leq t$ holds. If it does not, we have

$$\begin{aligned} &\mathbf{Q}\big(e^{(1+\overline{p})\zeta_{\tau \wedge t}} \mathbf{1}\big\{\zeta_s \leq g_s \ \forall s \leq \tau \wedge t \ \text{ and } \ \tau > t\big\}\big) \\ &\qquad = \ \mathbf{Q}\big(e^{(1+\overline{p})\zeta_t} \mathbf{1}\big\{\zeta_s \leq g_s \ \forall s \leq t \ \text{ and } \ \tau > t\big\}\big) \\ &\qquad \leq \ e^{(1+\overline{p})g_t} \mathbf{Q}\left(\zeta_s \leq g_s \ \forall s \leq t \ \text{ and } \ \tau > t\right) \\ &\qquad = \ e^{(1+\overline{p})g_t} \mathbf{Q}\big(g_s - f_s < \zeta_s \leq g_s \ \forall s \leq t\big). \end{aligned}$$

If it does, we have

$$\begin{aligned} &\mathbf{Q}\big(e^{(1+\overline{p})\zeta_{\tau \wedge t}} \mathbf{1}\big\{\zeta_s \leq g_s \ \forall s \leq \tau \wedge t \ \text{ and } \ \tau \leq t\big\}\big) \\ &\qquad = \ \mathbf{Q}\big(e^{(1+\overline{p})\zeta_\tau} \mathbf{1}\big\{\zeta_s \leq g_s \ \forall s \leq \tau \ \text{ and } \ \tau \leq t\big\}\big) \\ &\qquad \leq \ \sum_{i=0}^{\lceil t \rceil - 1} \mathbf{Q}\big(e^{(1+\overline{p})\zeta_\tau} \mathbf{1}\big\{\zeta_s \leq g_s \ \forall s \leq \tau \ \text{ and } \ i \leq \tau \leq i+1\big\}\big). \end{aligned}$$

Now we show that the spectral positively of ζ provides the bound $\zeta_\tau \leq g_\tau - f_\tau$ on $\{\tau < \infty\}$. Indeed, suppose that $\Delta := \zeta_\tau - (g_\tau - f_\tau)$ is positive. By the continuity of the function $g - f$ we can find $\delta = \delta(\Delta)$ so small that the line L of gradient $-c_{\overline{p}}$ rooted at ζ_τ does not intersect the graph of $g - f$ on $[\tau, \tau + \delta]$. But the path of ζ on $[\tau, \tau + \delta]$ lies almost surely above L. This means that $\zeta_s > g_s - f_s$ for all $s \in [\tau, \tau + \delta]$. We deduce that $\mathbf{P}(\Delta > 0) = 0$, else the definition of τ is contradicted with positive probability.

Since the function $g - f$ increases, we conclude that $\zeta_\tau \le g_\tau - f_\tau \le g_{i+1} - f_{i+1}$ up to a null set on the event $\{\tau \le i+1\}$. It follows that final expression in the previous display is bounded by

$$\sum_{i=0}^{\lceil t\rceil -1} e^{(1+\overline{p})(g_{i+1}-f_{i+1})}\mathbf{Q}\big(\zeta_s \le g_s \ \ \forall s \le \tau, \ \text{ and } \ \tau \ge i\big),$$

which in turn is trivially bounded above by

$$\sum_{i=0}^{\lceil t\rceil -1} e^{(1+\overline{p})(g_{i+1}-f_{i+1})}\mathbf{Q}\big(g_s - f_s < \zeta_s \le g_s \ \ \forall s \le i\big).$$

Putting the two bounds together gives the required statement. □

Now we move on to the main goal of this section—calculating second moments of random variables of the form

$$\sum_{[x]_t} F(I_s^x : s \in [0,t]). \tag{2.1}$$

for appropriate elements $F \in m^+\mathcal{D}([0,t])$. The first step is to translate the problem into the language of $\mathcal{P}$–valued fragmentations. Throughout this section we fix an interval-valued fragmentation U. We then write Π for the $\mathcal{P}$–valued fragmentation $\Pi(U)$ introduced in §1.2.7. We write Γ for the Poissonian fragmentation $\Gamma(\rho\nu)$, which, according to Theorem 1.13, is equal in law to Π.

Recall that we defined Π using a collection of independent uniform random variables $\{X_i : i \in \mathbb{N}\}$. The first step is to relabel the summands in (2.1) using this collection. To this end write $\mathcal{I}_s^n$ for the component of $U(s)$ that contains X_n, and I_s^n for its length. Then we have the almost sure equality

$$\sum_{[x]_t} F(I_s^x : s \in [0,t]) \;\; = \;\; \sum_{n\in\mathbb{N}} F(I_s^n : s \in [0,t]) \cdot \mathbf{1}(n = \min\{j : X_j \in \mathcal{I}_t^n\}).$$

Now we want to replace I_s^n with $|\Pi_s^n|$ and $\mathcal{I}_s^n$ with Π_s^n. We claim that

Lemma 2.2. *For each $n \in \mathbb{N}$,*

- $\mathbf{P}(|\Pi_t^n|$ *exists for all* $n \in \mathbb{N}$ *and* $t \ge 0) = 1$.
- *The process* $(-\log|\Pi_t^n| : t \ge 0)$ *is a subordinator for each* $n \in \mathbb{N}$.

Proof. We aim to apply Theorem 3 of [10]. At first glance this theorem is immediately applicable, because Π is a homogeneous fragmentation process (it is equal in law to Γ). Since we want the càdlàg property (part of the definition of 'subordinator'), however, we need to check that the regularized version of Π that Bertoin works with is indistinguishable from Π as we have defined it. We now show that this follows from the fact that we work with càdlàg interval-fragmentations.

Bertoin's regularisation (applied to our Π) is defined as follows. For each t introduce a new partition $\Pi(t+)$ defined by writing

$$(i \sim j, \Pi(t+)) \qquad \iff \qquad (i \sim j,\, \Pi(t')) \text{ for some } t' \in (t, \infty) \cap \mathbb{Q}.$$

We will show that on the part of the sample space where Π is well-defined, and where $t \mapsto U(t)$ is right-continuous ('for all t simulataneously'), the functions $t \mapsto \Pi(t)$ and $t \mapsto \Pi(t+)$ coincide. So let us assume that U is right-continuous at t, and show that $\Pi(t+) = \Pi(t)$. (The following argument is path-wise; all random variables should be read as realizations of random variables.) $\Pi(t+)$ is plainly finer than $\Pi(t)$ so we need to show that $(i \sim j, \Pi(t))$ implies $(i \sim j, \Pi(t+))$. To this end, suppose $(i \sim j, \Pi(t))$, and fix a sequence (q_n) of rationals decreasing strictly to t. Define $V := U(t)$ and $V_n := U(q_n)$. By hypothesis, $V_n \to V$ as $n \to \infty$. Let $\mathcal{I}$ stand for the component of V containing X_i and X_j. By Lemma 2(i) of [11] we conclude that there are components $\mathcal{I}_n$ of V_n with the property that $\mathcal{I}_n \to \mathcal{I}$. Write $\mathcal{I}_n = (a_n, b_n)$ and $\mathcal{I} = (a, b)$. We deduce that $|a_n - a| \to 0$ and that $|b_n - b| \to 0$. Since $\mathcal{I}$ is open and contains X_i and X_j, we deduce that X_i and X_j both lie in $\mathcal{I}_n$ whenever n is large enough. That is, $(i \sim j, \Pi(q_n))$ for all large n. □

Returning to our second moment problem, we conclude that for each $n \in \mathbb{N}$, the process $|\Pi^n| := (|\Pi^n_t| : t \geq 0)$ is well-defined, and right-continuous. Clearly the process $I^n := (I^n_t : t \geq 0)$ inherits right-continuity from U, and, by the law of large numbers, is a version of $|\Pi^n|$. Right continuous versions are indistinguishable. We conclude that

$$\sum_{n \in \mathbb{N}} F(I^n_s : s \in [0,t]) \cdot \mathbf{1}_{(n = \min\{j : X_j \in \mathcal{I}^n_t\})} \;\; = \;\; \sum_{n \in \mathbb{N}} F(|\Pi^n_s| : s \in [0,t]) \cdot \mathbf{1}_{(n = \min \Pi^n_t)}.$$

We now want to replace Π with Γ on the right-hand side. So far, we have worked with indistinguishable random elements of the space $\mathcal{D}([0,t])$, so have not need to use any special properties of F. Since Π and Γ are only equal in the sense of finite-dimensional distributions, we now need to show that the functions F we are interested in are, in some sense, determined in a countable way. We now make this idea precise.

Let $(q_i : i \in \mathbb{N})$ be a sequence with entries $[0, \infty)$. For each $k \in \mathbb{N}$ fix a measurable function $A_k : \mathbb{R}^k \to [0, \infty)$, and fix a measurable function $A : \mathbb{R}^{\mathbb{N}} \to [0, \infty)$. For a map $f \in \mathcal{D}([0,t])$ and $k \in \mathbb{N}$ write f_k for the sequence $(f(q_1), ..., f(q_k))$, and write f_∞ for the sequence $(f(q_1), f(q_2), ...)$. We call $((q_i), (A_k), A)$ an *approximation scheme* if for each $f \in \mathcal{D}([0,t])$ we have $\lim_k A_k(f_k) = A(f_\infty)$.

Definition 2.3. *We call a function $F : \mathcal{D}([0,t]) \to \mathbb{R}_+$ nice if for some approximation scheme $((q_k), (A_k), A)$ we have $A(f_\infty) = F(f)$ for all $f \in \mathcal{D}([0,t])$.*

The point is that if F is nice and the almost surely càdlàg stochastic processes X and Y are equal in law, then so are the random variables $F(X_s : 0 \leq s \leq t)$ and $F(X_s : 0 \leq s \leq t)$. Indeed, $A_k(X_k) = A_k(X_{q_1}, ..., X_{q_k})$ is equal to $A_k(Y_k) = A_k(Y_{q_1}, ..., X_{Y_k})$ in law. The former converges almost surely (on the part of the sample space where X is càdlàg) to $A(X_{q_1}, ...) = F(X_s : 0 \leq s \leq t)$ and the latter converges almost surely to $A(Y_{q_1}, ...) = F(Y_s : 0 \leq s \leq t)$.

In short, provided F is nice, we can replace Π with Γ in the previous display. This completes the first step:

Lemma 2.4. *Suppose that $F : \mathcal{D}([0,t]) \to \mathbb{R}_+$ is nice. Then*

$$\sum_{[x]_t} F(I_s^x : s \in [0,t]) \stackrel{L}{=} \sum_{n\in\mathbb{N}} F(|\Gamma_s^n| : s \in [0,t]) \cdot \mathbf{1}_{(n=\min \Gamma_t^n)}.$$

Next we give the only examples of nice functions we need.

Lemma 2.5. *Fix right-continuous functions $a : [0,t] \to \mathbb{R}$ and $b : [0,t] \to \mathbb{R} \cup \infty$ with $a < b$, and a measurable map $g : \mathbb{R} \to [0,\infty)$. The following maps from $\mathcal{D}([0,t])$ to $\mathbb{R}_+$ are nice:*

- *$F(f) := \mathbf{1}(f(s) \in K_s \;\; \forall s \in [0,t))$, where $K_s := [a_s, b_s]$.*
- *With F as above, $G(f) := F(f) \cdot g(f(t))$.*

Proof. The fact that F is nice is witnessed by any enumeration $(q_i : i \in \mathbb{N})$ of $[0,t) \cap \mathbb{Q}$, the functions $A_k(x_1, ..., x_k) := \mathbf{1}(x_i \in K(q_i) \;\; \forall 1 \le i \le k)$, and the function $A(x_1, ...) := \mathbf{1}(x_i \in K(q_i) \;\; \forall i \in \mathbb{N})$. To see that G is nice, let $q_1 := t$ and fix an enumeration $(q_i : i \ge 2)$ of $[0,t) \cap \mathbb{Q}$. Define $B_1 : \mathbb{R} \to [0,\infty)$ arbitrarily. For $k \ge 2$, define (with A_k and A as before) the maps $B_k(x_1, ..., x_k) := g(x_1) \cdot A_{k-1}(x_2, ..., x_k)$, and set $B_k(x_1, ...) := g(x_1) \cdot A(x_2, ...)$. Then $((q_i), (B_k), B)$ witnesses how nice G is. □

In particular, the second example of a nice function shows that the interval $[0,t)$ in the definition of F can be replaced with $[0,t]$ to obtain another nice function.

We are now ready to present the second moment calculation. Before proceeding, we recall (c.f. §1.2.7) that the following result, and its corollary, could be interestingly approached from a different angle, by formulating a Many-to-Two Lemma in the sense of Harris and Roberts [26].

Let us first introduce some notation to stream-line the proof. Recall that Γ is the Poissonian fragmentation corresponding to our interval fragmentation U. For $s \ge 0$ and $n \in \mathbb{N}$, we define

$$\delta_{s,n} \;:=\; \mathbf{1}(n = \min \Gamma_s^n) \qquad \text{and} \qquad \Lambda_s^n \;:=\; |\Gamma_s^n|,$$

where Γ_s^n stands for the block of Γ_s which contains $n \in \mathbb{N}$. We will write $\mathcal{A} = (\mathcal{A}_t)$ for the natural filtration associated with Γ, and $\mathbb{E}$ for the law of Γ.

Proposition 2.6. *Fix a fragmentation process U with dislocation measure ν. Fix $t \ge 0$, right-continuous functions $a : [0,t] \to \mathbb{R}$ and $b : [0,t] \to \mathbb{R}$ with $a < b$, and define $K_s := [a_s, b_s]$. Define the random variable Z by*

$$Z \;:=\; \sum_{[x]_t} \mathbf{1}(I_s^x \in K_s \;\; \forall s \in [0,t]).$$

Define the function $F : [0,t] \times [0,1] \to [0,\infty]$ *by*

$$F(r,\alpha) \;:=\; \mathbf{E} \sum_{[x]_{t-r}} \mathbf{1}(\alpha\, I^x_s \in K_{s+r} \;\; \forall s \in [0, t-r])$$

and the function $G : [0,\infty) \times [0,1] \times \mathcal{U} \to [0,\infty]$ *by*

$$G(r,\alpha,u) \;:=\; \sum_{i \neq j} F(r, \alpha \cdot |u_i|)\; F(r, \alpha \cdot |u_j|).$$

Then $\mathbf{E}Z^2 = \mathbf{E}Z + \lambda$*, where*

$$\lambda \;:=\; \int_0^t dr \,\cdot\, \mathbf{E} \sum_{[x]_r} \mathbf{1}_{(I^x_s \in K_s \;\; \forall s \in [0,r])} \int_{\mathcal{U}} G(r, I^x_r, u) \nu(du).$$

Proof. We have seen that the functional $f \mapsto \mathbf{1}(f_s \in K_s \;\; \forall s \in [0,t])$ is nice, so we can use Lemma 2.4 to write

$$Z \;\stackrel{L}{=}\; \sum \delta_{t,n} \cdot \mathbf{1}(\Lambda^n_s \in K_s \;\; \forall s \in [0,t]) \;\;=:\;\; Z_{\mathcal{P}}.$$

For Borel sets $\alpha \subseteq [0,t]$ and $n \in \mathbb{N}$, we define the event random variable A^n_α by

$$A^n_\alpha \;\;:=\;\; \mathbf{1}(\Lambda^n_s \in K_s \;\; \forall s \in \alpha).$$

We are now going to calculate $\mathbb{E}\, Z_{\mathcal{P}}$, and we start by separating "diagonal" and "non-diagonal" terms:

$$\begin{aligned} Z^2_{\mathcal{P}} \;&=\; \Big(\sum_{i \in \mathbb{N}} \delta_{t,i}\, A^i_{[0,t]} \Big) \Big(\sum_{j \in \mathbb{N}} \delta_{t,j} A^j_{[0,t]} \Big) \\ &=\; Z_{\mathcal{P}} \;+\; \sum_{i \neq j} \delta_{t,i}\, \delta_{t,j}\, A^i_{[0,t]}\, A^j_{[0,t]} \\ &=:\; Z_{\mathcal{P}} + Y. \end{aligned}$$

We now make crucial use of the Poissonian construction of Γ. Let us write $\mathcal{D}_\infty$ for the set of times at which Γ is discontinuous, and let us define $\mathcal{D}_t := \mathcal{D}_\infty \cap [0,t]$. Unravelling notation, we see that all non-zero elements in the sum $\sum_{i \neq j}$ in the previous display correspond to pairs of distinct blocks of Γ_t. Let us fix $i, j \in \mathbb{N}$ with $i \neq j$. We make the following observations. The process $\pi^{(i \vee j)}$ used in the construction of Γ is piecewise constant and right-continuous, and its jump times are a subset of (the projection onto the time axis of) the atoms of the Poisson random measure $M^{(i \vee j)}$ (the restricted random measure used to define $\pi^{(i \vee j)}$). These atoms arrive at a finite rate. As a result, there is a unique $r = r_{i,j} \in \mathcal{D}_\infty$ with the following properties: i and j lie in the same block of Γ_{r-}, but in distinct blocks of Γ_r. It therefore makes sense to call $\Gamma^i(r_{i,j}-) = \Gamma^j(r_{i,j}-)$ the *most recent common ancestor of* i *and* j. (Our statement assumes that i and j are eventually separated; this is true.)

We use these comments to give an ancestral decomposition of Y. Given $(i \nsim j, \Gamma_t)$, we find their most recent common ancestor Γ^i_{r-}. We then find the siblings Γ^k_r and

Γ^l_r born to the parent Γ^i_{r-} at time r which are the ancestors of i and j (respectively) alive at time r. This verbal description gives rise to the following expression:

$$Y \;=\; \sum_{r\in\mathcal{D}_t}\sum_{n\in\mathbb{N}} \delta_{r-,n}A^n_{[0,r)}\,\mathbf{1}_{(\Gamma^n_{r-}\neq\Gamma^n_r)} \sum_{\substack{k\neq l\\ k,l\in\Gamma^n_{r-}}} \delta_{r,k}\,\delta_{r,l}\sum_{\substack{i\in\Gamma^k_r\\ j\in\Gamma^l_r}} \delta_{t,i}\,\delta_{t,j}A^i_{[r,t]}\,A^j_{[r,t]}.$$

Now we give an explicit enumeration of $\mathcal{D}_\infty$ in which each element is an $\mathcal{A}$–stopping time. Recall that the Poissonian fragmentation process Γ corresponds to a Poisson point process M on $[0,\infty)\times\mathbb{N}\times\mathcal{P}$. For each $n,k\in\mathbb{N}$, let us write $\mathcal{P}^{(n,k)} := \{k\}\times(\mathcal{P}^*_{n+1}-\mathcal{P}^*_n)$, and $M^{(n,k)}$ for the restriction of the Poisson point process M to $[0,\infty)\times\mathcal{P}^{(n,k)}$. We write $\mathcal{D}^{(n,k)}_M$ for the projection onto the time axis of the set of atoms of $M^{(n,k)}$. These atoms arrive at a finite rate, so $\mathcal{D}^{(n,k)}_M$ almost surely has no limit points. We can therefore enumerate $\mathcal{D}^{(n,k)}_M\cap\mathcal{D}_\infty$ by writing its elements in increasing size: $(r^{(n,k)}_i : i\in\mathbb{N})$. We note that $r^{(n,k)}_i$ is an $\mathcal{A}$–stopping time—it is the i'th time a dislocation occurs corresponding to an element of $\mathcal{P}^{(n,k)}$. This enumeration allows us to write

$$\mathbb{E}Y \;=\; \mathbb{E}\sum_{r\in\mathcal{D}_t}\sum_{n\in\mathbb{N}} \delta_{r-,n}A^n_{[0,r)}\,\mathbf{1}_{(\Gamma^n_{r-}\neq\Gamma^n_r)} \sum_{\substack{k\neq l\\ k,l\in\Gamma^n_{r-}}} \delta_{r,k}\,\delta_{r,l}\,\mathbb{E}_{\mathcal{A}_r}\Bigg(\sum_{\substack{i\in\Gamma^k_r\\ j\in\Gamma^l_r}} \delta_{t,i}\,\delta_{t,j}A^i_{[r,t]}\,A^j_{[r,t]}\Bigg),$$

and to use the strong fragmentation property to calculate the conditional expectations. Indeed, whenever r is an $\mathcal{A}$–stopping time, we can use the independent evolution of distinct blocks to write

$$\mathbb{E}_{\mathcal{A}_r}\Bigg(\sum_{\substack{i\in\Gamma^k_r\\ j\in\Gamma^l_r}} \delta_{t,i}\,\delta_{t,j}A^i_{[r,t]}\,A^j_{[r,t]}\Bigg) \;=\; \Bigg(\mathbb{E}_{\mathcal{A}_r}\sum_{i\in\Gamma^k_r}\delta_{t,i}\,A^i_{[r,t]}\Bigg)\Bigg(\mathbb{E}_{\mathcal{A}_r}\sum_{j\in\Gamma^l_r}\delta_{t,j}\,A^j_{[r,t]}\Bigg)$$

whenever $k\neq l$ and $\delta_{r,k}\delta_{r,l}=1$ (which means in particular that Γ^k_r and Γ^l_r are distinct blocks of Γ_r). The fragmentation property tells us that

$$\mathbb{E}_{\mathcal{A}_r}\sum_{i\in\Gamma^k_r}\delta_{t,i}\,A^i_{[r,t]} \;=\; \mathbb{E}_{\mathcal{A}_r}\Bigg(\sum_{i\in\mathbb{N}}\widetilde{\delta}_{t-r,i}\,\mathbf{1}(\Lambda^k_r\cdot\widetilde{\Lambda}^i_s\in K_{s+r}\;\;\forall s\in[0,t-r])\Bigg)$$

where the $\widetilde{\Lambda}$– and $\widetilde{\delta}$–values are derived from an independent fragmentation process with the same law as Γ. We can therefore write

$$\mathbb{E}_{\mathcal{A}_r}\sum_{i\in\Gamma^k_r}\delta_{t,i}\,A^i_{[r,t]} \;=\; F^{\mathcal{P}}(r,\Lambda^k_r)$$

where we define the function $F^{\mathcal{P}} : [0,\infty)\times[0,1]\to[0,\infty]$ by

$$F^{\mathcal{P}}(r,\alpha) \;:=\; \mathbb{E}\sum_{i\in\mathbb{N}}\delta_{t-r,i}\mathbf{1}(\alpha\,\Lambda^i_s\in K_{s+r}\;\;\forall s\in[0,t-r]).$$

By Lemma 2.4, F and $F^{\mathcal{P}}$ are identical. Applying this work to our ancestral decomposition of Y yields the following formula:

$$\mathbb{E}\,Y \;=\; \mathbb{E}\sum_{r\in\mathcal{D}_t}\sum_{n\in\mathbb{N}} \delta_{r-,n}\,A^n_{[0,r)}\,\mathbf{1}_{(\Gamma^n_{r-}\neq\Gamma^n_r)} \sum_{\substack{k\neq l\\ k,l\in\Gamma^n_{r-}}} \delta_{r,k}\,\delta_{r,l}\,F(r,\Lambda^k_r)F(r,\Lambda^l_r).$$

The next step is to write all dislocation activity in terms of the atoms of M. There is a slight technicality to be addressed, which is effectively induced by the fact that we are working with the collection $(\Gamma^n : n \in \mathbb{N})$ which contains duplicate blocks (whereas the Poissonian construction uses the ordering of blocks by least element, which contains no duplicates). To deal with this, write $\Gamma_s = (B_1(s), ...)$ where the $(B_i(s) : i \in \mathbb{N})$ are the blocks of Γ_s written in order of least element. If $n \in B_k(s)$, then we write $f(s,n) = k$. Writing (r, k_r, π_r) for the atoms of M, we can say that $\mathbb{E}\, Y$ is equal to

$$\mathbb{E} \sum_{r \in \mathcal{D}_t} \sum_{n \in \mathbb{N}} \delta_{r-,n}\, A^n_{[0,r)}\ \mathbf{1}_{(f(r-,n)=k_r)} \sum_{i \neq j} \delta^M_{r,i}\ \delta^M_{r,j}\ F(r, |\pi^i_r| \cdot \Lambda^n_{r-}) F(r, |\pi^j_r| \cdot \Lambda^n_{r-}),$$

where $\delta^M_{r,i} := \mathbf{1}(i = \min \pi^i_r)$. We are now in fantastic shape to apply the compensation formula (see [9, pg. 7]), which tells us that $\mathbb{E}\, Y$ equals

$$\int_0^t dr \cdot \mathbb{E} \int_{\mathbb{N} \times \mathcal{P}} (\# \times \mu)(dk, d\pi) \cdot$$
$$\sum_{n \in \mathbb{N}} \delta_{r-,n}\, A^n_{[0,r)}\ \mathbf{1}_{(f(r-,n)=k)} \sum_{i \neq j} \delta^\pi_i\ \delta^\pi_j\ F(r, |\pi^i| \cdot \Lambda^n_{r-})\, F(r, |\pi^j| \cdot \Lambda^n_{r-}),$$

where $\delta^\pi_i := \mathbf{1}(i = \min \pi^i)$, and $\mu := \mu_{\rho\nu}$, the mixture of paint-boxes directed by the Lévy measure $\rho\nu$. Now we note that for each $n \in \mathbb{N}$ and $r > 0$ there is precisely one $k \in \mathbb{N}$ for which $f(r-, n) = k$ (this says nothing more or less than "Γ_{r-} is a partition of $\mathbb{N}$"). Performing the sum in k (viz. the integral $\#(dk)$), we conclude that $\mathbb{E}\, Y$ is equal to

$$\int_0^t dr \cdot \mathbb{E} \int_{\mathcal{P}} \mu(d\pi) \cdot \sum_{n \in \mathbb{N}} \delta_{r-,n}\, A^n_{[0,r)} \sum_{i \neq j} \delta^\pi_i\ \delta^\pi_j\ F(r, |\pi^i| \cdot \Lambda^n_{r-}) F(r, |\pi^j| \cdot \Lambda^n_{r-}).$$

Now we translate the integral in π to an integral in u, recalling that μ is equal to the $(\rho\nu)$–mixture of paint-boxes. For any functions measurable functions $f, g : [0,1] \to \mathbb{R}_+$, and writing ι_i for the map that sends a sequence to its i'th element, we have

$$\begin{aligned}
\int_{\mathcal{P}} \mu(d\pi) \cdot \sum_{i \neq j} \delta^\pi_i\, \delta^\pi_j\, f(|\pi^i|) g(|\pi^j|) &= \int_{\mathcal{S}^\downarrow} \rho\nu(ds) \cdot\ P^s\Big(\sum_{i \neq j} \delta^\pi_i\, \delta^\pi_j\, f(|\pi^i|) g(|\pi^j|) \Big) \\
&= \int_{\mathcal{S}^\downarrow} \rho\nu(ds) \cdot \sum_{i \neq j} f(s_i) g(s_j) \\
&= \int_{\mathcal{S}^\downarrow} \rho\nu(ds) \cdot \sum_{i \neq j} f(\iota_i s) g(\iota_j s) \\
&= \int_{\mathcal{U}} \nu(du) \cdot \sum_{i \neq j} (f \iota_i \rho u)(f \iota_j \rho u).
\end{aligned}$$

In the first line we just use the definition of μ as a mixture of paint-boxes; the second equality is a simple consequence of the definition of the s–paint-box; the third is trivial; and in the fourth we make a change of variables. It remains to note that $\iota_i \rho u$ is precisely what we've been calling $|u_i|$ all along. We conclude that

$$\mathbb{E}\, Y \ = \ \int_0^t dr \cdot \mathbb{E} \sum_{n \in \mathbb{N}} \delta_{r-,n}\, A^n_{[0,r)} \int_{\mathcal{U}} \nu(du) \cdot \sum_{i \neq j}\ F(r, |u_i| \cdot \Lambda^n_{r-}) F(r, |u_j| \cdot \Lambda^n_{r-}).$$

Sample paths of the fragmentation Γ have at most a countable number of discontinuities. Lebesgue measure assigns zero mass to countable sets. We can therefore replace $r-$ with r in the previous expression, which, together with an application of Fubini's theorem, yields

$$\mathbb{E}Y \;=\; \int_0^t dr \int_{\mathcal{U}} \nu(du) \cdot \mathbb{E} \sum_{n\in\mathbb{N}} \delta_{r,n}\, A^n_{[0,r]}\; G(r, \Lambda^n_r, u)$$

where G is defined in the statement of the proposition. We know that, for fixed $r \geq 0$ and $u \in \mathcal{U}$, that the functional

$$\mathcal{D}([0,t]) \ni f \mapsto \mathbf{1}_{(f(s)\in K(s) \;\; \forall s\in[0,r])}\, G(r, f(r), u)$$

is nice, so Lemma 2.4 tells us that

$$\mathbb{E}\sum_{n\in\mathbb{N}} \delta_{r,n}\, A^n_{[0,r]}\; G(r, \Lambda^n_r, u) \;=\; \mathbf{E}\sum_{[x]_r} \mathbf{1}_{(I^x_s\in K_s \;\; \forall s\in[0,r])} G(r, I^x_r, u).$$

Returning to the previous display, we conclude that $\mathbb{E}Y$ is equal to λ (as defined in the statement of the proposition). According to proposition Lemma 2.4, Z and $Z_{\mathcal{P}}$ are equal in law. We conclude that

$$\mathbf{E}Z^2 \;=\; \mathbb{E}Z^2_{\mathcal{P}} \;=\; \mathbb{E}Z_{\mathcal{P}} + \mathbb{E}Y \;=\; \mathbf{E}Z + \lambda,$$

as required. □

The following corollary is obtained from the proposition above by applying the Many-to-One Lemma. We state the corollary in terms of the ζ^x–processes since we will only use it in this form.

Corollary 2.7. *Fix a fragmentation process U with dislocation measure ν. Fix $t \geq 0$, right-continuous functions $a : [0,t] \to \mathbb{R}$ and $b : [0,t] \to \mathbb{R} \cup \infty$ with $a < b$, and define $K_s := [a_s, b_s]$. Define the random variable Z by*

$$Z \;:=\; \sum_{[x]_t} \mathbf{1}(\zeta^x_s \in K_s \;\; \forall s \in [0,t]).$$

Define the function $F : [0,t] \times \mathbb{R} \to [0,\infty]$ by

$$F(r,\alpha) \;:=\; \mathbf{Q}\big(e^{(1+\overline{p})\zeta_{t-r}}\, \mathbf{1}_{(\alpha+\zeta_s\in K_{s+r} \;\; \forall s\in[0,t-r])}\big)$$

and the function $G : [0,\infty) \times \mathbb{R} \times \mathcal{U} \to [0,\infty]$ by

$$G(r,\alpha,u) \;:=\; \sum_{i\neq j} F(r, \alpha - \log|u_i|)\;\; F(r, \alpha - \log|u_j|).$$

Then $\mathbf{E}Z^2 = \mathbf{E}Z + \lambda$, where

$$\lambda \;:=\; \int_0^t dr \;\cdot\; \mathbf{Q}\left(e^{(1+\overline{p})\zeta_r}\; \mathbf{1}_{(\zeta_s\in K_s \;\; \forall s\in[0,r])} \int_{\mathcal{U}} G(r, \zeta_r, u)\;\, \nu(du)\right).$$

At a certain point in the proof of the lower bounds of Theorem 1.15 and Theorem 1.16, we will need to recognize the Lévy measure of $(\zeta, \mathbf{Q})$ hidden in a rather complicated expression. The next two results contain the required information.

We remind the reader that, given a measurable space $(\Sigma_1, \mathcal{G}_1)$, a measure space $(\Sigma_2, \mathcal{G}_2, \mu)$ and a measurable function $f : \Sigma_1 \to \Sigma_2$, the image measure $f\mu$ on $(\Sigma_1, \mathcal{G}_1)$ is defined by $f\mu(\cdot) := \mu f^{-1}(\cdot)$.

Lemma 2.8. *For any non-negative measurable function g on $(0,1]$, we have*

$$\begin{aligned}\int_{\mathcal{U}} \nu(du) \cdot \sum |u_i| g(|u_i|) &= \int_0^\infty g(e^{-x}) e^{-x} \sum \nu(-\log|u_i| \in dx) \\ &= \int_0^\infty g(e^{-x}) L(dx)\end{aligned}$$

where L is the Lévy measure of $(\zeta, \mathbf{P})$, and where the sums are finite when the decomposition of u contains finitely many fragments.

Proof. The proof is trivially completed by making two changes of variable. Let us introduce, for each $i \in \mathbb{N}$, the projection map $\pi_i : \mathcal{U} \to [0,1]$, which maps the open set $u \in \mathcal{U}$ to $|u_i|$, the length of its i'th largest interval component. Let's also write $F := -\log$ and $G(x) := F^{-1}(x) = \exp(-x)$. Using the change of variables formula under the maps $\mathcal{U} \cap \{|u_i| > 0\} \xrightarrow{\pi_i} (0,1] \xrightarrow{F} [0,\infty)$ we obtain

$$\begin{aligned}\int_{\mathcal{U}} |u_i| g(|u_i|) \nu(du) = \int_{\mathcal{U}} (\pi_i u) g(\pi_i u) \nu(du) &= \int_0^1 x g(x) (\pi_i \nu)(dx) \\ &= \int_0^1 GF(x) g(GF(x)) (\pi_i \nu)(dx) \\ &= \int_0^\infty G(x) g(G(x)) (F\pi_i \nu)(dx).\end{aligned}$$

By definition, for a Borel set $A \subset \mathbb{R}$, we have

$$F\pi_i \nu(A) = \nu(u \in \mathcal{U} : -\log|u_i| \in A).$$

The first equality in the statement of the lemma then follows by summing over i, and the second follows from the explicit identification of L contained in §1.2.3. $\square$

In Lemma 1.6, we saw that the Lévy measure Π of $(\zeta, \mathbf{Q})$ is $e^{-\overline{p}x} L(dx)$. Applying the previous lemma to the function $x \mapsto x^{\overline{p}} g(x)$ for a given non-negative measurable function g on $(0,1]$, we deduce the following corollary:

Corollary 2.9. *For any non-negative measurable function g on $(0,1]$, we have*

$$\int_{\mathcal{U}} \nu(du) \cdot \sum u_i^{1+\overline{p}} g(u_i) = \int_0^\infty g(e^{-x}) \Pi(dx)$$

where Π is the Lévy measure of $(\zeta, \mathbf{Q})$.

The next two results in this section will allow us, later on, to move ζ–particle killing barriers of the form $t \mapsto f(t)$ to killing barriers $t \mapsto f(t) + \epsilon t^{1/3}$. We will need to do this when proving Theorem 1.15 and Theorem 1.16, in order to apply our version of Mogulskii's Theorem.

Lemma 2.10. *There exists $\eta > 0$ such that*

$$\mathbf{P}\big(\exists x \in (0,1) : \quad \zeta_s^x < 0 \ \ \forall s \in (0,1], \quad \zeta_1^x \leq -\eta\big) \ > \ 0.$$

Proof. Define the random variable

$$Y \ := \ \sum_{[x]_1} \mathbf{1}\big\{\zeta_s^x < 0 \ \ \forall s \in (0,1], \quad \zeta_1^x \leq -\eta\big\}.$$

We want to show that $\mathbf{P}(Y \geq 1) > 0$ for some $\eta > 0$. Since Y is integer-valued, it suffices to show that $\mathbf{E}\,Y > 0$. An application of the Many-to-One Lemma gives

$$\mathbf{E}\,Y \ = \ \mathbf{Q}\big(e^{(1+\overline{p})\zeta_1}\,; \ \zeta_s < 0 \ \ \forall s \in (0,1], \ \ \zeta_1 \leq -\eta\big)$$

where of course $\mathbf{Q}(X; A) := \mathbf{Q}(X\mathbf{1}_A)$ for a random variable X and an event A. Because $(\zeta, \mathbf{Q})$ only jumps up and only drifts down (at rate $c_{\overline{p}}$), c.f. Lemma 1.6, we know that ζ_1 is at least $-c_{\overline{p}}$. Consequently,

$$\mathbf{E}\,Y \ \geq \ \gamma\,\mathbf{Q}(\zeta_s < 0 \ \ \forall s \in (0,1], \ \ \zeta_1 \leq -\eta)$$

where $\gamma := e^{-c_{\overline{p}}(1+\overline{p})}$. We will be done if we can show that the probability on the right-hand side is positive for some $\eta > 0$.

Let us fix some $\lambda > c_{\overline{p}}$. We proved in §1.2.2 that $\mathbf{Q}(\tau_{-\lambda}^- < \tau_0^+) > 0$ (here the τ's are defined relative to ζ). Again using the structure of $(\mathbf{Q}, \zeta)$ as summarized in Lemma 1.6, we know that $\tau_{-\lambda}^- \geq \lambda / c_{\overline{p}} > 1$. In consequence,

$$\big\{\tau_{-\lambda}^- < \tau_0^+\big\} \ \subseteq \ \{\zeta_s < 0 \ \ \forall s \in (0,1]\}.$$

The result follows immediately. $\square$

The next result is a simple corollary of the previous lemma and the fragmentation property.

Corollary 2.11. *Fix η as in the previous lemma, and denote the positive probability there by p. For all $t \geq 0$ and $\epsilon > 0$, we have*

$$\mathbf{P}(\exists x \in (0,1) : \ \zeta_s^x < 0 \ \ \forall s \in (0,\alpha], \ \ \zeta_\alpha^x \leq -\epsilon t^{1/3}) \ \geq \ p^\alpha$$

where $\alpha = \alpha(t, \epsilon, \eta) := \lceil \frac{\epsilon t^{1/3}}{\eta} \rceil$.

The key point, as will become clear later, is that $\lim_{t\to\infty} t^{-1/3} \log p^{\alpha(t,\epsilon,\eta)} = o(\epsilon)$. This will give us just enough wiggle room.

Proof. Using the previous lemma there exists, with probability at least p, a point $x_1 \in (0,1)$ such that ζ^{x_1} stays below 0 on $(0,1]$ and ends below $-\eta$ at time 1. Using the fragmentation property and the previous lemma we can pick a point $x_2 \in \mathcal{I}_1^{x_1}$, again with probability at least p, such that ζ^{x_2} remains below $\zeta_1^{x_1}$ on $(1,2]$ and ends below $\zeta_1^{x_1} - \eta \leq -2\eta$ at time 2. Iterating this procedure $\alpha = \lceil \frac{\epsilon t^{1/3}}{\eta} \rceil$ times, we find (with probability at least p^α) a point $x^\alpha \in (0,1)$ such that $\zeta_s^{x^\alpha} < 0$ on $(0,\alpha]$, and $\zeta_\alpha^{x^\alpha} \leq -\epsilon t^{1/3}$. □

We are now going to present several results of John Biggins. The theorem to follow is an amalgamation of the discussion preceding Theorem A in [17], and the contents of Theorem B in [18]. First we set up the required notation. Given is a branching random walk in discrete time, in which individuals alive at time a given time reproduce independently of one another and the history of the process, and in the same way as the initial ancestor, which is located at the origin. This process can be encoded as a point process on the real line, which we denote by $\mathcal{V}$. We let V stand for the intensity measure of $\mathcal{V}$, defined by $V(A) := \mathbf{E}\,\mathcal{V}(A)$ for Borel sets $A \subset \mathbb{R}$. We also write $\mathcal{V}^n$ for the point process describing the positions of particles in generation n, and write V^n for its intensity. In a standard abuse of notation we write $V^n(y)$ for $V^n((-\infty, y])$.

The Laplace transform m of V is defined by

$$m(\theta) \ := \ \int_{\mathbb{R}} e^{-\theta y}\, V(dy)$$

whenever it exists and is finite. For all such values of θ, the process

$$W^{(n)}(\theta) \ := \ m(\theta)^{-n} \int_{\mathbb{R}} e^{-\theta y}\, \mathcal{V}^n(dy)$$

is a non-negative martingale; its almost sure limit is denoted by $W(\theta)$.

Next we define

$$\begin{aligned} \theta_1 \ &:= \ \inf\{\theta \in \mathbb{R}: \ m(\theta) < \infty\} \qquad \text{and} \\ \theta_2 \ &:= \ \sup\{\theta \in \mathbb{R}: \ m(\theta) < \infty\}. \end{aligned}$$

For $\theta \in (\theta_1, \theta_2)$, we define

$$b(\theta) \ := \ -\frac{m'(\theta)}{m(\theta)} \qquad \text{and then} \qquad v(\theta) \ := \ m(\theta)\exp\left(\theta\, b(\theta)\right).$$

According to the discussion preceding Theorem A in [17], the set $(\theta_1, \theta_2) \cap v^{-1}(1,\infty)$ is an open (possibly empty) interval, which we denote by $(\vartheta_1, \vartheta_2)$. For $\theta \in (\theta_1, \theta_2)$, we also define

$$w(\theta) \ := \ \theta\, b(\theta) + \log m(\theta).$$

Given $d > 0$, we write ℓ_d for the measure on the Borel sets of $\mathbb{R}$ which gives weight d to points on the lattice $d\mathbb{Z}$, and assigns no mass elsewhere. We call a function $f : \mathbb{R} \to \mathbb{R}$ directly Riemann integral with respect to ℓ_d if $\int_{\mathbb{R}} |f|\, \ell_d < \infty$. When $d = 0$, we use the notation $d\mathbb{Z}$ to stand for $\mathbb{R}$, and ℓ_d to stand for the Lebesgue measure. In this case, the usual definition of direct Riemann integrability applies.

Theorem 2.12. *Suppose that $\mathcal{V}$ is a point process on $\mathbb{R}$, and define*

$$d \quad := \quad \max\{h \geq 0 : \mathcal{V} \text{ is concentrated on } h\mathbb{Z}\}.$$

Suppose, in the notation introduced above, that the following hypotheses hold:

1. *The function $y \mapsto V(y)$ is finite for all $y \in \mathbb{R}$, has more than one point of increase, and satisfies $V(\infty) > 1$.*
2. *The value θ lies in the (possibly empty) interval $(\vartheta_1, \vartheta_2)$, and that, for some $\epsilon > 0$,*
$$\mathbf{E}\Big(W^{(1)}(\theta)\,(\log_+ W^{(1)}(\theta))^{5/2+\epsilon}\Big) \quad < \quad \infty.$$
3. *The function $g : \mathbb{R} \to \mathbb{R}$ is directly Riemann integrable with respect to ℓ_d.*

Then $w(\theta) > 0$, the inequalities $0 < W(\theta) < \infty$ hold almost surely, and

$$B(n,\theta) \int_{\mathbb{R}} g(y - nb(\theta))\, \mathcal{V}^{(n)}(dy) \quad \longrightarrow \quad W(\theta) \int_{\mathbb{R}} e^{\theta y} g(y)\, \ell_d(dy)$$

almost surely as $n \to \infty$, where

$$B(n,\theta) \quad := \quad \sigma\sqrt{2\pi n}\, \exp\left(-nw(\theta)\right)$$

and

$$\sigma^2 \quad := \quad \frac{m''(\theta)}{m(\theta)} - b(\theta)^2 \quad \in \quad (0,\infty).$$

In particular, the following limits hold almost surely as $n \to \infty$:

$$B(n,\theta)\mathcal{V}^n(\{nb(\theta) + y\}) \quad \longrightarrow \quad d\, e^{\theta y}\, W(\theta)$$

whenever $d > 0$ and $y \in d\mathbb{Z}$, and

$$B(n,\theta)\mathcal{V}^n\big([nb(\theta) + y, nb(\theta) + y + x)\big) \quad \longrightarrow \quad \theta^{-1}\, e^{\theta y}\,(e^{\theta x} - 1)\, W(\theta)$$

whenever $d = 0$, $y \in \mathbb{R}$ and $x > 0$.

Bertoin and Rouault provide an analogue of this result for fragmentation processes in [15]. We prefer to rely on Biggins' far earlier work, however, as it is more than powerful enough for our purposes. In addition, Bertoin and Rouault restrict their attention to non-geometric dislocation measures. This restriction excludes, for instance, fragmentation processes in which each dislocation event produces a fixed number of intervals equal in length. We want to avoid ruling out this very natural class of fragmentation processes in our proof.

With this collection of useful fragmentation results in hand, we move on to our discussion of the fluctuation theory of spectrally one-sided Lévy processes with bounded variation.

2.2 Fluctuations of Lévy Processes

As we saw in the introduction, the Many-to-One Lemma allows us to relate functionals of fragmentation processes to their corresponding spines, denoted by $(\zeta, \mathbf{Q})$. We will be exclusively interested in functionals that count the number of particles lying in certain intervals at given times. This corresponds, after applying the Many-to-One Lemma, to studying a weighted $\mathbf{Q}$–probability that ζ lies in certain intervals as time passes. In this section we will develop large-time asymptotic relations for such probabilities, which we derive from well-trodden random walk theory. Although the process of translating these results from discrete to continuous time is simple, to the best of our knowledge the results we derive here are not explicitly stated in the existing literature.

First we show that a "triangular" version of Mogulskii's Theorem holds for Lévy processes like $(\zeta, \mathbf{Q})$. The non-triangular result first appeared in [40], and has been fruitfully applied ever since. We derive our version from the one contained in [25].

Lemma 2.13. *For each $t \geq 0$, let $X^{(t)}$ be a spectrally positive Lévy process with bounded variation. Suppose that the drift coefficient of all these processes is the same, and call it $-c < 0$. Let $a : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying $a_t \to \infty$ and $a_t = o(t^{1/2})$ as $t \to \infty$. Suppose that for some $\eta, \sigma^2 > 0$ the family $(X^{(t)} : t \geq 0)$ has the following properties:*

1. $\sup_{t\geq 1} \mathbf{E}|X_1^{(t)}|^{2+\eta} < \infty$;
2. $\mathbf{E}X_1^{(t)} = o(a_t/t)$;
3. $\mathrm{var}X_1^{(t)} \to \sigma^2$ *as* $t \to \infty$.

Let $g_1 < g_2$ be continuous functions on $[0,1]$ with $g_1(0) < 0 < g_2(0)$, and define the events

$$E_t := \left\{ g_1\left(\frac{s}{t}\right) \leq \frac{X_s^{(t)}}{a_t} \leq g_2\left(\frac{s}{t}\right) \quad \forall s \in [0,t] \right\}.$$

Then

$$\lim_{t\to\infty} \frac{a_t^2}{t} \log \mathbf{P}(E_t) = -\frac{\pi^2\sigma^2}{2} \int_0^1 \frac{dt}{(g_2(t) - g_1(t))^2}.$$

Moreover, for any $b > 0$,

$$\lim_{t\to\infty} \frac{a_t^2}{t} \log \mathbf{P}\left(E_t, \;\; \frac{X_t^{(t)}}{a_t} \geq g_2(1) - b \right) = -\frac{\pi^2\sigma^2}{2} \int_0^1 \frac{dt}{(g_2(t) - g_1(t))^2}.$$

Proof. Denote the proposed limit by $\gamma < 0$. Assume that we can find $\epsilon > 0$ and a sequence of positive numbers $t_k \uparrow \infty$ such that

$$\frac{a_{t_k}^2}{t_k} \log \mathbf{P}(E_{t_k}) \;\;\leq\;\; \gamma(1+\epsilon) \;\;<\;\; \gamma \qquad \forall k \in \mathbb{N}.$$

Define $n_k := \lceil t_k \rceil$, $b_{n_k} := a_{t_k}$, and $S_i^{(n_k)} := X_i^{(t_k)}$ for $1 \leq i \leq n_k$. For $n \in \mathbb{N} \setminus \{n_k : k \in \mathbb{N}\}$, let $S_i^{(n)} := Y_1 + \cdots + Y_i$, where the Y_i iid copies of some random variable with variance σ^2 and zero mean. Also set $b_n := n^{1/3}$ for for $n \in \mathbb{N} \setminus \{n_k : k \in \mathbb{N}\}$. Now fix $\delta > 0$ so small that $g_2 - g_1 - 2\delta > 0$ on $[0,1]$ and that $g_1(0) + \delta < 0 < g_2(0) - \delta$ (recall that g_1 and g_2 are continuous), and define

$$G_n(\delta) := \left\{ g_1\left(\frac{i}{n}\right) + \delta \leq \frac{S_i^{(n)}}{b_n} \leq g_2\left(\frac{i}{n}\right) - \delta \ \forall 1 \leq i \leq n \right\}.$$

By construction, the array $(S_i^{(n)})$ satisfies the hypotheses of the triangular version of Mogulskii's Theorem in [25]. This allows us to conclude that

$$\lim_{n\to\infty} \frac{b_n^2}{n} \log \mathbf{P}(G_n(\delta)) \quad = \quad -\frac{\pi^2\sigma^2}{2} \int_0^1 \frac{dt}{(g_2(t) - g_1(t) - 2\delta)^2}. \tag{2.2}$$

Now for each $\theta \geq 0$ we introduce the seminorm

$$\Delta(f, \theta) \quad := \quad \sup\{|f(x) - f(y)| : \ x, y \in [0,1], \ |x - y| \leq \theta\}$$

defined on the space $C([0,1])$. For a fixed $f \in C([0,1])$, the uniform continuity of f is equivalent to the statement $\lim_{\theta\downarrow 0} \Delta(f, \theta) = 0$. Since $t_k, a_{t_k} \to \infty$ as $k \to \infty$, it is therefore possible to choose $K = K(\delta) \in \mathbb{N}$ so large that $k \geq K$ forces

$$2\Delta(g_1, t_k^{-1}) + 2\Delta(g_2, t_k^{-1}) + c a_{t_k}^{-1} \quad \leq \quad \delta,$$

where we recall that c is the common drift of the Lévy processes $(X^{(t)} : t \geq 0)$. The point of all this is that

$$k \geq K(\delta) \quad \Longrightarrow \quad G_{n_k}(\delta) \subset E_{t_k}. \tag{2.3}$$

Before checking that this is the case, let us see how to conclude the argument. Using this implication, we conclude that, for arbitrary (small) $\delta > 0$, we have

$$\limsup_{k\to\infty} \frac{b_{n_k}^2}{n_k} \log \mathbf{P}(G_{n_k}(\delta)) \quad \leq \quad \limsup_{k\to\infty} \frac{a_{t_k}^2}{t_k} \log \mathbf{P}(E_{t_k}) \quad \leq \quad \gamma(1+\epsilon) \quad < \quad \gamma,$$

recalling that $b_{n_k} = a_{t_k}$ by definition. Using (2.2), we deduce that

$$-\frac{\pi^2\sigma^2}{2} \int_0^1 \frac{dt}{(g_2(t) - g_1(t) - 2\delta)^2} \leq \gamma(1+\epsilon) < \gamma,$$

for arbitrary (small) $\delta > 0$. Letting $\delta \downarrow 0$, we arrive at the contradiction $\gamma < \gamma$. It remains to check (2.3), which is a simple consequence of the definition of $K(\delta)$ and the fact that the Lévy processes under consideration decrease by at most $c\lambda$ over an interval of length λ. Let us suppose $k \geq K(\delta)$ and that $\omega \in G_{n_k}(\delta)$; that is,

$$g_1\left(\frac{i}{n_k}\right) + \delta \quad \leq \quad \frac{S_i^{(n_k)}(\omega)}{b_{n_k}} \quad \leq \quad g_2\left(\frac{i}{n_k}\right) - \delta \qquad \forall 1 \leq i \leq n_k.$$

For all $n \in \mathbb{N}$ we define $S_0^{(n)} := 0$. This doesn't affect the probabilities because $g_1(0) < 0 < g_2(0)$, so $g_1(0) < S_0^{(n)} < g_2(0)$ for all $n \in \mathbb{N}$ as a matter of deterministic fact. We want to show that

$$g_1\left(\frac{s}{t_k}\right) \quad \leq \quad \frac{X_s^{(t_k)}(\omega)}{a_{t_k}} \quad \leq \quad g_2\left(\frac{s}{t_k}\right) \qquad \forall s \in [0, t_k].$$

So fix $s \in [0, t_k]$. Then $s \in [i, i+1]$ for some $0 \leq i < n_k$. We have (all random variables are evaluated at ω)

$$\begin{aligned}
\frac{X_s^{(t_k)}}{a_{t_k}} \geq \frac{S_i^{(n_k)} - c}{b_{n_k}} &\geq g_1\left(\frac{i}{n_k}\right) + \delta - \frac{c}{b_{n_k}} \\
&\geq g_1\left(\frac{s}{n_k}\right) - \Delta(g_1, n_k^{-1}) + \delta - \frac{c}{b_{n_k}} \\
&= g_1\left(\frac{s}{t_k}\right) + \left(g_1\left(\frac{s}{n_k}\right) - g_1\left(\frac{s}{t_k}\right)\right) - \Delta(g_1, n_k^{-1}) + \delta - \frac{c}{b_{n_k}}.
\end{aligned}$$

Now we note that

$$\left|\frac{s}{n_k} - \frac{s}{t_k}\right| = \frac{s(n_k - t_k)}{n_k t_k} \leq \frac{s}{n_k t_k} \leq \frac{1}{n_k},$$

so that, returning to the previous display, we have

$$\frac{X_s^{(t_k)}}{a_{t_k}} \geq g_1\left(\frac{s}{t_k}\right) - 2\Delta(g_1, n_k^{-1}) + \delta - \frac{c}{b_{n_k}}.$$

It remains to note that, since $k \geq K(\delta)$, we have

$$-2\Delta(g_1, n_k^{-1}) + \delta - \frac{c}{b_{n_k}} \geq 0.$$

To prove the other inequality in the definition of E_{t_k} (referring to $X_s^{t_k}$ and g_2), we bound $X_s^{(t_k)}$ above by $S_{i+1}^{(n_k)} + c$ for $s \in [i, i+1]$, and proceed as above.

Altogether, we've so far shown that

$$\liminf_{t\to\infty} \frac{a_t^2}{t} \log \mathbf{P}(E_t) \geq \gamma.$$

The proof of the upper bound uses the same discretization, but is essentially trivial, as the set inclusion corresponding to the one we just proved will be reversed; and, if a Leˊvy process lies in some intervals for all times in $[0, t]$, it certainly does at times $i \in [0, t] \cap \mathbb{N}$. The proof of the second statement is similar. □

In fact we will only use the 'non-triangular' version of the previous two results, which follows as a simple corollary:

Lemma 2.14. *Let X be a spectrally positive Lévy process with bounded variation. Suppose X is centered and has finite moment of order $2+\eta$ for some $\eta > 0$. Then X has finite variance which we denote by σ^2. Let $a : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying $a_t \to \infty$ and $a_t = o(t^{1/2})$ as $t \to \infty$, and suppose that g_1 and g_2 are continuous functions on $[0, 1]$ satisfying $g_1 < g_2$ on $[0, 1]$ and $g_1(0) < 0 < g_2(0)$. Define events E_t for $t \geq 0$ by*

$$E_t := \left\{ g_1\left(\frac{s}{t}\right) \leq \frac{X_s}{a_t} \leq g_2\left(\frac{s}{t}\right) \quad \forall s \in [0, t] \right\}.$$

Then the two conclusions of Lemma 2.13 obtain.

Our next result combines several results from [32], slightly adapted to our particular needs. In particular, we direct the reader to Theorem 2.7(ii) of this review article, which identifies the resolvent measure of a spectrally positive Lévy process killed on passing below a fixed level.

Lemma 2.15. *Let X be a spectrally positive Lévy process with zero mean and finite variance σ^2. There exists a strictly increasing, absolutely continuous function W on $\mathbb{R}$ that vanishes on the negative half-line, and has the following properties:*

1. $W(x) \sim \dfrac{x}{\sigma^2}$ *as* $x \to \infty$.
2. *For all real numbers $a \le x$ and all non-negative measurable functions f on $[a, \infty)$, we have*

$$\mathbf{E}_x \int_0^{\tau_a^-} f(X_t)\, dt \;=\; \int_a^\infty dv \cdot f(v)\big[W(v-a) - W(v-x)\big].$$

The remaining results in this section are proved with reference to the corresponding random walk results contained in the appendix of [3]. Our approach is simple: we assume that the result in question does not hold for a given Lévy process, and obtain a random walk contradicting the corresponding result in [3] by looking at the Lévy process on a sufficiently fine mesh. First we state two elementary lemmas which will be of use in carrying out such arguments. The first is a topological lemma whose proof can be found in [41]. The second is a simple observation, recorded for convenience.

Lemma 2.16. *Let $U \subseteq [0, \infty)$ be open and unbounded. Then there exists $h > 0$ such that $nh \in U$ for infinitely many $n \in \mathbb{N}$.*

Lemma 2.17. *Let X be a real-valued stochastic process issued from zero which is right-continuous at the origin. Then*

$$\forall \epsilon > 0 \quad \forall \delta > 0 \quad \exists a > 0 \qquad \textit{such that} \qquad \mathbf{P}\big(\|X\|_{[0,a]} > \delta\big) \;<\; \epsilon,$$

where $\|X\|_{[0,a]} := \sup_{0 \le t \le a} |X_t|$.

The following result corresponds to statement (A.1) in [3].

Proposition 2.18. *Let X be a Lévy process with zero mean and finite variance. Then*

$$\exists C > 0 \quad \textit{such that} \quad \forall h \ge C \;\; \forall t \ge 0 \quad \sup_{r \in \mathbb{R}} \mathbf{P}\big(r \le X_t \le r + h\big) \;\le\; C\,\frac{h}{t^{1/2}} \;.$$

Proof. Assume the above statement is not true, i.e. for some such Lévy process X, the following statement holds:

$$\forall n \in \mathbb{N} \;\; \exists h_n \ge n \;\; \exists t_n > 0 \;\; \exists r_n \in \mathbb{R} \quad \text{such that} \quad \mathbf{P}\big(r_n \le X_{t_n} \le r_n + h_n\big) \;>\; n\,\frac{h_n}{t_n^{1/2}} \;.$$

Now select an $a > 0$ corresponding to the choices $\epsilon = \frac{1}{2}$ and $\delta = 1$ in Lemma 2.17. Evidently, for all $n \in \mathbb{N}$,

$$\begin{aligned} \mathbf{P}\big(r_n - 1 \le X_t \le & r_n + h_n + 1 \ \ \forall t \in [t_n, t_n + a]\big) \\ & \ge \ \mathbf{P}\big(r_n \le X_{t_n} \le r_n + h_n, \ \ \|X_t - X_{t_n}\|_{t \in [t_n, t_n+a]} \ < \ 1\big) \\ & \ge \ \frac{1}{2}\mathbf{P}\big(r_n \le X_{t_n} \le r_n + h_n\big) \ \ \ge \ \ \frac{n}{2}\frac{h_n}{t_n^{1/2}} \ , \end{aligned}$$

where in the second inequality we have used the Markov property of the Lévy process at time t_n. Let $U := \bigcup_{n=1}^{\infty}(t_n, t_n + a)$, which is an open set. Note that, to prevent the probability in the first display of the proof from exceeding one, we must have $t_n \ge n^4$, proving that U is unbounded. Lemma 2.16 therefore supplies an $h > 0$ and two strictly increasing sequences (m_j) and (n_j) of natural numbers with the property that, for all $j \in \mathbb{N}$ we have $m_j h \in [t_{n_j}, t_{n_j} + a]$. Note that $t_{n_j}/m_j \to h$ as $j \to \infty$. In particular, there exists $K > 0$ such that $K/m_j^{1/2} < 1/t_{n_j}^{1/2}$ for all $j \in \mathbb{N}$. Now define a random walk on $\mathbb{R}$ by $S_n := X_{nh}$, and note that this random walk has zero mean and finite variance. We estimate

$$\begin{aligned} \mathbf{P}\big(r_{n_j} - 1 \le S_{m_j} \ \ \le \ & r_{n_j} + h_{n_j} + 1\big) \\ & \ge \ \mathbf{P}\big(r_{n_j} - 1 \le X_t \le r_{n_j} + h_{n_j} + 1 \quad \forall t \in [t_{n_j}, t_{n_j} + a]\big) \\ & \ge \ \frac{K}{2} n_j \frac{h_{n_j}}{m_j^{1/2}}. \end{aligned}$$

Taking suprema and assuming without loss of generality that $h_{n_j} \ge 2$ for all $j \in \mathbb{N}$, we find that, for all $j \in \mathbb{N}$,

$$\sup_{r \in \mathbb{R}} \mathbf{P}(r \le S_{m_j} \le r + h_{n_j} + 2) \quad \ge \quad \frac{K}{4} n_j \frac{h_{n_j} + 2}{m_j^{1/2}},$$

contradicting (A.1) in [3]. ☐

Our next proposition corresponds to statement (A.3) in [3].

Proposition 2.19. *Let X be a Lévy process with zero mean and finite variance. Then, with $\underline{X}_t := \inf_{0 \le s \le t} X_s$, we have*

$$\limsup_{t \to \infty} \ t^{1/2} \ \sup_{u \ge 0} \ \frac{1}{u+1} \ \mathbf{P}\big(\underline{X}_t \ge -u\big) < \infty \ .$$

Proof. The statement in the proposition is equivalent to the following statement:

$$\exists C > 0 \quad \exists T > 0 \quad \text{ such that } \quad t \ge T \Rightarrow \sup_{u \ge 0} \ \frac{1}{u+1}\mathbf{P}\big(\underline{X}_t \ge -u\big) \ \le \ \frac{C}{t^{1/2}} \ .$$

For a contradiction, let us assume the converse of this statement holds. Then

$$\forall n \in \mathbb{N} \quad \exists t_n \ge n \quad \exists u_n \ge 0 \quad \text{ such that } \quad \frac{1}{u_n+1}\mathbf{P}\big(\underline{X}_{t_n} \ge -u_n\big) \ \ge \ \frac{n}{t_n^{1/2}} \ .$$

As in Proposition 2.18, select $a > 0$ with the following property:

$$\frac{1}{u_n + 1}\mathbf{P}\big(\underline{X}_t \geq -u_n - 1 \quad \forall t \in [t_n, t_n + a]\big) \;\geq\; \frac{n}{2t_n^{1/2}}\;.$$

Now choose sequences (m_j) and (n_j), and $K > 0$ precisely as in the proof of Proposition 2.18. Select furthermore an $M > 0$ with the property that $\frac{1}{u} \leq \frac{M}{u+1} \quad \forall u \geq 1$. Defining the random walk $(S_n)_{n\in\mathbb{N}}$ as in Proposition 2.18, we estimate

$$\begin{aligned}
\frac{K}{2} n_j \frac{1}{m_j^{1/2}} \quad &\leq \quad \frac{1}{u_{n_j} + 1}\; \mathbf{P}\big(\underline{X}_t \geq -u_{n_j} - 1 \quad \forall t \in [t_{n_j}, t_{n_j} + a]\big) \\
&\leq \quad \frac{1}{u_{n_j} + 1}\; \mathbf{P}\big(\underline{S}_{m_j} \geq -u_{n_j} - 1\big) \\
&\leq \quad \sup_{u\geq 0}\; \frac{1}{u+1}\; \mathbf{P}\big(\underline{S}_{m_j} \geq -u - 1\big) = \sup_{u\geq 1}\; \frac{1}{u}\; \mathbf{P}\big(\underline{S}_{m_j} \geq -u\big) \\
&\leq \quad M \sup_{u\geq 1}\; \frac{1}{u+1}\; \mathbf{P}\big(\underline{S}_{m_j} \geq -u\big) \;\leq\; M \sup_{u\geq 0}\; \frac{1}{u+1}\; \mathbf{P}\big(\underline{S}_{m_j} \geq -u\big).
\end{aligned}$$

This contradicts (A.3) in [3]. $\square$

With Proposition 2.18 and Proposition 2.19 in hand, the proof of the following corollary follows verbatim from the proof of Lemma A.1 of [3].

Corollary 2.20. *Let C be the constant whose existence is guaranteed by Proposition 2.18. For all $C' \geq C$ there exists $\gamma = \gamma(C') > 0$ such that, for all $a \geq 0$ and $b \geq -a$, the following inequality holds for all $t \geq 0$:*

$$\mathbf{P}\big(\,\underline{X}_t \geq -a,\;\; b \leq X_t \leq b + C'\,\big) \;\leq\; \gamma\, \frac{\big\{(1+a) \wedge t^{1/2}\big\}\big\{(1+a+b) \wedge t^{1/2}\big\}}{t^{3/2}}.$$

In particular, there exists $\tilde{\gamma} > 0$ such that for all $a \geq 0$ and $b \geq -a$ the following inequality holds for all $t \geq 0$:

$$\mathbf{P}\big(\underline{X}_t \geq -a,\;\; X_t \leq b\big) \;\leq\; \tilde{\gamma}\, \frac{\big\{(1+a) \wedge t^{1/2}\big\}\big\{(1+a+b)^2 \wedge t\big\}}{t^{3/2}}.$$

Our final result concerning fluctuation theory corresponds to Lemma A.3 in [3]. Like our version of Mogulskii's Theorem, this lemma is only stated for Lévy process that are spectrally positive and have bounded variation.

Proposition 2.21. *Let X be a spectrally positive Lévy process with bounded variation, zero mean, and finite variance. We write $c > 0$ for the coefficient characterizing the downward drift. For $\alpha > 0$ let $X_t^\alpha := X_t + \alpha$. Then there exists $C > 0$ such that, for any $f\colon [0,\infty) \to \mathbb{R}$ satisfying $\limsup_{t\to\infty} t^{-1/2} f(t) < \infty$ and $f(t) \geq \alpha$, for all large t, we have*

$$\liminf_{t\to\infty}\; t^{3/2}\, \mathbf{P}\Big(\underline{X}_t^\alpha \geq 0,\;\; \inf_{t\leq s\leq 2t} X_s^\alpha \geq f(t),\;\; f(t) \leq X_{2t}^\alpha < f(t) + C\Big) \;>\; 0. \qquad (2.4)$$

Proof. Let us assume that there exists no such constant $C > 0$, and fix an $\alpha > 0$. Select an $a > 0$ corresponding to the choices $\epsilon = \frac{1}{2}$ and $\delta = 1$ in Lemma 2.17. Finally, choose an $h \in (0, \frac{1}{4}\min\{a, \frac{\alpha}{c}\})$. Define a random walk (S_n) by $S_n := X_{nh}$ and note that (S_n) satisfies the hypotheses of Lemma A.3 in [3]. Let K denote the positive constant corresponding to (S_n) whose existence is guaranteed by Lemma A.3 in [3] (there, K is called $2C$), and pick $\tilde{C} > K+1+\alpha$. Since, in particular, we are assuming that (2.4) does not hold for $C = \tilde{C}$, we infer the existence of a sequence $(t_k) \subseteq [0, \infty)$ such that $\lim_{k\to\infty} t_k = \infty$, and the existence of a function f with the properties given in the statement of the proposition, which together satisfy

$$\left(\frac{t_k}{h}\right)^{3/2} \mathbf{P}\Big(\underline{X}^{\alpha}_{t_k} \geq 0, \inf_{t_k \leq s \leq 2t_k} X^{\alpha}_s \geq f(t_k), \; f(t_k) \leq X^{\alpha}_{2t_k} < f(t_k) + \tilde{C}\Big) \; < \; \frac{1}{k} \quad (2.5)$$

for all $k \in \mathbb{N}$. Now define $n_k := \lfloor \frac{t_k}{h} \rfloor - 1$. Note in particular that $(n_k+1)h \in [t_k - h, t_k]$; this will allow us to ensure that $X^{\alpha}_{t_k} \geq f(t_k)$ in the following computation. Define $a_{n_k} := f(t_k) + \alpha$ for each $k \in \mathbb{N}$, and $a_n := 0$ whenever there is no k such that $n = n_k$. The important thing to note is that for any $j, k \in \mathbb{N}$ with $j \leq k$ and all $r \geq 0$ we have

$$\min_{s\in[jh, jh+r]} X^{\alpha}_s \;\geq\; S_j - rc + \alpha \;\geq\; \underline{S}_k - rc + \alpha \;\geq\; \underline{S}_k \qquad \text{whenever} \qquad r \leq \frac{\alpha}{c}.$$

Consequently, whenever $r \leq \frac{\alpha}{c}$ we find, for any $k \in \mathbb{N}$, that $\underline{X}^{\alpha}_{kh+r} \geq \underline{S}_k$ Recalling that $n_k h \in [t_k - 2h, t_k - h]$, we can write $t_k = n_k h + r_k$ for some $r_k \in [h, 2h]$. Consequently, we deduce that

$$\underline{X}^{\alpha}_{t_k} \;=\; \underline{X}^{\alpha}_{n_k h + r_k} \;\geq\; \underline{S}_{n_k} \qquad \text{provided} \qquad r_k \leq \frac{\alpha}{c},$$

and the condition on r_k holds because we have selected $h < \frac{\alpha}{2c}$. We will use this in the computation below, where we require $\{\underline{S}_{n_k} \geq 0\} \subseteq \{\underline{X}^{\alpha}_{t_k} \geq 0\}$. By the same considerations, we also have the inclusion

$$\left\{\inf_{t_k \leq s \leq 2t_k} X^{\alpha}_s \geq f(t_k)\right\} \;\subseteq\; \left\{\min_{n_k < j \leq 2n_k} S_j \geq \; f(t_k) + \alpha\right\},$$

since we have in fact picked $h < \frac{\alpha}{4c}$. We can therefore make the estimate

$$\begin{aligned}
\left(\tfrac{t_k}{h}\right)^{3/2} & \mathbf{P}\Big(\underline{X}^{\alpha}_{t_k} \geq 0, \inf_{t_k \leq s \leq 2t_k} X^{\alpha}_s \geq f(t_k), \; f(t_k) \leq X^{\alpha}_{2t_k} < f(t_k) + \tilde{C}\Big) \\
&\geq \; n_k^{3/2} \, \mathbf{P}\Big(\underline{S}_{n_k} \geq 0, \; \min_{n_k < j \leq 2n_k} S_j \geq \; f(t_k) + \alpha, \\
&\qquad\qquad f(t_k) + \alpha \leq S_{2n_k} < f(t_k) + \tilde{C} - 1, \|X_t - X_{2n_k h}\|_{t\in[2n_k h, 2t_k]} < 1\Big) \\
&\geq \; \tfrac{1}{2} n_k^{3/2} \, \mathbf{P}\Big(\underline{S}_{n_k} \geq 0, \; \min_{n_k < j \leq 2n_k} S_j \geq a_{n_k}, \; a_{n_k} \leq S_{2n_k} < a_{n_k} + K\Big). \qquad (2.6)
\end{aligned}$$

In the second inequality we used the fact that $h < \frac{a}{4}$ and the Markov property of X^{α} at time $2n_k h$. Combining (2.5) and (2.6), we find that, for all $k \in \mathbb{N}$, we have

$$n_k^{3/2} \, \mathbf{P}\Big(\underline{S}_{n_k} \geq 0, \; \min_{n_k < j \leq 2n_k} S_j \geq a_{n_k}, \; a_{n_k} \leq S_{2n_k} < a_{n_k} + K\Big) \;\leq\; \frac{2}{k},$$

contradicting Lemma A.3 in [3]. $\square$

We now have all the technical tools required for the proofs of our three main results, which we embark on now. We begin with our theorem on the size of the largest fragment.

CHAPTER 3

THE LARGEST FRAGMENT OF A HOMOGENEOUS FRAGMENTATION PROCESS

In this chapter we prove Theorem 1.14. This result is joint work with Andreas Kyprianou and Peter Mörters, and has been published in *The Journal of Statistical Physics* [33].

Recall that the largest fragment decays exponentially at rate $\Phi'(\overline{p})$, almost surely. As discussed in the introduction, our theorem, which we state again now, refines this result:

Theorem. *Starting from any initial configuration in $\mathcal{U}$, the following convergence holds in probability, as $t \to \infty$:*

$$\frac{\min_{x\in(0,1)} \xi_t^x - c_{\overline{p}} t}{\log t} \quad \longrightarrow \quad \frac{3}{2}(1+\overline{p})^{-1}\,.$$

At this point let's re-orientate ourselves with some very simple remarks. First, recall that $\xi_t^x := -\log|\mathcal{I}_t^x|$. Since $-\log$ is a decreasing function, the largest particle at time t corresponds to the smallest value of ξ_t^x. Second, the reason we can write "min" instead of "inf" is because there are always finitely many particles whose sizes are greater than a given positive number.

Before giving the proof in full detail, we provide a brief sketch to explain the main ideas. First let us emphasize our debt to Aïdékon and Shi [3], on whose proof the following work is based.

We start by breaking the result into an upper and lower bound. Throughout this chapter the proposed limit is denoted by l. We will show that the following two statements hold:

$$\mathbf{P}_u\left(\min_{x\in(0,1)} \zeta_t^x \le \alpha \log t\right) \to 0 \quad \text{as} \quad t\uparrow\infty \quad \text{for all} \quad \alpha < l\,; \quad \text{and} \tag{3.1}$$

$$\limsup_{t\to\infty} \frac{\min_{x\in(0,1)} \zeta_t^x}{\log t} \le l \quad \mathbf{P}_u\text{–almost surely.} \tag{3.2}$$

Since the second bound holds almost surely, it's natural to ask whether we can strengthen our theorem to an almost sure statement. In the discrete-time branching random walk case, this is not possible (see [27, Theorem 1.2]), so we have good reason to believe to believe that it isn't possible here either.

The proof of the lower bound (3.1) is the easy part, and could be derived directly from the random walk result using Lemma 2.1. We will prove it from scratch, however, as the proof is short and consolidates the notation and ideas introduced in the previous chapters. Fix $\alpha < l$ and $k \in \mathbb{N}$. Let Z_t^k stand for the number of x–labelled particles that satisfy the inequality $\zeta_t^x \leq \alpha \log t$ and the inequality $\zeta_s^x \geq -k$ for all $s \in [0, t]$. Using the Many-to-One Lemma and Corollary 2.20, we will show that $\mathbf{E}Z_t^k \to 0$. The reason why we introduce the truncation in k is so we can apply Corollary 2.20. To remove the truncation, we will use the convergence of the additive martingale introduced in §1.2.4 to show that the entire $-\log$ process almost surely lies above some fixed level.

Now let's discuss our approach to proving (3.2), which is significantly more complicated. We will show, using the second moment method, that for any $x \in (0, 1)$, $t \geq 0$, and sufficiently large s, we have

$$\mathbf{P}(\exists y \in \mathcal{I}_t^x : \ \zeta_{t+s}^y \leq \zeta_t^x + l \log s) \ \geq \ (\log s)^{-3}. \tag{3.3}$$

That is, a ζ–parent alive at time s will have a ζ–child exceeding its position at time $t + s$ by at most $l \log t$ with a probability that, as we will see below, is sufficiently large. (In fact there will be constants all over the place, which we omit in this sketch for clarity.)

Now fix $\epsilon > 0$. We will see that at a random time $T(n, \epsilon)$ satisfying

$$\limsup_n \frac{T(n, \epsilon)}{\log n} \ \leq \ K\epsilon$$

for some constant $K > 0$ independent of ϵ and n, there are at least n^ϵ ζ–particles alive whose positions do not exceed $K\epsilon \log n$. Let's denote this set of n^ϵ ζ–particles by $\mathcal{L}_n$. Let's also call a ζ–particle, alive at time t, s–good if at least one of its descendants alive at time $t + s$ exceeds the value of its parent by at most $l \log s$, and s–bad otherwise. Using (3.3), we find that

$$\mathbf{P}(\text{all particles in } \ \mathcal{L}_n \ \text{ are } n\text{–bad}) \ \leq \ \left(1 - (\log n)^{-3}\right)^{n^\epsilon}.$$

It is simple to show that

$$\sum_{n \geq 4} \left(1 - (\log n)^{-3}\right)^{n^\epsilon} \ < \ \infty.$$

By the Borel-Cantelli lemma, it follows that the event

$$\liminf_n \left\{\text{some particle in } \ \mathcal{L}_n \ \text{ is } n\text{–good}\right\} \tag{3.4}$$

has probability 1. This means that with probability 1 we can find a sequence of tags $(x_n) \subset (0, 1)$ such that $\mathcal{I}_{T_n}^{x_n} \in \mathcal{L}_n$ and

$$\zeta_{T_n + n}^{x_n} \ \leq \ \zeta_{T_n}^{x_n} + l \log n \tag{3.5}$$

for all sufficiently large n. Since $\mathcal{I}_{T_n}^{x_n} \in \mathcal{L}_n$, we know that

$$\zeta_{T_n}^{x_n} \;\leq\; K\epsilon \log n.$$

Combining this with (3.4) and (3.5) we find that almost surely, for all large n we have

$$\min_{x\in(0,1)} \zeta_{T_n+n}^{x} \;\leq\; (K\epsilon + l)\log n.$$

For any $x \in (0,1)$ we know that

$$\zeta_{T_n+n}^{x} \;=\; \xi_{T_n+n}^{x} - c_{\overline{p}}(T_n+n) \;\geq\; \zeta_n^x - c_{\overline{p}} T_n$$

where the inequality holds because $\xi_{T_n+n}^{x} \geq \xi_n^x$. It follows that almost surely the following inequality holds for all large n:

$$\inf_{x\in(0,1)} \zeta_n^x \;\leq\; (K\epsilon + l)\log n + c_{\overline{p}} T_n,$$

almost surely. Dividing by $\log n$ and taking $\limsup$'s we deduce that

$$\limsup_{\mathbb{N}\ni n\to\infty} \frac{\zeta_n^x}{\log n} \;\leq\; l + (1 + c_{\overline{p}})K\epsilon.$$

Letting ϵ go to zero, we arrive at *nearly* what we want, just with the delimiter of the $\limsup$ running through $\mathbb{N}$. But this isn't a problem because $\inf_{x\in(0,1)} \zeta_s^x$ is bounded above on $s \in [n, n+1]$ by $\inf_{x\in(0,1)} \zeta_{n+1}^x + c_{\overline{p}}$, and $c_{\overline{p}} = o(\log n)$.

We hope that this sketch will help the reader navigate their way through our detailed proof, which we turn to now.

3.1 Proof of the lower bound

Fix an arbitrary $\alpha \in (0, l)$, $k \in \mathbb{N}$ and $u \in \mathcal{U}$. Define, for $t \geq 0$, the random variable

$$Z_t^k := \sum_{[x]_t} \mathbf{1}\Big(\zeta_t^x \leq \alpha \log t,\ \underline{\zeta}_t^x \geq -k\Big), \tag{3.6}$$

where $\underline{\zeta}_t^x := \inf_{0\leq s\leq t} \zeta_s^x$. This random variable counts the number of 'bad' particles (with a truncation we will remove later).

We estimate the mean of Z_t^k under $\mathbf{E}_u$ as follows:

$$\begin{aligned}
\mathbf{E}_u Z_t^k \;&=\; \sum_{i=1}^{\infty} \mathbf{Q}\Big(e^{\zeta_t(\overline{p}+1)} \mathbf{1}\big(\zeta_t - \log|u_i| \leq \alpha \log t,\ \underline{\zeta}_t - \log|u_i| \geq -k\big)\Big) \\
&\leq\; t^{\alpha(\overline{p}+1)} \sum_i |u_i|^{\overline{p}+1} \mathbf{Q}\Big(\zeta_t - \log|u_i| \leq \alpha \log t,\ \underline{\zeta}_t - \log|u_i| \geq -k\Big). \quad (3.7)
\end{aligned}$$

In the first line we use MT1 (Lemma 1.7), and in the second we bound the exponential factor using the indicator. Recalling that $(\zeta, \mathbf{Q})$ is a spectrally positive Lévy process with zero mean and finite variance (c.f. Lemma 1.6), we can estimate a typical probability on the right-hand side of the previous inequality using Corollary 2.20:

$$\begin{aligned}\mathbf{Q}\Big(\zeta_t - \log|u_i| \le \alpha\log t,\ \underline{\zeta}_t - \log|u_i| \ge -k\Big) \\ &\le\ \gamma\, t^{-3/2}(k-\log|u_i|+1)(k+\alpha\log t)^2 \\ &\le\ \gamma_k\, t^{-3/2}(\log t)^2(1-\log|u_i|),\end{aligned}$$

for some constants $\gamma, \gamma_k > 0$ (where the latter depends on k). Putting this back into (3.7), we find that, for all sufficiently large t,

$$\mathbf{E}_u Z_t^k \ \le\ \gamma_k t^{\alpha(\overline{p}+1)} t^{-3/2}(\log t)^2 \sum_i |u_i|^{\overline{p}+1}(1-\log|u_i|).$$

Since $\overline{p} > 0$, the function $x \mapsto x^{\overline{p}}(1-\log x)$ has an upper bound $K > 0$ on $(0,1)$, so the sum on the right-hand side is bounded by $K\sum|u_i| = K$. We deduce that

$$\mathbf{E}_u Z_t^k \ \le\ K\gamma_k\, t^{\alpha(\overline{p}+1)} t^{-3/2}(\log t)^2.$$

Since $\alpha(\overline{p}+1) < l(\overline{p}+1) = 3/2$, this quantity goes to zero as $t \to \infty$.

To complete this part of the proof we must remove the truncation $\underline{\zeta}_t^x \ge -k$ in (3.6). To this end, we recall the *intrinsic additive martingale* corresponding to $\overline{p}$, which we introduced in §1.2.4:

$$M_t \ :=\ e^{\Phi(\overline{p})t}\sum_{[x]_t} I^x(t)^{1+\overline{p}} \ =\ \sum_{[x]_t}\exp\big(-(1+\overline{p})\,\zeta_t^x\big).$$

By the martingale convergence theorem, M_t converges to a finite limit $\mathbf{P}_u$–almost surely as $t \to \infty$. Noting that $\overline{p} > 0$, we find that

$$\inf_{t\ge 0}\inf_{x\in(0,1)} \zeta_t^x \ >\ -\infty \quad \mathbf{P}_u\text{–almost surely.}$$

Letting $B_k := \big\{\inf_{t\ge0}\inf_{x\in(0,1)}\zeta_t^x \ge -k\big\}$ for each $k \in \mathbb{N}$, it follows that

$$\lim_{k\to\infty}\mathbf{P}_u(B_k) = 1. \tag{3.8}$$

Next fix an arbitrary $\epsilon > 0$, and (using (3.8)) select $k = k(\epsilon) \in \mathbb{N}$ so large that $\mathbf{P}_u(B_k) \ge 1-\epsilon$. Observing that $Z_t^k \ge \mathbf{1}_{B_k}\sum_{[x]_t}\mathbf{1}(\zeta_t^x \le \alpha\log t)$ for all $t \ge 0$, we may then write,

$$\begin{aligned}\mathbf{P}_u(Z_t^k = 0) \ &\le\ \mathbf{P}_u\Big[B_k \cap \Big\{\sum_{[x]_t}\mathbf{1}\big(\zeta_t^x \le \alpha\log t\big) = 0\Big\}\Big] \ +\ \mathbf{P}_u(B_k^c) \\ &\le\ \mathbf{P}_u\Big[\sum_{[x]_t}\mathbf{1}\big(\zeta_t^x \le \alpha\log t\big) = 0\Big] \ +\ \epsilon,\end{aligned} \tag{3.9}$$

for all $t \ge 0$. We have already shown that $\mathbf{E}_u(Z_t^k) \to 0$ as $t \to \infty$, and so, since Z_t^k takes values in $\{0,1,2,...\}$, we deduce that $\mathbf{P}_u(Z_t^k = 0) \to 1$ as $t \uparrow \infty$. Combining this observation with (3.9) we conclude that

$$1 \;=\; \liminf_{t\to\infty} \mathbf{P}_u(Z_t^k = 0) \;\leq\; \liminf_{t\to\infty} \mathbf{P}_u\Big[\sum_{[x]_t} \mathbf{1}\big(\zeta_t^x \leq \alpha \log t\big) \;= 0\Big] \;+\; \epsilon\,.$$

Since $\epsilon > 0$ was arbitrary, we deduce that $\lim_{t\to\infty} \mathbf{P}_u\Big[\sum_{[x]_t} \mathbf{1}\big(\zeta_t^x \leq \alpha \log t\big) \;= 0\Big] = 1$. Finally, observe that

$$\Big\{\sum_{[x]_t} \mathbf{1}(\zeta_t^x \leq \alpha \log t) \;= 0\Big\} \;\subseteq\; \Big\{\inf_{x\in(0,1)} \zeta_t^x > \alpha \log t\Big\},$$

so that $\mathbf{P}_u\big(\inf_{x\in(0,1)} \zeta_t^x > \alpha \log t\big) \to 1$ as $t \uparrow \infty$, for arbitrary $\alpha < l$. This is equivalent to (3.1) . □

3.2 Proof of the upper bound

In this part of the proof, we can work under $\mathbf{P}$ without loss of generality. To see why, note that we are now trying to show the existence of 'big' particles (in the sense made precise by (3.2)). This means that, starting the fragmentation from general $u \in \mathcal{U}$, we can immediately look only at the largest particle at time t descending from u_1, whose size we call $B_t^{u_1}$. Let B_t denote the size of the largest fragment at time t in a fragmentation issued from $(0,1)$. The fragmentation property implies that $(B_t^{u_1}, \mathbf{P}_u)$ is equal in law to $(|u_1|B_t, \mathbf{P})$. The numerator in (3.2) corresponding to these two processes will therefore only differ by the additive constant $-\log|u_1|$, which, upon division by $\log t$, goes to zero in the limit.

3.2.1 Step 1: The second moment method

As mentioned in our sketch, we first want to show (up to some constants) that for all $x \in (0,1)$, $t \geq 0$ and sufficiently large s we have

$$\mathbf{P}(\exists y \in \mathcal{I}_t^x : \;\; \zeta_{t+s}^y \leq \zeta_t^x + l \log t) \;\;\geq\;\; (\log s)^{-3}. \tag{3.10}$$

Let $C > 0$ be the larger of the two constants provided by Proposition 2.18 and Proposition 2.21. Introduce the following intervals:

$$J_s(t) \;:=\; \begin{cases} [-1, \infty) & \text{if } \; 0 \leq s \leq t \\ [l \log t, \infty) & \text{if } \; t < s < 2t \\ [l \log t, l \log t + C] & \text{if } \; s = 2t. \end{cases}$$

The reason we work on the interval $[0, 2t]$ and not on $[0, t]$ is simply to avoid messy fractional indices like $t/2$.

For $x \in (0,1)$ and $u, v \in [0, 2t]$, define the events $A_{[u,v]}^x := \{\zeta_s^x \in J_s(t) \;\forall s \in [u,v]\}$, and write $A_{2t}^x := A_{[0,2t]}^x$. In what follows, $A_{[u,v]}$ (with no superscript) means $A_{[u,v]}^\chi$, where χ is the uniformly distributed random tag in $(0,1)$ in the definition of ζ. Finally, define for $t \geq 0$ the random variable

$$Z_t \;:=\; \sum_{[x]_{2t}} \mathbf{1}_{A^x_{2t}}.$$

We want to show that $\mathbf{P}(Z_t > 0)$ is eventually bounded below by some constant multiple of $(\log t)^{-3}$. The key to doing this is the so-called Paley-Zygmund inequality, which is a trivial consequence of the Cauchy-Schwarz inequality:

$$\mathbf{P}(Z_t > 0) \;\geq\; \frac{(\mathbf{E}\, Z_t)^2}{\mathbf{E}(Z_t^2)}.$$

In view of this inequality, we need to bound the first moment of Z_t from above and the second moment from below. In fact, we will show the following: for some $\gamma_1, \gamma_2 > 0$, we have

$$\begin{aligned}
\mathbf{E}(Z_t) &\;\geq\; \gamma_1\,; && \text{and} && (3.11)\\
Z_t^2 &\;=\; Z_t + \Lambda_t\,, && \text{with} && (3.12)\\
\mathbf{E}\Lambda_t &\;\leq\; \gamma_2(\log t)^3, &&&& (3.13)
\end{aligned}$$

for all large t. Before proving these facts, let's see how we can use them to complete the second moment argument. First we make the following simple calculation, valid for all large t:

$$\begin{aligned}
\mathbf{E}(Z_t^2) \;\leq\; \gamma_2(\log t)^3 + \mathbf{E}(Z_t) \;&\leq\; \left[\tfrac{\gamma_2}{\gamma_1}(\log t)^3 + 1\right]\mathbf{E}(Z_t)\\
&\leq\; \left[\tfrac{\gamma_2}{\gamma_1}(\log t)^3 + 1\right]\tfrac{1}{\gamma_1}\mathbf{E}(Z_t)^2\,. \qquad (3.14)
\end{aligned}$$

In the first inequality we use (3.12) and (3.13), and in the next two inequalities we use (3.11). First making use of the Paley-Zygmund inequality, and then of (3.14), we find that

$$\mathbf{P}(Z_t > 0) \;\geq\; \frac{(\mathbf{E}Z_t)^2}{\mathbf{E}(Z_t)^2} \;\geq\; \frac{\gamma}{(\log t)^3}\;.$$

for some $\gamma > 0$ and all large t. The point is that

$$\left\{\min_{x\in(0,1)} \zeta^x_t \;\leq\; l\log\tfrac{t}{2} + C\right\} \;\supseteq\; \left\{Z_{t/2} > 0\right\}$$

so that, for all sufficiently large t, we have

$$\begin{aligned}
\mathbf{P}\left\{\min_{x\in(0,1)} \zeta^x_t \;\leq\; l\log t + C\right\} \;&\geq\; \mathbf{P}\left\{\min_{x\in(0,1)} \zeta^x_t \;\leq\; l\log\tfrac{t}{2} + C\right\}\\
&\geq\; \mathbf{P}(Z_{t/2} > 0) \;\geq\; \frac{\gamma}{(\log(t/2))^3}\\
&\geq\; \frac{\gamma}{(\log t)^3}
\end{aligned}$$

for some γ and all large t. By the fragmentation property, this is equivalent (modulo the constants C and γ) to our aim, (3.10).

Let's now complete this part of the proof by proving (3.11), (3.12) and (3.13).

Proof of (3.11): This is the easy part. Using MT1 we obtain

$$\mathbf{E}Z_t \;=\; \mathbf{Q}\big(e^{\zeta_{2t}(\overline{p}+1)}\,\mathbf{1}_{A_{2t}}\big) \;\geq\; \gamma\, t^{3/2}\,\mathbf{Q}(A_{2t}) \;\geq\; \gamma' \;>\; 0,$$

for some $\gamma, \gamma' > 0$ and all large t. In the first inequality we have used the indicator to bound the exponential factor from below; the second uses Proposition 2.21. □

Proof of (3.12) *and* (3.13): Using Corollary 2.7 we find that $\mathbf{E}Z_t^2 = EZ_t + \Lambda_t$, where

$$\Lambda_t \;=\; \int_0^{2t} dr \,\cdot\, \mathbf{Q}\left(e^{(1+\overline{p})\zeta_r}\mathbf{1}_{A_{[0,r]}}\int_{\mathcal{U}} G(\zeta_r, u)\nu(du)\right)$$

with

$$G(\alpha, u) \;:=\; \sum_{i\neq j} F(\alpha - \log|u_i|)F(\alpha - \log|u_j|),$$

where, for $\alpha \in \mathbb{R}$,

$$F(\alpha) \;:=\; \mathbf{Q}\, e^{\zeta_{2t-r}(\overline{p}+1)}\mathbf{1}_{\left(\alpha+\zeta_s\in J_{s+r}(t)\;\;\forall s\in[0,2t-r]\right)}.$$

For $r \in [0, t]$, let's write λ_r for the integrand of the integral defining Λ_t.

It remains to show that: $\quad \mathbf{E}\Lambda_t = \displaystyle\int_0^{2t} \lambda(r)dr = O\big((\log t)^3\big) \quad$ as $\quad t \uparrow \infty.$

Notation: *In the remainder of this section, positive constants (independent of t) will be denoted by $\gamma > 0$, the value of which will change from one inequality to another.*

First we estimate $F(\alpha - \log|u_i|)$ for interval components u_i of $u \in \mathcal{U}$ and $\alpha \in \mathbb{R}$: using the indicator to bound the exponent we have

$$\begin{aligned} F(\alpha - \log|u_i|) \;&=\; \mathbf{Q}\, e^{\zeta_{2t-r}(\overline{p}+1)}\mathbf{1}_{\left(\alpha-\log|u_i|+\zeta_s\in J_{s+r}(t)\;\;\forall s\in[0,2t-r]\right)} \\ &\leq\; \gamma\, t^{3/2}\; |u_i|^{\overline{p}+1}e^{-\alpha(\overline{p}+1)}f(\alpha - \log|u_i|), \end{aligned}$$

for some $\gamma > 0$, with

$$f(\theta) \;:=\; \mathbf{Q}\Big(\theta + \zeta_s \in J_{s+r}(t)\;\;\forall s \in [0, 2t-r]\Big), \qquad \text{for } \theta \in \mathbb{R}.$$

We estimate f in two different ways, depending on the value of r. For $r \in [t, 2t]$, Proposition 2.18 provides the estimate

$$f(\theta) \;\leq\; \mathbf{Q}\Big(\zeta_{2t-r} \in [l\log t - \theta, l\log t - \theta + 2C]\Big) \;\leq\; \gamma\, n_{2t-r},$$

with $n_\theta := \theta^{-1/2} \wedge 1$ for $\theta \geq 0$. This leads to the bound

$$\int_t^{2t} \lambda(r)\,dr \;\leq\; \gamma\, I_1\, t^3 \int_t^{2t} dr\,\cdot n_{2t-r}^2\mathbf{Q}\left(e^{-(1+\overline{p})\zeta_r}\mathbf{1}_{A_{[0,r]}}\right), \tag{3.15}$$

where

$$I_1 \quad := \quad \int_{\mathcal{U}} \nu(du) \, \cdot \sum_{i \neq j} |u_i|^{\overline{p}+1} |u_j|^{\overline{p}+1} \, .$$

Let us check that I_1 is finite. Indeed,

$$\begin{aligned}
\sum_{i \neq j} |u_i|^{\overline{p}+1} |u_j|^{\overline{p}+1} \quad &\leq \quad \sum_{i \neq j} |u_i||u_j| \quad = \quad \sum_i |u_i|(1 - |u_i|) \\
&\leq \quad (1 - |u_1|) + \sum_{i \geq 2} |u_i| \\
&= \quad 2(1 - |u_1|).
\end{aligned}$$

In the first inequality we use the facts that $|u_i|, |u_j| < 1$ and $\overline{p} > 0$; in the first equality we fix an interval component u_i of $u \in \mathcal{U}$ and sum over the interval components $u_j \neq u_i$ of u; and in the second inequality we use the fact that $|u_i| \in (0, 1)$. The finiteness of I_1 then follows from the integrability condition in the definition of dislocation measures, Definition 1.2. It remains to estimate the expectation in (3.15):

$$\begin{aligned}
\mathbf{Q}\big(e^{-\zeta_r(\overline{p}+1)} \mathbf{1}_{A_{[0,r]}}\big) \quad &\leq \quad t^{-3/2} \, \mathbf{Q}\big(\mathbf{1}_{A_{[0,r]}} \mathbf{1}_{(\zeta_r \leq 2\, l \log t)}\big) + \mathbf{Q}\big(e^{-\zeta_r(\overline{p}+1)} \mathbf{1}_{A_{[0,r]}} \mathbf{1}_{(\zeta_r > 2\, l \log t)}\big) \\
&\leq \quad t^{-3/2} \mathbf{Q}\big(\underline{\zeta}_r \geq -1, \; \zeta_r \leq 2l \log t\big) \, + \, t^{-3} \\
&\leq \quad \gamma \, t^{-3/2} \, r^{-3/2} (\log t)^2 \, + \, t^{-3} \\
&\leq \quad \gamma \, t^{-3} (\log t)^2.
\end{aligned}$$

In the first line we split the event $\{\zeta_r \geq l \log t\} \subset A_{[0,r]}$ into the events $\{\zeta_r > 2l \log t\}$ and $\{l \log t \leq \zeta_r \leq 2l \log t\}$. In the second line, we discard some information from the indicator on the interval $[t, r]$ and estimate the exponential factor in the second term using the indicator $\mathbf{1}_{(\zeta_r > 2\, l \log t)}$. In the penultimate line, we use Corollary 2.20 to estimate the remaining expectation. Returning to (3.15), we conclude that

$$\int_t^{2t} \lambda(r) \, dr \quad \leq \quad \gamma (\log t)^2 \int_t^{2t} n_{2t-r}^2 dr \quad = \quad O\big((\log t)^3\big),$$

as required.

Now we look at $\lambda(r)$ for $r \in [0, t]$. This time we make the estimate, valid for $\theta \geq -1$:

$$\begin{aligned}
f(\theta) \quad &\leq \quad \mathbf{Q}\left(\underline{\zeta}_{2t-r} \geq -1 - \theta, \;\; \zeta_{2t-r} \in [l \log t - \theta, l \log t - \theta + 2C]\right) \\
&\leq \quad \gamma \, (2 + \theta) \, (\log t) \, (2t - r)^{-3/2} \\
&\leq \quad \gamma \, (2 + \theta) \, (\log t) \, t^{-3/2}.
\end{aligned}$$

In the first inequality we throw away some information from the indicator on the interval $[t, 2t - r)$; in the second we use Corollary 2.20; and the final inequality uses the fact that $r \in [0, t]$. Making the substitution $\theta = \alpha - \log |u_i|$, we arrive at

$$\begin{aligned}
f(\alpha - \log |u_i|) \quad &\leq \quad \gamma \, (2 + \alpha - \log |u_i|) \, (\log t) \, t^{-3/2} \\
&\leq \quad 2\gamma \, (2 + \alpha) \, (1 - \log |u_i|) \, (\log t) \, t^{-3/2}
\end{aligned}$$

for $\alpha \geq -1$ (recall we intend to make the substitution $\alpha = \zeta_r \geq -1$). This leads to the bound

$$\lambda(r) \;\; \leq \;\; \gamma\, I_2\, (\log t)^2\, \mathbf{Q}\left(e^{-(1+\overline{p})\zeta_r}(2+\zeta_r)^2 \mathbf{1}_{A_{[0,r]}}\right),$$

where

$$I_2 \;\; := \;\; \int_{\mathcal{U}} \nu(du) \cdot \sum_{i \neq j} |u_i|^{\overline{p}+1}|u_j|^{\overline{p}+1}(1-\log|u_i|)(1-\log|u_j|).$$

This time we note that the function $x \mapsto x^{\overline{p}}(1-\log x)$ is bounded on $[0,1]$, since $\overline{p} > 0$. This allows us to write $I_2 \leq K \int_{\mathcal{U}} \nu(du) \cdot \sum |u_i||u_j|$ (for some $K > 0$), which is finite by the same arguments we used for I_1. It remains to show that

$$\int_0^t dr \cdot \mathbf{Q}\left((2+\zeta_r)^2 e^{-\zeta_r(\overline{p}+1)} \mathbf{1}_{A_{[0,r]}}\right) \;\; < \;\; \infty.$$

We start by noting that the function $x \mapsto (1+x)^2 e^{-(1+\overline{p})x}$ is bounded above on $[-1, \infty)$ by Ke^{-x}, for some finite constant $K > 0$ (since $\overline{p} > 0$). First using this fact, and then Lemma 2.15, we find that

$$\begin{aligned}
\int_0^t dr \cdot \mathbf{Q}\left((2+\zeta_r)^2 e^{-\zeta_r(\overline{p}+1)} \mathbf{1}_{A_{[0,r]}}\right) \;\; &\leq \;\; K\,\mathbf{Q} \int_0^{\tau_{-1}^-} \exp(-\zeta_t)\; dt \\
&= \;\; K \int_{-1}^{\infty} \exp(-v)\big[W(v+1) - W(v)\big]\, dv,
\end{aligned}$$

where W is the function discussed in Lemma 2.15 corresponding to the spectrally positive process $(\zeta, \mathbf{Q})$. Since W is continuous and asymptotically linear, the rightmost expression in the previous display is finite. □

Let us summarize what we have shown so far:

Proposition 3.1. *There exist constants $C, \gamma > 0$ such that for all sufficiently large t,*

$$\mathbf{P}\left\{\min_{x \in (0,1)} \zeta_t^x \;\; \leq \;\; l \log t + C\right\} \;\; \geq \;\; \frac{\gamma}{(\log t)^3}.$$

3.2.2 Step 2: The proliferation of large particles

As per our sketch, we now estimate how many particles are alive at integer times k whose sizes are close to $\exp(-c_{\overline{p}}k)$, which is roughly the size of the largest particle. First we introduce the following notation:

$$\mathcal{N}(k, \delta) \;\; := \;\; \{\mathcal{I}_k^x : \; \zeta_t^x \leq \delta k\}.$$

We also define a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ as follows. Recall that $v_{\max} := \Phi'(0+) \in (c_{\overline{p}}, \infty]$. Also recall that the image of $(\underline{p}, \overline{p})$ under the continuous map Φ' is $(c_{\overline{p}}, v_{\max})$. Now fix $\delta > 0$. If $c_{\overline{p}} + \delta < v_{\max}$, we find the unique $p_\delta \in (\underline{p}, \overline{p})$ satisfying $\Phi'(p_\delta) = c_{\overline{p}} + \delta$ and then set $\rho(\delta) := (1+p_\delta)\Phi'(p_\delta) - \Phi(p_\delta)$. Now suppose $c_{\overline{p}} + \delta \geq v_{\max}$. This is only possible in case $v_{\max} < \infty$, so we can set $\delta_* := \frac{1}{2}(v_{\max} - c_{\overline{p}})$. We then set $\rho(\delta) := \rho(\delta_*)$, which we have already defined (note that $c_{\overline{p}} + \delta_* < v_{\max}$). Note that $\rho(\delta) > 0$ for all $\delta > 0$, by Lemma 1.5.

We will use Theorem 2.12 to prove the following proposition:

Proposition 3.2. *For all $\delta > 0$ the following inequality holds almost surely:*

$$\liminf_{\mathbb{N}\ni k\to\infty} \frac{1}{k} \log \#\mathcal{N}(k,\delta) \;\; \geq \;\; \rho(\delta).$$

Proof. We will apply Biggins' result, Theorem 2.12, to the branching random walk in discrete time generated by the point process

$$\mathcal{V} \;\; := \;\; \sum_{[x]_1} \delta_{\xi_1^x},$$

which is concentrated on $[0,\infty)$. We note that, by the fragmentation property, the process $(\mathcal{V}^n : n \in \mathbb{N})$ is equal in law to the process obtained by sampling ξ–values at times in $\mathbb{N}$. We also note that the branching random walk generated by $\mathcal{V}$ is concentrated on a lattice if and only if the dislocation measure ν of the fragmentation process is geometric. We will treat both cases in what follows.

First we need to identify identify the various functions and values discussed before the statement of Theorem 2.12. We have

$$m(\theta) \;\; = \;\; \mathbf{E}\left(\int_{[0,\infty)} e^{-\theta y} \sum_{[x]_1} \delta_{\xi_1^x}(dy) \right) \;\; = \;\; \mathbf{E} \sum_{[x]_1} (I_1^x)^\theta \;\; = \;\; \exp(-\Phi(\theta-1))$$

yielding the identifications $\theta_1 = 1 + \underline{p}$ and $\theta_2 = \infty$. Similarly, we find that the martingale $(W^{(n)}(\theta))$ coincides with the intrinsic additive martingale (see §1.2.4) corresponding to the value $p = \theta - 1$ and sampled at integer times. That is, for all $n \in \mathbb{N}$, there is the equality $W^{(n)}(\theta) = M(n, \theta-1)$ in law.

Next, we see that

$$b(\theta) \;\; = \;\; \Phi'(\theta-1) \quad \text{and} \quad v(\theta) \;\; = \;\; \exp\left(\theta\Phi'(\theta-1) - \Phi(\theta-1)\right).$$

It follows that the inequality $v(\theta) > 1$ is equivalent to the inequality $\theta\Phi'(\theta-1) - \Phi(\theta-1) > 0$, which, according to Lemma 1.5, is in turn equivalent to $\theta \in (1+\underline{p}, 1+\overline{p})$. We can therefore make the identification $(\vartheta_1, \vartheta_2) = (1+\underline{p}, 1+\overline{p})$. We also have

$$w(\theta) \;\; = \;\; \theta\Phi'(\theta-1) - \Phi(\theta-1) \quad \text{and} \quad \sigma^2(\theta) = -\Phi''(\theta-1).$$

Now we verify that hypotheses 1 and 2 of Theorem 2.12 hold. Since $\mathcal{V}((-\infty, y])$ is bounded deterministically by $\exp(y)$ by conservation of mass, $V(y)$ is finite for all $y \geq 0$. Since with positive probability at least one fragmentation event occurs before time 1 (and fragmentation events always produce at least two particles), $V(\infty) > 1$. Next we check that V has at least two points of increase. In fact, we show V has infinitely many. Suppose that V is constant on $[\alpha,\infty)$ for some $\alpha > 0$. Then for $y > \alpha$ we have

$$0 \;=\; V(y) - V(\alpha) \;=\; \mathbf{E}\sum_{[x]_1} \mathbf{1}_{(\xi_1^x \in (\alpha,y])} \;\to\; \mathbf{E}\sum_{[x]_1} \mathbf{1}_{(\xi_1^x \in (\alpha,\infty))}$$

as $y \to \infty$, by monotone convergence. But the right-hand side equals

$$\mathbf{Q}\big(e^{(1+\overline{p})\zeta_1}\mathbf{1}_{(\xi_1 \in (\alpha,\infty))}\big) \;\geq\; e^{-(1+\overline{p})c}\,\mathbf{Q}(\xi_1 > \alpha).$$

We conclude that $\mathbf{Q}(\xi_1 > \alpha) = 0$. Unless the fragmentation process is trivial, however, there exists some $a > 0$ such that jumps of ξ (under $\mathbf{Q}$) of size exceeding a arrive according to a Poisson process with positive rate. The probability that more than α/a such jumps arrive before time 1 is positive, giving a contradiction.

Now we verify hypothesis 2 of Theorem 2.12: we fix $p > \underline{p}$ and show that, for all $q > 0$,

$$\mathbf{E}\Big(M(1,p)\,(\log_+ M(1,p))^q\Big) \;<\; \infty.$$

Recall that

$$M(1,p) \;=\; e^{\Phi(p)}\sum_{[x]_1}(I_1^x)^{1+p} \;=\; \gamma\, A_p,$$

where we've written $\gamma := \exp(\Phi(p)) < \infty$ and $A_p := \sum_{[x]_1}(I_1^x)^{1+p}$. Then note that $\log_+(ab) \leq \log_+(a) + \log_+(b)$ for all $a, b > 0$, and that $(a+b)^q \leq 2^q(a^q + b^q)$ for all $a, b, q > 0$. We deduce that

$$M(1,p)\,(\log_+ M(1,p))^q \;\leq\; 2^q\gamma A_p((\log_+\gamma)^q + (\log_+ A_p)^q)$$

Now $\mathbf{E}A_p = \exp(-\Phi(p)) < \infty$, since $p > \underline{p}$, so we just need to show that

$$\mathbf{E}\left(A_p(\log_+ A_p)^q\right) \;<\; \infty$$

whenever $p > \underline{p}$ and $q > 0$. Since $(\log_+(y))^q = o(y^\epsilon)$ as $y \to \infty$, for all $q, \epsilon > 0$, it further suffices to show that for all $p > \underline{p}$ there exists $\epsilon > 0$ so small that $\mathbf{E}A_p^{1+\epsilon} < \infty$; i.e., that

$$\mathbf{E}\bigg(\sum_{[x_1]}(I_1^x)^{1+p}\bigg)^{1+\epsilon} \;<\; \infty.$$

But this follows immediately from Lemma 4.5 of [36].

We will only need the conclusions (2.1) and (2.2) of Theorem 2.12, which now read as follows. Define, for $n \in \mathbb{N}$ and $p > \underline{p}$,

$$C(k,p) \;:=\; -\Phi''(p)\sqrt{2\pi k}e^{-k\big((1+p)\Phi'(p)-\Phi(p)\big)}.$$

Then, in the lattice case (that is, when ν is geometric),

$$C(k,p)\,\mathcal{V}^k(\{k\Phi'(p)+y\}) \;\longrightarrow\; d\,e^{(1+p)y}\,M(\infty,p)$$

almost surely as $k \to \infty$, for all $p \in (\underline{p},\overline{p})$ and $y \in d\mathbb{Z}$.

In the non-lattice case,

$$C(k,p)\,\mathcal{V}^k\big([k\Phi'(p)+y, k\Phi'(p)+y+x)\big) \quad\longrightarrow\quad (1+p)^{-1}e^{(1+p)y}(e^{(1+p)x}-1)M(\infty,p)$$

almost surely as $k\to\infty$, for all $p\in(\underline{p},\overline{p})$, all $y\in\mathbb{R}$ and all $x>0$.

In both cases, the martingale limit $M(\infty,p)$ satisfies the inequalities $0<M(\infty,p)<\infty$ almost surely. In the lattice case we set $y=0$ and take logarithms, to deduce that, almost surely,

$$\lim_{k\to\infty}\frac{1}{k}\log\,\mathcal{V}^k(\{k\Phi'(p)\}) \;=\; (1+p)\Phi'(p)-\Phi(p).$$

In the non-lattice case we start with the trivial observation that

$$\mathcal{V}^k\big((-\infty,\,k\Phi'(p)+y+x)\big) \;\geq\; \mathcal{V}^k\big([k\Phi'(p)+y, k\Phi'(p)+y+x)\big).$$

We set $y=-1$ and $x=1/2$. Taking logarithms then allows us to conclude that, almost surely,

$$\liminf_{k\to\infty}\frac{1}{k}\log\,\mathcal{V}^k\big((-\infty,\,k\Phi'(p)]\big) \;\geq\; (1+p)\Phi'(p)-\Phi(p).$$

Now let us define $\mathcal{N}^\xi(t,\beta):=\{\mathcal{I}^x_t : \xi^x_t\leq\beta t\}$. We can summarize our conclusions above, in both the lattice and non-lattice case, by asserting that for all $\underline{p}<p<\overline{p}$, we have

$$\liminf_{k\to\infty}\frac{1}{k}\log\#\mathcal{N}^\xi(k,\Phi'(p)) \;\geq\; (1+p)\Phi'(p)-\Phi(p)$$

almost surely.

It remains to note that $\mathcal{N}^\xi(k,c_{\overline{p}}+\delta)=\mathcal{N}(k,\delta)$, and to use the definition of ρ. Indeed, suppose that $c_{\overline{p}}+\delta<v_{\max}$. Then

$$\#\mathcal{N}(k,\delta) \;=\; \#\mathcal{N}^\xi(k,c_{\overline{p}}+\delta) \;=\; \#\mathcal{N}^\xi(k,\Phi'(p_\delta)),$$

so that

$$\liminf_{k\to\infty}\frac{1}{k}\log\#\mathcal{N}(k,\delta) \;\geq\; (1+p_\delta)\Phi'(p_\delta)-\Phi(p_\delta) \;=\; \rho(\delta).$$

If $c_{\overline{p}}+\delta\geq v_{\max}$, we have

$$\#\mathcal{N}(k,\delta) \;\geq\; \#\mathcal{N}(k,\delta_*).$$

As above, we deduce that

$$\liminf_{k\to\infty}\frac{1}{k}\log\#\mathcal{N}(\delta,k) \;\geq\; \liminf_{k\to\infty}\frac{1}{k}\log\#\mathcal{N}(\delta_*,k) \;\geq\; \rho(\delta_*) \;=\; \rho(\delta),$$

completing the proof. □

Now we want to see how long it takes for $\#\mathcal{N}(k,\delta)$ to exceed n^ϵ for $\epsilon>0$ and $n\in\mathbb{N}$. So for all such ϵ and n we define

$$T(n,\epsilon,\delta) \;:=\; \inf\{k\in\mathbb{N}:\; \#\mathcal{N}(k,\delta)\geq n^\epsilon\}.$$

The following result follows from Proposition 3.2.

Corollary 3.3. *The following inequality holds almost surely:*

$$\limsup_{n\to\infty} \frac{1}{\log n} T(n,\epsilon,\delta) \;\leq\; \frac{\epsilon}{\rho(\delta)}.$$

Proof. We will show that

$$\left\{\liminf_{k\to\infty} \frac{1}{k}\ \log \#\mathcal{N}(k,\delta) \;\geq\; \rho(\delta)\right\} \;\subseteq\; \left\{\limsup_{n\to\infty} \frac{1}{\log n} T(n,\epsilon,\delta) \;\leq\; \frac{\epsilon}{\rho(\delta)}\right\}.$$

So let us fix a sample point ω in the left-hand side and show that ω belongs to the right-hand side (all random variables below are evaulated at ω). For $\eta \in (0,\rho(\delta))$, there exists some integer $M \geq 0$ such that $k \geq M$ forces

$$\#\mathcal{N}(k,\delta) \;\geq\; e^{(\rho(\delta)-\eta)k}.$$

We have

$$\begin{aligned}
T(n,\epsilon,\delta) &= \inf\{k\in\mathbb{N}:\ \#\mathcal{N}(k,\delta)\geq n^{\epsilon}\} \\
&\leq \inf\{k\geq M:\ \#\mathcal{N}(k,\delta)\geq n^{\epsilon}\} \\
&\leq \inf\{k\geq M: e^{(\rho(\delta)-\eta)k}\geq n^{\epsilon}\} \\
&= \max\left(\frac{\epsilon}{\rho(\delta)-\eta}\ \log n + 1, M\right) \\
&= \frac{\epsilon}{\rho(\delta)-\eta}\ \log n + 1
\end{aligned}$$

for all large n. We conclude that

$$\limsup_{n\to\infty} \frac{1}{\log n} T(n,\epsilon,\delta) \;\leq\; \frac{\epsilon}{\rho(\delta)-\eta}.$$

Since η was arbitrary, the claim follows. □

3.2.3 Step 3: Completing the argument

We are now ready to complete the argument using the Borel Cantelli Lemma. First, fix arbitrary $\epsilon > 0$ and choose the $\lfloor n^{\epsilon}\rfloor$ largest elements of $\mathcal{N}(1,T_n)$ (we write T_n for $T(n,\epsilon,1)$), labelling them $(\mathcal{I}^{n,j} : 1 \leq j \leq \lfloor n^{\epsilon}\rfloor)$. Since distinct particles evolve independently, we know that

$$\mathbf{P}\Big(\min_{x\in\mathcal{I}^{n,j}} \zeta^{x}_{T_n+n} > -\log|\mathcal{I}^{n,j}| - c_{\overline{p}}T_n + l\log n + C\ \ \forall 1\leq j\leq \lfloor n^{\epsilon}\rfloor\Big)$$

is bounded above by

$$\mathbf{P}\Big(\min_{x\in(0,1)} \zeta^{x}_{n} > l\log n + C\Big)^{\lfloor n^{\epsilon}\rfloor}$$

which, by Proposition 3.1, is bounded above by

$$\left(1-\frac{\gamma}{(\log n)^3}\right)^{\lfloor n^{\epsilon}\rfloor}$$

whenever n is sufficiently large. We now claim that this expression is summable in n. It clearly suffices to show that for any $\alpha > 0$ and $k \in \mathbb{N}$ we have

$$\sum_{n=4}^{\infty} \left(1 - \frac{1}{(\log n)^k}\right)^{n^\alpha} \;<\; \infty.$$

We will show that

$$\int_4^\infty \left(1 - (\log x)^{-k}\right)^{x^\alpha} dx \;=\; \int_{\log 4}^\infty e^x \left(1 - x^{-k}\right)^{e^{\alpha x}} dx \;<\; \infty.$$

by proving that the second integrand is $o(e^{-x})$ as $x \to \infty$, or, equivalently, that

$$e^{\alpha x}\left(\log(x^k) - \log(x^k - 1)\right) - 2x \;\to\; \infty \quad \text{as} \quad x \to \infty.$$

But for all $t > 1$ we have $\log'(s) \geq \frac{1}{t}$ $\forall s \in [t-1, t]$, so $\log(x^k) - \log(x^k - 1) \geq \frac{1}{x^k}$ for all $x > 1$. It remains to note that $x^{-k}e^{\alpha x} - 2x \to \infty$ as $x \to \infty$.

By the Borel Cantelli Lemma we deduce that, almost surely, for all sufficiently large n there exists a $j_n \in \{1, ..., \lfloor n^\epsilon \rfloor\}$ such that

$$\min_{x \in \mathcal{I}^{n,j_n}} \zeta^x_{T_n+n} \;\leq\; -\log|\mathcal{I}^{n,j_n}| - c_{\overline{p}}T_n + l\log n + C$$

where $\mathcal{I}^{n,j_n}$ is an element of $\mathcal{N}(1, T_n)$. But this means that

$$-\log|\mathcal{I}^{n,j_n}| - c_{\overline{p}}T_n \;\leq\; T_n.$$

On the other hand,

$$\begin{aligned}
\min_{x \in \mathcal{I}^{n,j_n}} \zeta^x_{T_n+n} \;\geq\; \min_{x \in (0,1)} \zeta^x_{T_n+n} \;&=\; \min_{x \in (0,1)} \xi^x_{T_n+n} - c_{\overline{p}}(T_n + n) \\
&\geq\; \min_{x \in (0,1)} \xi^x_n - c_{\overline{p}}(T_n + n) \\
&=\; \min_{x \in (0,1)} \zeta^x_n - c_{\overline{p}}T_n.
\end{aligned}$$

Altogether, we deduce that, almost surely, for all large enough n we have

$$\min_{x \in (0,1)} \zeta^x_n \;\leq\; (1 + c_{\overline{p}})T_n + l\log n + C.$$

Diving by $\log n$ and sending $n \to \infty$, we deduce that

$$\limsup_{\mathbb{N} \ni n \to \infty} \frac{1}{\log n} \min_{x \in (0,1)} \zeta^x_n \;\leq\; (1 + c_{\overline{p}})\frac{\epsilon}{\rho(1)} + l.$$

Letting $\epsilon \to 0$ gives

$$\limsup_{\mathbb{N} \ni n \to \infty} \frac{1}{\log n} \min_{x \in (0,1)} \zeta^x_n \;\leq\; l,$$

almost surely.

For $t \in [n, n+1]$ we have $\zeta_t^x = \xi_t^x - c_{\overline{p}} t \leq \xi_{n+1}^x - c_{\overline{p}} n = \zeta_{n+1}^x + c_{\overline{p}}$. We deduce that, whenever $t \in [n, n+1]$, we have

$$\begin{aligned}\frac{1}{\log t} \min_{x \in (0,1)} \zeta_t^x &\leq \frac{1}{\log s} \Big(\min_{x \in (0,1)} \zeta_{n+1}^x + c_{\overline{p}} \Big) \\ &\leq \frac{\log(n+1)}{\log n} \frac{1}{\log(n+1)} \min_{x \in (0,1)} \zeta_{n+1}^x + \frac{c_{\overline{p}}}{\log(n+1)}.\end{aligned}$$

This is enough to show that

$$\left\{ \limsup_{\mathbb{N} \ni n \to \infty} \frac{1}{\log n} \min_{x \in (0,1)} \zeta_n^x \leq l \right\} \subseteq \left\{ \limsup_{\mathbb{R}_+ \ni t \to \infty} \frac{1}{\log t} \min_{x \in (0,1)} \zeta_t^x \leq l \right\}.$$

Since the smaller of these events has probability 1, the proof is complete. □

CHAPTER 4

SURVIVAL OF SUPERCRITICALLY KILLED FRAGMENTATION PROCESSES

In this chapter we will prove Theorem 1.15, which concerns the asymptotic behaviour of the survival probability of a $(c_{\overline{p}} + \epsilon)$–killed fragmentation process as $\epsilon \downarrow 0$. We refer the reader to §1.2.6 for a summary of the relevant definitions. Let's state Theorem 1.15 again:

Theorem. *The survival probability $\rho(\epsilon)$ of the $(c_{\overline{p}} + \epsilon)$–killed fragmentation process satisfies the following asymptotic identity:*

$$\lim_{\epsilon \downarrow 0} \epsilon^{1/2} \log \rho(\epsilon) \;=\; -\sqrt{\frac{\pi^2 (1+\overline{p})|\Phi''(\overline{p})|}{2}}.$$

Our proof of this theorem is based on a paper by Gantert, Hu and Shi [25], in which the authors address the analogous question in the context of branching random walk. In fact, we will use Lemma 2.1 to extract the upper bound from this paper, without much further work. The lower bound, on the other hand, must be proved from scratch, and will rely on several applications of our version of Mogulskii's Theorem, Lemma 2.14.

4.1 Proof of the upper bound

In this section, we will prove that

$$\limsup_{\epsilon \downarrow 0} \epsilon^{1/2} \log \rho(\epsilon) \;\leq\; -\sqrt{\frac{\pi^2 (1+\overline{p})|\Phi''(\overline{p})|}{2}}\,.$$

We start by noting that, for all $n \in \mathbb{N}$, we have

$$\rho(\epsilon) \;\leq\; \mathbf{P}\big(\exists x \in (0,1) : \;\; \zeta_s^x \leq \epsilon s \quad \forall s \in [0,n]\big). \tag{4.1}$$

Next we fix an array $(b_i^{(n)} : 1 \leq i \leq n)_{n \in \mathbb{N}}$ with $b_i^{(n)} \leq b_j^{(n)}$ whenever $i \geq j$. For each $n \in \mathbb{N}$, we let $f^{(n)}$ be the function on $[0,n]$ obtained by applying linear interpolation to the collection of coordinates $\{(i,(1+\overline{p})^{-1}b_i^{(n)}) : 0 \le i \le n\}$, where we define $b_0^{(n)} := b_1^{(n)}$ for all $n \in \mathbb{N}$. The function $f^{(n)}$ is continuous and decreasing. Applying Lemma 2.1 to (4.1) for each $n \in \mathbb{N}$ leads to the bounds

$$\rho(\epsilon) \;\le\; e^{\epsilon(1+\overline{p})n} I_n^{(n)} \;+\; \sum_{i=0}^{n-1} e^{(1+\overline{p})\left(\epsilon(j+1)-f_{j+1}^{(n)}\right)} I_j^{(n)} \qquad \forall n \in \mathbb{N} \tag{4.2}$$

where, for each $n \in \mathbb{N}$,

$$I_j^{(n)} \;:=\; \mathbf{Q}\big(\epsilon r - f_r^{(n)} < \zeta_r \le \epsilon r \quad \forall r \in [0,j]\big) \qquad \text{for } j \in [0,n] \cap \mathbb{N}.$$

Now we define a random walk S by setting $S_i := (1+\overline{p})\zeta_i$ for $i \in \mathbb{N}$. First discretizing in time and then using the definition of $f^{(n)}$ gives

$$\begin{aligned} I_j^{(n)} &\le \mathbf{Q}\big(\epsilon i - f_i^{(n)} < \zeta_i \le \epsilon i \quad \forall i \le j\big) \\ &= \mathbf{Q}\big((1+\overline{p})\epsilon i - b_i^{(n)} < S_i \le (1+\overline{p})\epsilon i \quad \forall i \le j\big). \end{aligned}$$

Let us define

$$J_j^{(n)} \;:=\; \mathbf{Q}\big(\epsilon i - b_i^{(n)} < S_i \le \epsilon i \quad \forall i \le j\big).$$

Returning to (4.2), we deduce that

$$\rho\left(\frac{\epsilon}{1+\overline{p}}\right) \;\le\; e^{\epsilon n} J_n^{(n)} + \sum_{j=0}^{n-1} e^{\epsilon(j+1)-b_{j+1}^{(n)}} J_j^{(n)} \;=:\; F(n, b^{(n)}, \epsilon).$$

The argument $b^{(n)} = \big(b_1^{(n)}, \cdots, b_n^{(n)}\big)$ on the right-hand side means that F is a function of the whole (finite) sequence $b^{(n)}$.

Now we fix $\beta > 0$ and make the explicit choice $b_i^{(n)} := \beta(n-i)^{1/3}$. It is shown in [25] that for any $\theta > 0$ and $N \in \mathbb{N}$, the following inequality holds:

$$\limsup_{k\to\infty} \frac{\theta^{1/2}}{(Nk)^{1/3}} \log F\left(Nk, b^{(Nk)}, \frac{\theta}{(Nk)^{2/3}}\right) \;\le\; G(N,\theta,\beta),$$

where

$$\begin{aligned} G(N,\theta,\beta) \;:=\; \theta^{1/2} \max_{1\le l\le N-1} \Bigg\{ &\theta - \frac{3\pi^2\sigma_S^2}{2\beta^2},\; \frac{\theta}{N} - \beta\left(1-\frac{1}{N}\right)^{1/3}, \\ &\frac{\theta(l+1)}{N} - \beta\left(1-\frac{l+1}{N}\right)^{1/3} - \frac{3\pi^2\sigma_S^2}{2\beta^2}\frac{N^{1/3}-(N-l)^{1/3}}{N^{1/3}} \Bigg\}. \end{aligned}$$

Here, σ_S stands for the variance of S, which equals $(1+\overline{p})^2|\Phi''(\overline{p})|$. We immediately deduce that for all $\theta > 0$ and $N \in \mathbb{N}$, we have

$$\limsup_{k\to\infty} \frac{\theta^{1/2}}{(Nk)^{1/3}} \log \rho\left(\frac{\theta(Nk)^{-2/3}}{1+\overline{p}}\right) \;\le\; G(N,\theta,\beta).$$

On the other hand, regardless of the values of θ and N, the left-hand side of the inequality above is equal to

$$\limsup_{\epsilon\downarrow 0} \epsilon^{1/2} \log \rho\left(\frac{\epsilon}{1+\overline{p}}\right),$$

because $\epsilon \mapsto \rho(\epsilon)$ is a monotone function. Altogether we deduce that

$$\limsup_{\epsilon \downarrow 0} \epsilon^{1/2} \log \rho \left(\frac{\epsilon}{1+\overline{p}} \right) \;\; \leq \;\; G(N, \theta, \beta)$$

for all $N \in \mathbb{N}$ and $\theta, \beta > 0$. Now set

$$\theta \;\; := \;\; \theta_0(\beta) \;\; := \;\; \frac{\pi^2 \sigma^2}{2\beta^2} - \frac{\beta}{3}.$$

Then

$$\limsup_{\epsilon \downarrow 0} \epsilon^{1/2} \log \rho \left(\frac{\epsilon}{1+\overline{p}} \right) \;\; \leq \;\; \lim_{\beta \downarrow 0} \limsup_{N \to \infty} G(N, \theta_0(\beta), \beta).$$

But, according to [25], we have

$$\lim_{\beta \downarrow 0} \limsup_{N \to \infty} G(N, \theta_0(\beta), \beta) \;\; \leq \;\; -\sqrt{\frac{\pi^2 \sigma_S^2}{2}} \;\; = \;\; -\sqrt{\frac{\pi^2 (1+\overline{p})^2 |\Phi''(\overline{p})|}{2}}.$$

So

$$\limsup_{\epsilon \downarrow 0} \epsilon^{1/2} \log \rho \left(\frac{\epsilon}{1+\overline{p}} \right) \;\; \leq \;\; -\sqrt{\frac{\pi^2 (1+\overline{p})^2 |\Phi''(\overline{p})|}{2}}.$$

The required result follows immediately. □

4.2 Proof of the lower bound

In this section we will show that

$$\liminf_{\epsilon \downarrow 0} \epsilon^{1/2} \log \rho(\epsilon) \;\; \geq \;\; -\sqrt{\frac{\pi^2 (1+\overline{p}) |\Phi''(\overline{p})|}{2}}\,. \tag{4.3}$$

First we note that the method of the previous section is not applicable. The reason why is essentially contained in the inequality "$J_j^{(n)} \geq I_j^{(n)}$" that we used there. Probabilities of the form $\mathbf{Q}(f(s) \leq \zeta_s \leq g(s) \;\; \forall s \in [0,t])$ are dominated by probabilities of the form $\mathbf{Q}(f(s) \leq \zeta_s \leq g(s) \;\; \forall s \in [0,t] \cap \mathbb{N})$, but not vice versa. It is also unclear how to apply the discretization scheme of Bertoin and Rouault [15], for the reasons briefly outlined at the end of the introduction. With these comments in mind, we will now proceed to prove (4.3) from scratch, though our proof is still based on the one contained in [25].

The central idea, as in [25], is to construct a Galton-Watson process whose survival probability is dominated by $\rho(\epsilon)$. Galton-Watson processes being easier to study than fragmentation processes, this represents a significant reduction of our problem. The method of constructing the appropriate Galton-Watson process used in [25] will not work for us; the interested reader is referred to the first infimum in display (4.3) of that paper, which in the fragmentation case equals zero, no matter how large their M is.

4.2.1 Constructing the Galton-Watson tree

For $0 \le u < v \le t$, we say a particle $\mathcal{I}_t^x$ alive at time t remained β–low on $[u, v]$ if

$$\zeta_s^x \;\; \le \;\; \zeta_u^x + \beta(s-u) \qquad \text{for all } \; s \in [u, v].$$

We say a particle $\mathcal{I}_t^x$ alive at time t was a–good on $[u, v]$ if

$$\text{for all integers } \; 0 \le i \le \lceil v-u \rceil - 1 \quad \text{and for all } \; \theta \in [0,1], \quad \zeta_{u+i+\theta}^y \;\; \le \;\; \zeta_{u+i}^y + a\theta.$$

Now we fix constants $\epsilon, M, \alpha > 0$ and integers $n > L \ge 1$. Write $\mathcal{G}_1(\epsilon, M, \alpha, n, L)$ for the set of all particles alive at time n which remained $(\alpha\epsilon)$–low on $[0, L]$, and were M–good on $[L, n]$ (see Figure 4.1, page 71).

We iterate this definition relative to birth position to form a set of particles $\mathcal{G}_k = \mathcal{G}_k(\epsilon, M, \alpha, n, L)$ alive at time kn for each $k \in \mathbb{N}$. To be precise, for $k \ge 1$ let $\mathcal{G}_{k+1}$ denote the set particles alive at time $(k+1)n$ which descend from particles in $\mathcal{G}_k$, remained $(\alpha\epsilon)$–low on $[kn, kn+L]$, and were M–good on $[kn+L, kn]$. Clearly, the fragmentation property implies that $G_k := \#\mathcal{G}_k$ defines a Galton-Watson process $(G_k : k \in \mathbb{N})$. We will write $\mathbb{G} = \mathbb{G}(\epsilon, M, \alpha, n, L)$ for the Galton-Watson tree generated by discarding spatial information contained in the collection of particles $(\mathcal{G}_k : k \in \mathbb{N})$.

Before studying the Galton-Watson tree $\mathbb{G}$, we introduce a little bit more notation. We let X_M denote the descendants of the initial particle $(0, 1)$ which are alive at time 1 and which were M–good on $[0, 1]$. That is,

$$X_M \;\; := \;\; \#\{\mathcal{I}_1^x : \;\; \zeta_s^x \le Ms \;\; \forall s \in (0, 1]\}.$$

We then use X_M as an offspring distribution to generate another Galton-Watson process, independent of the fragmentation process, and defined on the same probability space. We call this process Θ_M, and write Z_k^M for the size of its k'th generation. The point is this. Fix $j > i \ge 0$ and a particle $\mathcal{I}_i^x$ alive at time i. The number of descendants (alive at time j) of $\mathcal{I}_i^x$ which were M–good on $[i, j]$ is equal in law to Z_{j-i}^M.

For $\epsilon, \alpha > 0$ and $L \in \mathbb{N}$, we let $\mathcal{A}(\epsilon, \alpha, L)$ stand for the set of particles alive at time L which remained $(\alpha\epsilon)$–low on $[0, L]$:

$$\mathcal{A}(\epsilon, \alpha, L) \;\; := \;\; \big\{\mathcal{I}_L^x : \;\; \zeta_s^x \le \alpha\epsilon s \;\; \forall s \in [0, L]\big\}.$$

The following lemma is a simple consequence of the construction of $\mathcal{G}_1$, but will be very useful; see Figure 4.2, page 72, for an illustration.

Lemma 4.1. *$G_1 = \#\mathcal{G}_1$ is equal in law to the sum of $\#\mathcal{A}(\epsilon, \alpha, L)$ independent copies of Z_{n-L}^M.*

Now we turn to studying the properties of $\mathbb{G}$, and its relationship to the original problem of estimating $\rho(\epsilon)$ from below. Let us write $S(\epsilon)$ for the event that the $(c_{\overline{p}} + \epsilon)$–killed fragmentation process survives. We have the following:

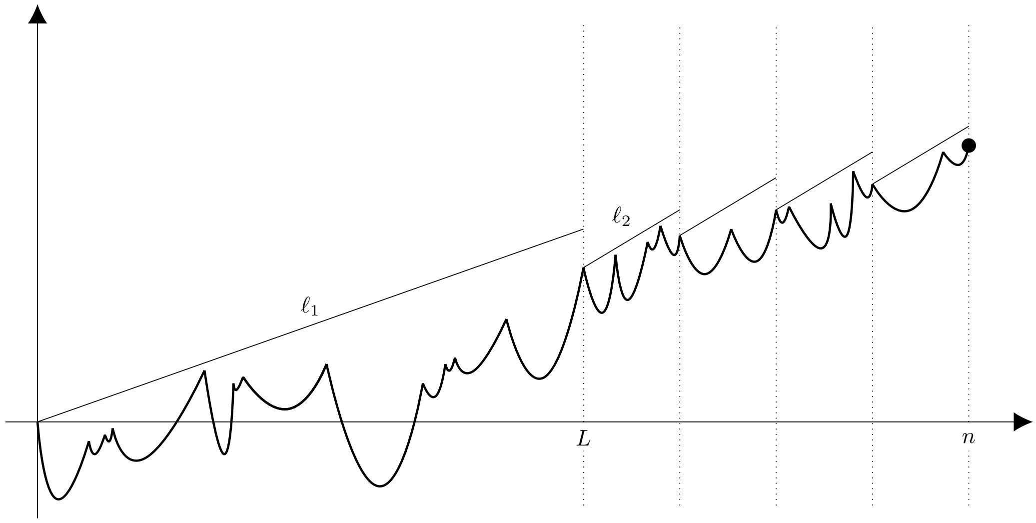

Figure 4.1. Symbolic sketch of a sample path $t \mapsto \zeta_t^x$ for $x \in (0,1)$ such that $\mathcal{I}_n^x \in \mathcal{G}_1$. The line ℓ_1 is the graph of $t \mapsto \epsilon\alpha t$. The vertical dotted lines are one unit of time apart. The line ℓ_2 has gradient M, as do the similar lines following it. The circle is located at (n, ζ_n^x).

Lemma 4.2. *Fix constants $M, \epsilon > 0$, $\alpha \in (0,1)$ and integers $n > L \geq 1$ satisfying the inequality*

$$(n - L)M \;\; \leq \;\; (1-\alpha)\epsilon L. \tag{4.4}$$

The following inclusion then holds:

$$S(\epsilon) \;\; \supset \;\; \big\{\mathbb{G}(\epsilon, M, \alpha, n, L) \;\; \textit{survives}\big\}.$$

Of course, this inclusion implies that $\rho(\epsilon)$ dominates the survival probability of $\mathbb{G}(\epsilon, M, \alpha, n, L)$, provided the parameters satisfy the conditions in the statement of the lemma.

Proof. By induction it suffices to show that, whenever $\mathcal{I}_n^x \in \mathcal{G}_1$, the inequality $\zeta_t^x \leq \epsilon t$ holds for all $t \in [0, n]$. For $t \in [0, L]$ we know that $\zeta_t^x \leq \alpha\epsilon t \leq \epsilon t$, since $\alpha \in (0,1)$. For $t \in [L, n]$, we know that

$$\zeta_t^x \;\; \leq \;\; \alpha\epsilon L + (t - L)M \;\; \leq \;\; \alpha\epsilon L + (n-L)M \;\; \leq \;\; \alpha\epsilon L + (1-\alpha)\epsilon L \;\; = \;\; \epsilon L \;\; \leq \;\; \epsilon t.$$

In the first inequality we use the definition of the set $\mathcal{G}_1$; in the second we use the fact that n exceeds t; in the third we use (4.4); and in the final inequality we use the fact that t exceeds L. $\square$

Now we define, for $b \in \mathbb{R}$ and $t \geq 0$, the probability

$$\rho(b, t) \;\; := \;\; \mathbf{P}(\exists x \in (0,1) : \;\; \zeta_s^x \leq bs \;\; \forall s \in [0, t]).$$

The following lemma is the result of a simple application of the fragmentation property at time L:

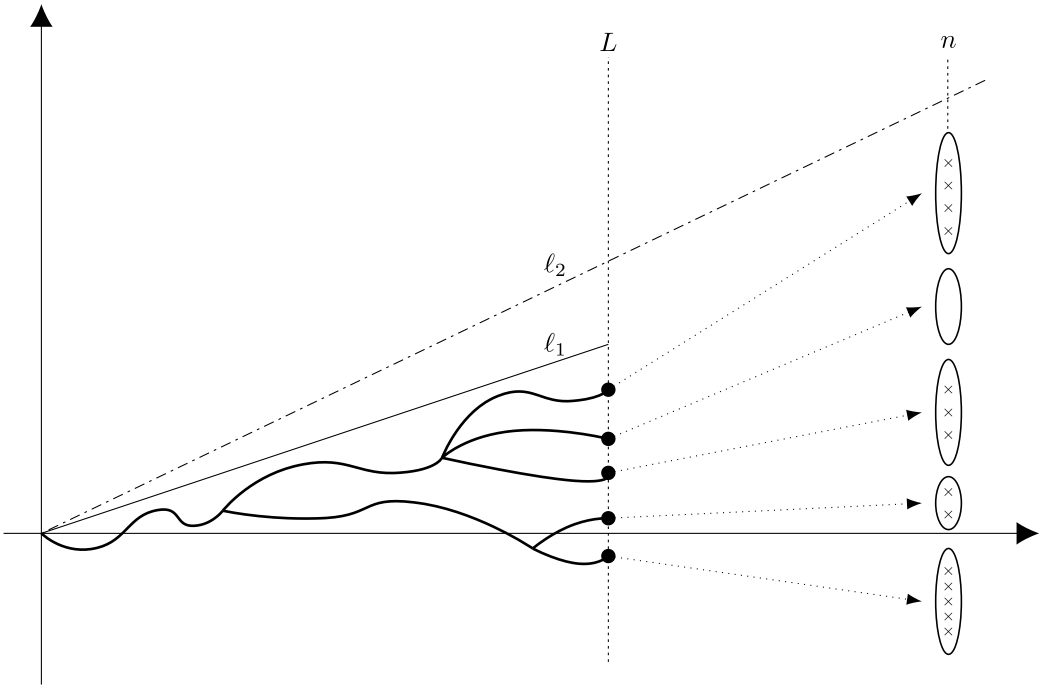

Figure 4.2. The ancestries of particles in $\mathcal{G}_1$. The circles at time L are the ζ–values of particles in $\mathcal{A}(\epsilon, \alpha, L)$. The curves illustrate their genealogical lines of ζ–descent, which always lie beneath the line $\ell_1 : s \mapsto \alpha\epsilon s$. We have omitted the necessary discontinuities at fragmentation events, where the curves branch. The arrow emanating from a given circle points to the collection of its its descendants alive at time n which were M–good on $[L, n]$—these descendants are represented by $\times$–marks. When the parameters ϵ, M, α, n and L satisfy certain conditions, the ζ–history of all $\times$–marked particles lies below the line $\ell_2 : s \mapsto \epsilon s$, as we show in Lemma 4.2.

Lemma 4.3. *For any $M, \epsilon, \alpha > 0$, and integers $n > L \geq 1$,*

$$\mathbf{P}\Big(\mathcal{G}_1(\epsilon, M, \alpha, n, L) \neq \emptyset\Big) \;\; \geq \;\; p_M \, \rho(\alpha\epsilon, L),$$

where p_M denotes the survival probability of the Galton-Watson process Θ_M.

Of course, this lemma is useless to us unless $p_M > 0$, so we now show that we can pick M large enough to ensure the supercriticality of Θ_M.

Lemma 4.4. *The Galton-Watson process Θ_M is supercritical whenever M is large enough.*

Proof. We note that for any $n \in \{1, 2, ...\}$ we have

$$\begin{aligned} \mathbf{E}\, X_M &= \sum_{[x]_1} \mathbf{1}_{(\zeta_s^x \leq Ms \;\; \forall s \in [0,1])} \\ &\geq \mathbf{Q}\left(e^{(1+\overline{p})\zeta_1}; \, \zeta_s < 0 \text{ on } (0, \tfrac{1}{n}], \;\; \zeta_s \leq \tfrac{M}{n} \text{ on } [\tfrac{1}{n}, 1]\right). \end{aligned}$$

Sending $M \to \infty$, we deduce that for any $n \in \{1, 2, ...\}$

$$\lim_{M\to\infty} \mathbf{E}\, X_M \;\; \geq \;\; \mathbf{Q}\left(e^{(1+\overline{p})\zeta_1}; \, \zeta_s < 0 \text{ on } (0, \tfrac{1}{n}]\right).$$

Taking limits in n, and using the fact that 0 is irregular for $[0,\infty)$ relative to ζ, we conclude that

$$\lim_{M\to\infty} \mathbf{E}\, X_M \;\geq\; \mathbf{Q}\left(e^{(1+\overline{p})\zeta_1}\right) \;=\; \mathbf{E} \sum_{[x]_1:(0,1)} 1 \in (1,\infty],$$

completing the proof. $\square$

In order to prove our next result about $\mathbb{G}$, we will need the following result of McDiarmid [39], quoted here as it appears in [25]:

Lemma 4.5. *Whenever M is so large that Θ_M is supercritical, there exists $q_M > 1$ such that*

$$\mathbf{P}\big(Z_n^M \leq q_M^n \big|\ Z_n^M > 0\big) \;\leq\; q_M^{-n}, \qquad \forall n \geq 1.$$

Now we use this lemma to provide some information about the size of the first generation of $\mathbb{G}$:

Lemma 4.6. *Whenever M is so large that Θ_M is supercritical, there exists a constant K_M such that for all $\epsilon, \alpha > 0$ and integers $n > L \geq 1$, we have*

$$\mathbf{P}\big(1 \leq G_1 \leq q_M^{n-L}\big) \;\leq\; \frac{K_M}{q_M^{n-L}}\,,$$

where $q_M > 1$ is the constant from Lemma 4.5.

Proof. In the notation of Lemma 4.1 and the discussion preceding it, we define $\mathcal{A} := \mathcal{A}(\epsilon,\alpha,L)$, and $Z := Z_{n-L}^M$. We write A for the collection of midpoints of the intervals in $\mathcal{A}$. Finally, we define $a := q_M^{n-L}$.

According to Lemma 4.1, G_1 is equal in law to the sum of $\#\mathcal{A}$ independent copies of Z. For the inequality $1 \leq G_1 \leq a$ to obtain, none of these copies can exceed a, and at least one must exceed zero; of course, $G_1 \geq 1$ forces $\mathcal{A} \neq \emptyset$. First conditioning on $\mathcal{F}_L$, these observations lead to the inequality

$$\begin{aligned}&\mathbf{P}\big(1 \leq G_1 \leq a\big)\\ &\qquad \leq\; \mathbf{E}\bigg(\mathbf{1}_{(\#A\geq 1)}\ \mathbf{E}_{\mathcal{F}_L}\Big(\exists x \in A \text{ such that } Z^x > 0, \quad \text{and } Z^x \leq a\ \ \forall x \in A\Big)\bigg),\end{aligned}$$

where, given $\mathcal{A}$, $(Z^x : x \in A)$ are independent copies of Z. The following calculation is elementary: we have

$$\begin{aligned}&\mathbf{E}_{\mathcal{F}_L}\Big(\exists x \in A \text{ such that } Z^x > 0, \quad \text{and } Z^x \leq a\ \ \forall x \in A\Big)\\ &\qquad\qquad\qquad \leq\; \#A \cdot \mathbf{P}(Z \leq a)^{\#A-1} \cdot \mathbf{P}(1 \leq Z \leq a).\end{aligned}$$

Consequently,

$$\mathbf{P}\big(1 \leq G_1 \leq a\big) \;\leq\; \mathbf{P}(1 \leq Z \leq a)\, \mathbf{E}\Big(\mathbf{1}_{(\#A\geq 1)}\ \#A \cdot \mathbf{P}(Z \leq a)^{\#A-1}\Big). \tag{4.5}$$

Now we need to make sure that $\mathbf{P}(Z \le a) = \mathbf{P}(Z^M_{n-L} \le a)$ can be bounded away from one, in a manner independent of n and L. But for all integers $n > L \ge 1$,

$$\begin{aligned}\mathbf{P}(Z > a) &= \mathbf{P}(Z > 0)\,\mathbf{P}(Z > a \,\big|\, Z > 0) \\ &\ge p_M \left(1 - q_M^{-(n-L)}\right) \\ &\ge p_M \left(1 - q_M^{-1}\right) \;>\; 0,\end{aligned}$$

where in the second line we use Lemma 4.5 (also recall that p_M stands for the survival probability of Θ_M, which we are assuming to be positive). Let us write $\kappa_M := 1 - p_M \left(1 - q_M^{-1}\right) < 1$. Using the previous display, we can now write

$$\mathbf{E}\Big(\mathbf{1}_{(\#A \ge 1)} \;\#A \cdot \mathbf{P}(Z^M_{n-L} \le a)^{\#A-1}\Big) \;\le\; \mathbf{E}\Big(\mathbf{1}_{(\#A\ge 1)} \;\#A \cdot \kappa_M^{\#A-1}\Big)$$

for all integers $n > L \ge 1$. Noting that the function $x \mapsto x\beta^{x-1}$ is bounded on $[1, \infty)$ whenever $\beta \in [0, 1)$, the previous display allows us to bound the second factor in (4.5) by a constant depending only on M. Applying Lemma 4.5 to the first factor finishes the proof. □

By fixing M so large that Θ_M is supercritical, we arrive at the following summary of what we have shown so far:

Proposition 4.7. *There exist constants $M, K, \gamma > 0$ and a constant $q > 1$ such that for all $\epsilon > 0$, all $\alpha \in (0, 1)$ and all integers $n > L \ge 1$ collectively satisfying the inequality $(n - L)M \le (1 - \alpha)\epsilon L$, we can construct a super-critical Galton-Watson tree $\mathbb{G} = \mathbb{G}(\epsilon, M, \alpha, n, L)$, with first generation size G, that has the following properties:*

1. $\mathbf{P}\big(\mathbb{G}$ *survives*$\big) \;\le\; \rho(\epsilon)$.
2. $\mathbf{P}(G \ge 1) \;\ge\; \gamma\, \rho(\epsilon\alpha, L)$.
3. $\mathbf{P}\big(1 \le G \le q^{n-L}\big) \;\le\; \dfrac{K}{q^{n-L}}$.

4.2.2 Survival below the ray $t \mapsto bt$ over finite time horizons

In light of Proposition 4.7, it shouldn't be too surprising that the next step in the proof of (4.3) is to provide quantitative information about the behaviour of $\rho(\cdot, \cdot)$. Proving the following proposition is the most technically challenging part of the proof of Theorem 1.15.

Proposition 4.8. *For any $\theta > 0$ we have*

$$\liminf_{t\to\infty} \frac{1}{t^{1/3}} \log \rho(\theta t^{-2/3}, t) \;\ge\; -\sqrt{\frac{\pi^2(1 + \overline{p})|\Phi''(\overline{p})|}{2\theta}}\,.$$

The proof of this proposition is long, so we break it up into several steps.

Step 1: Preliminaries. First we move the killing barrier $s \mapsto \theta t^{-2/3} s$ (for $s \in [0,t]$) implicit in the definition of $\rho(\theta t^{-2/3}, t)$ up a little, by the amount $\epsilon t^{1/3}$. To be precise, Corollary 2.11 yields the following statement: there exists $\eta > 0$ such that for all $\epsilon, \theta > 0$ and $t \geq 0$, we have

$$\rho(\theta t^{-2/3}, t) \;\; \geq \;\; p^{\alpha} \cdot \mathbf{P}\big(\exists x \in (0,1): \;\; \zeta_s^x \leq \theta t^{-2/3} s + \epsilon t^{1/3} \;\; \forall s \in [0,t]\big), \tag{4.6}$$

where $\alpha = \alpha(t, \epsilon, \eta) := \lceil \frac{\epsilon t^{1/3}}{\eta} \rceil$. Since

$$\lim_{t \to \infty} \frac{1}{t^{1/3}} \log p^{\alpha} \;\; = \;\; o_\epsilon(1) \qquad \text{as } \epsilon \downarrow 0,$$

we can focus on the second factor in (4.6) from now on. We define, for $b \in \mathbb{R}$, $\epsilon > 0$ and $t \geq 0$,

$$\rho_\epsilon(b, t) \;\; := \;\; \mathbf{P}\big(\exists x \in (0,1): \;\; \zeta_s^x \leq bs + \epsilon t^{1/3} \;\; \forall s \in [0,t]\big),$$

so that the second factor can be written as $\rho_\epsilon(\theta t^{-2/3}, t)$. The remainder of the proof will consist in estimating this quantity from below using a second moment argument.

We define, for any $\lambda > 0$, $t \geq 0$ and $s \in [0,t]$ the intervals

$$I_s(t) \;\; := \;\; \left[\theta \frac{s}{t^{2/3}} - \lambda t^{1/3}, \;\; \theta \frac{s}{t^{2/3}} + \epsilon t^{1/3}\right].$$

We note that $I_s(t)$ has nothing to do with the length of the randomly tagged fragment (which we called $I(t)$)—this length does not feature in what follows, so there is no opportunity for confusion.

Next we count those fragments that have remained inside these intervals until time $t \geq 0$:

$$Z_t \;\; := \;\; \sum_{[x]_t} \mathbf{1}_{(\zeta_s^x \in I_s(t) \;\; \forall s \in [0,t])}.$$

Then $\rho_\epsilon(\theta t^{-2/3}, t) \geq \mathbf{P}(Z_t > 0)$, and the Paley-Zygmund inequality tells us that

$$\mathbf{P}(Z_t > 0) \;\; \geq \;\; \frac{(\mathbf{E} Z_t)^2}{\mathbf{E}(Z_t^2)}.$$

We now rewrite the first and second moments of Z_t in terms of the spine $(\zeta, \mathbf{Q})$. Using the Many-to-One Lemma, we have

$$E_t \;\; := \;\; \mathbf{E} Z_t \;\; = \;\; \mathbf{Q}\big(e^{(1+\overline{p})\zeta_t} \mathbf{1}_{(\zeta_s \in I_s(t) \;\; \forall s \in [0,t])}\big) \;\; = \;\; \mathbf{Q}\big(e^{(1+\overline{p})\zeta_t} \mathbf{1}_{A_{[0,t]}}\big)$$

where for $0 \leq v \leq w \leq t$ we write $A_{[v,w]}$ for the event that $\zeta_s \in I_s(t)$ for all $v \leq s \leq w$. By Corollary 2.7 we have

$$\mathbf{E}(Z_t^2) \;\; = \;\; E_t + \Lambda_t,$$

where

$$\Lambda_t \ := \ \int_0^t dr \ \cdot \ \mathbf{Q}\left(e^{(1+\overline{p})\zeta_r}\mathbf{1}_{A_{[0,r]}}\int_{\mathcal{U}} G(\zeta_r, u)\nu(du)\right),$$

with

$$G(\alpha, u) \ := \ \sum_{i\neq j} F(\alpha - \log|u_i|)F(\alpha - \log|u_j|),$$

and

$$F(\alpha) \ := \ \mathbf{Q}\, e^{\zeta_{t-r}(\overline{p}+1)}\mathbf{1}_{\left(\alpha+\zeta_s\in I_{s+r}(t) \ \ \forall s\in[0,t-r]\right)}.$$

For future reference, we rewrite our application of the Paley-Zygmund inequality in these terms:

$$\rho_\epsilon(\theta t^{-2/3}, t) \ \geq \ \frac{E_t^2}{E_t + \Lambda_t}. \tag{4.7}$$

Our task now is to find upper and lower bounds on E_t and an upper bound on Λ_t. Unsurprisingly, the first two bounds are very easy; the third is where the difficulty lies.

Step 2: Bounding E_t from above and below. For the lower bound on E_t, we start with the following calculation:

$$\begin{aligned} E_t \ &\geq \ \mathbf{Q}\Big(e^{(1+\overline{p})\zeta_t}; \quad \zeta_s \in I_s(t) \ \forall s \in [0,t], \ \zeta_t \geq \theta t^{1/3}\Big) \\ &\geq \ e^{(1+\overline{p})\theta t^{1/3}}\mathbf{Q}\Big(\zeta_s \in I_s(t) \ \forall s\in[0,t], \ \zeta_t \geq \theta t^{1/3}\Big). \end{aligned} \tag{4.8}$$

In the first inequality we have added the extra condition $\zeta_t \geq \theta t^{1/3}$, and in the second we have used this condition to bound the exponential factor. Now we apply the second part of our version of Mogulskii's Theorem, Lemma 2.14, by noting that

$$\begin{aligned} &\mathbf{Q}\Big(\zeta_s \in I_s(t) \ \forall s\in[0,t], \ \zeta_t \geq \theta t^{1/3}\Big) \\ &\qquad = \ \mathbf{Q}\left(\theta\frac{s}{t} - \lambda \ \leq \ \frac{\zeta_t}{t^{1/3}} \ \leq \ \theta\frac{s}{t} + \epsilon \ \forall s\in[0,t], \ \frac{\zeta_t}{t^{1/3}} \geq \theta\right) \end{aligned}$$

so that

$$\lim_{t\to\infty} \frac{1}{t^{1/3}} \log \mathbf{Q}\Big(\zeta_s \in I_s(t) \ \forall s\in[0,t], \ \zeta_t \geq \theta t^{1/3}\Big) \ = \ -\frac{\pi^2\sigma^2}{2(\lambda+\epsilon)^2} \ \geq \ -\frac{\pi^2\sigma^2}{2\lambda^2}.$$

Returning to (4.8), we deduce that

$$\liminf_{t\to\infty} \frac{1}{t^{1/3}} \log E_t \ \geq \ (1+\overline{p})\theta - \frac{\pi^2\sigma^2}{2\lambda^2}.$$

Now we bound E_t from above in a similar fashion. We have

$$\begin{aligned} E_t \ &\leq \ e^{(1+\overline{p})(\theta+\epsilon)t^{1/3}}\mathbf{Q}\Big(\zeta_s \in I_s(t) \ \forall s\in[0,t]\Big) \\ &= \ e^{(1+\overline{p})(\theta+\epsilon)t^{1/3}}\mathbf{Q}\left(\theta\frac{s}{t} - \lambda \ \leq \ \frac{\zeta_t}{t^{1/3}} \ \leq \ \theta\frac{s}{t} + \epsilon \ \forall s\in[0,t]\right). \end{aligned}$$

Mogulskii's Theorem then allows us to deduce that

$$\limsup_{t\to\infty} \frac{1}{t^{1/3}} \log E_t \;\leq\; (1+\overline{p})(\theta+\epsilon) - \frac{\pi^2\sigma^2}{2(\lambda+\epsilon)^2}.$$

Step 3: A preliminary upper bound on Λ_t. We claim that for all $t \geq 0$, we have

$$\Lambda_t \;\leq\; \gamma\, t\, e^{(1+\overline{p})(\theta+\epsilon)t^{1/3}} \sup_{r\in[0,t]} h_{r,t} \cdot \sup_{r\in[0,t]} g_{r,t} \tag{4.9}$$

where γ is a positive constant,

$$h_{r,t} \;:=\; e^{(1+\overline{p})\left[\theta(t-r)t^{-2/3}+(\lambda+\epsilon)t^{1/3}\right]} \sup_{0\leq x\leq(\lambda+\epsilon)t^{1/3}} \mathbf{Q}\big(\zeta_s \in x + J_s(t)\ \ \forall s\in[0,t-r]\big)$$

and

$$g_{r,t} \;:=\; \mathbf{Q}\left(\zeta_s \in I_s(t)\ \ \forall s\in[0,r]\right) \sup_{0\leq x\leq(\lambda+\epsilon)t^{1/3}} \mathbf{Q}\big(\zeta_s \in x + J_s(t)\ \ \forall s\in[0,t-r]\big),$$

with

$$J_s(t) \;:=\; \left[\theta\frac{s}{t^{2/3}} - (\lambda+2\epsilon)t^{1/3},\ \theta\frac{s}{t^{2/3}} + \epsilon t^{1/3}\right].$$

Before proving this claim we make two remarks. First, the second factors in the definitions of $h_{r,t}$ and $g_{r,t}$ are the same, which will allow us to recycle some work when estimating them from above. Second, the interval $J_s(t)$ arises from widening the interval $I_s(t)$ by $\epsilon t^{1/3}$ at both endpoints. The reason for doing this will be explained during the proof.

Proof. We fix $r \in [0,t]$ and start by estimating the integrand in the definition of Λ_t at this value. First we estimate away the sum in u_j in the definition of $G(\alpha,u)$ (explanations follow the calculation):

$$\begin{aligned}
&\sum_{i\neq j} F(\alpha - \log|u_j|) \\
=\;& \sum_{i\neq j} |u_j|^{\overline{p}+1}\mathbf{Q}\left(e^{(1+\overline{p})(\zeta_{t-r}-\log|u_j|)};\quad \alpha-\log|u_j|+\zeta_s\in I_{s+r}(t)\ \ \forall s\in[0,t-r]\right) \\
\leq\;& e^{(1+\overline{p})(\theta+\epsilon)t^{1/3}-\alpha(1+\overline{p})} \sup_{\beta\in I_r(t)} \mathbf{Q}\left(\beta+\zeta_s\in I_{s+r}(t)\ \ \forall s\in[0,t-r]\right)\sum_{i\neq j}|u_j|^{\overline{p}+1} \\
\leq\;& (1-|u_i|)^{1+\overline{p}} e^{(1+\overline{p})(\theta+\epsilon)t^{1/3}-\alpha(1+\overline{p})} \sup_{\beta\in I_r(t)} \mathbf{Q}\left(\beta+\zeta_s\in I_{s+r}(t)\ \ \forall s\in[0,t-r]\right).
\end{aligned}$$

In the first line we introduce and remove the multiplicative term $|u_j|^{\overline{p}+1}$. In the second we bound the exponential factor using information following the semicolon. More precisely, on the event

$$\big\{\alpha - \log|u_j| + \zeta_s \in I_{s+r}(t)\ \ \forall s\in[0,t-r]\big\} \tag{4.10}$$

we know that (by setting $s = t-r$)

$$\zeta_{t-r} - \log|u_j| \;\in\; I_t(t)-\alpha \;=\; \left[(\theta-\lambda)t^{1/3},\,(\theta+\epsilon)t^{1/3}\right]-\alpha,$$

yielding the upper bound that we used in the calculation. We also know that unless $\alpha - \log|u_j| \in I_r(t)$, the event in (4.10) is null (this comes from setting $s = 0$ in the definition of the event). This explains the presence of the uniform bound $\sup_{\beta \in I_r(t)} \mathbf{Q}(\cdots)$ appearing above.

In the third line we use the elementary inequality $\sum a_i^p \leq (\sum a_i)^p$ for non-negative sequences (a_i) and $p > 1$, and the fact that, ν–almost everywhere, we have $\sum_{i \neq j} |u_j| = 1 - |u_i|$.

So far we have shown that, for $r \geq 0$ and $u \in \mathcal{U}$, $G(\zeta_r, u)$ is bounded above, almost surely, by

$$e^{(1+\overline{p})(\theta+\epsilon)t^{1/3} - (1+\overline{p})\zeta_r} \sup_{\beta \in I_r(t)} \mathbf{Q}\left(\beta + \zeta_s \in I_{s+r}(t) \;\; \forall s \in [0, t-r]\right) \cdot \sum_i (1 - |u_i|)^{1+\overline{p}} F(\zeta_r - \log|u_i|). \quad (4.11)$$

But we are working on the event that $\zeta_s \in I_s(t)$ for all $s \in [0, r]$ (note the indicator $\mathbf{1}_{A_{[0,r]}}$ in the definition of Λ_t). On this event, in particular, $\zeta_r \geq \theta \frac{r}{t^{2/3}} - \lambda t^{1/3}$. This means that we can bound the exponent in the previous display from above by

$$(1+\overline{p})(\theta+\epsilon)t^{1/3} - \left(\theta \frac{r}{t^{2/3}} - \lambda t^{1/3}\right)(1+\overline{p}) \;\; = \;\; (1+\overline{p})\left[\theta(t-r)t^{-2/3} + (\lambda+\epsilon)t^{1/3}\right]. \quad (4.12)$$

By shifting the origin to the coordinate (r, right endpoint of $I_r(t)$) and using the fact that the intervals $I_r(t)$ have constant width $L_t := |I_r(t)| = (\lambda + \epsilon)t^{1/3}$ (for fixed $t \geq 0$ and varying $r \in [0, t]$), we note that

$$\begin{aligned} &\sup_{\beta \in I_r(t)} \mathbf{Q}\left(\beta + \zeta_s \in I_{s+r}(t) \;\; \forall s \in [0, t-r]\right) \\ &= \sup_{0 \leq x \leq L_t} \mathbf{Q}\left(\zeta_s \in x + \left[\theta \frac{s}{t^{2/3}} - (\lambda+\epsilon)t^{1/3}, \;\; \theta \frac{s}{t^{2/3}}\right] \;\; \forall s \in [0, t-r]\right). \end{aligned} \quad (4.13)$$

When $x = 0$, the right endpoint of the interval corresponding to $s = 0$ in the right-hand side of the previous display equals 0. Similarly, when $x = L_t$, the left endpoint of the interval corresponding to $s = 0$ equals 0. In order to be able to apply Mogulskii's Theorem in one fell swoop, therefore, we widen all the intervals by the amount $\epsilon t^{1/3}$. That is, we introduce the intervals

$$J_s(t) \;\; := \;\; \left[\theta \frac{s}{t^{2/3}} - (\lambda + 2\epsilon)t^{1/3}, \;\; \theta \frac{s}{t^{2/3}} + \epsilon t^{1/3}\right],$$

for $t \geq 0$ and $s \in [0, t]$, and make the trivial observation that, by (4.13), we have

$$\begin{aligned} &\sup_{\beta \in I_r(t)} \mathbf{Q}\left(\beta + \zeta_s \in I_{s+r}(t) \;\; \forall s \in [0, t-r]\right) \\ &\qquad \leq \sup_{0 \leq x \leq L_t} \mathbf{Q}\big(\zeta_s \in x + J_s(t) \;\; \forall s \in [0, t-r]\big). \end{aligned}$$

Putting this estimate, and the estimate (4.12), into (4.11), we conclude that, on the event $A_{[0,r]}$,

$$G(\zeta_r, u) \;\; \leq \;\; h_{r,t} \;\; \sum_i (1 - |u_i|)^{1+\overline{p}} F(\zeta_r - \log|u_i|) \quad (4.14)$$

where

$$h_{r,t} \;:=\; e^{(1+\overline{p})\left[\theta(t-r)t^{-2/3}+(\lambda+\epsilon)t^{1/3}\right]} \sup_{0\le x\le L_t} \mathbf{Q}\big(\zeta_s \in x + J_s(t) \;\; \forall s\in[0,t-r]\big).$$

The t–dependence on the right-hand side of (4.14) looks a bit odd at first glance, but this is because the event $A_{[0,r]}$ where the estimate is valid depends on t. Now we use our bound on G to make the following estimate (with the obvious interpretation in case $h_{r,t}=0$):

$$h_{r,t}^{-1}\int_{\mathcal{U}} G(\zeta_r,u)\nu(du) \;\le\; \int_{\mathcal{U}} \nu(du)\cdot \sum_i (1-|u_i|)^{1+\overline{p}} F(\zeta_r - \log|u_i|).$$

By the definition of F, the right-hand side of this inequality is equal to

$$\int_{\mathcal{U}} \nu(du)\cdot \sum_i (1-|u_i|)^{1+\overline{p}} \left(\mathbf{Q}\, e^{\zeta_{t-r}(\overline{p}+1)} \mathbf{1}_{\left(\alpha-\log|u_i|+\zeta_s\in I_{s+r}(t)\;\;\forall s\in[0,t-r]\right)}\right)\Bigg|_{\alpha=\zeta_r}.$$

Next we introduce and remove the factor $|u_i|^{1+\overline{p}}$: the expression in the previous display is equal to

$$\int_{\mathcal{U}} \nu(du)\cdot \sum_i (1-|u_i|)^{1+\overline{p}} |u_i|^{1+\overline{p}} \;\cdot$$
$$\cdot\;\left(\mathbf{Q}\, e^{(\zeta_{t-r}-\log|u_i|)(\overline{p}+1)} \mathbf{1}_{\left(\alpha-\log|u_i|+\zeta_s\in I_{s+r}(t)\;\;\forall s\in[0,t-r]\right)}\right)\Bigg|_{\alpha=\zeta_r}.$$

Now we apply Corollary 2.9 to deduce that this expression is equal to

$$\int_0^\infty \Pi(dx)\cdot (1-e^{-x})^{1+\overline{p}} \left(\mathbf{Q}\, e^{(\zeta_{t-r}+x)(\overline{p}+1)} \mathbf{1}_{\left(\alpha+x+\zeta_s\in I_{s+r}(t)\;\;\forall s\in[0,t-r]\right)}\right)\Bigg|_{\alpha=\zeta_r},$$

which, by the elementary inequality $1-\exp(-x)\le 1\wedge x$ for $x\ge 0$, and the fact that $\overline{p}>0$, is bounded above by

$$\int_0^\infty \Pi(dx)\cdot (1\wedge x) \left(\mathbf{Q}\, e^{(\zeta_{t-r}+x)(\overline{p}+1)} \mathbf{1}_{\left(\alpha+x+\zeta_s\in I_{s+r}(t)\;\;\forall s\in[0,t-r]\right)}\right)\Bigg|_{\alpha=\zeta_r}.$$

Substituting this bound into the definition of Λ_t, we deduce that

$$\Lambda_t \;\le\; \int_0^t dr \cdot h_{r,t} \int_0^\infty \Pi(dx)\cdot(1\wedge x)\, f(t,r,x) \tag{4.15}$$

where, for $t\ge 0$, $r\in[0,t]$ and $x\ge 0$,

$$f(t,r,x) \;:=\; \mathbf{Q}\left(e^{(1+\overline{p})\zeta_r}\mathbf{1}_{A_{[0,r]}}\left[\mathbf{Q}e^{(1+\overline{p})(\zeta_{t-r}+x)}\mathbf{1}_{\left(\alpha+x+\zeta_s\in I_{s+r}(t)\;\;\forall s\in[0,t-r]\right)}\right]\Bigg|_{\alpha=\zeta_r}\right).$$

Clearly, $f(t,r,x)$ is ripe for simplification via the Markov property at time r (explanations follow the calculation):

$$
\begin{aligned}
f(t,&r,x) \\
&= \mathbf{Q}\left(e^{(1+\overline{p})\zeta_r}\mathbf{1}_{A_{[0,r]}}\left[\mathbf{Q}_{\mathcal{G}_r}e^{(1+\overline{p})(\zeta_t-\zeta_r+x)}\mathbf{1}_{\left(\zeta_r+x+\zeta_{s+r}-\zeta_r\in I_{s+r}(t)\ \ \forall s\in[0,t-r]\right)}\right]\right) \\
&= \mathbf{Q}\left(e^{(1+\overline{p})(\zeta_t+x)}\mathbf{1}_{A_{[0,r]}}\mathbf{1}_{\left(x+\zeta_{s+r}\in I_{s+r}(t)\ \ \forall s\in[0,t-r]\right)}\right) \\
&= \mathbf{Q}\left(e^{(1+\overline{p})(\zeta_t+x)}\mathbf{1}_{A_{[0,r]}}\mathbf{1}_{\left(x+\zeta_{s}\in I_{s}(t)\ \ \forall s\in[r,t]\right)}\right) \\
&\le e^{(1+\overline{p})(\theta+\epsilon)t^{1/3}}\mathbf{Q}\left(\zeta_s\in I_s(t)\ \ \forall s\in[0,r],\quad x+\zeta_s\in I_s(t)\ \ \forall s\in[r,t]\right) \\
&\le e^{(1+\overline{p})(\theta+\epsilon)t^{1/3}}\mathbf{Q}\left(\zeta_s\in I_s(t)\ \ \forall s\in[0,r]\right)\sup_{0\le x\le L_t}\mathbf{Q}\big(\zeta_s\in x+J_s(t)\ \ \forall s\in[0,t-r]\big) \\
&=: e^{(1+\overline{p})(\theta+\epsilon)t^{1/3}}g_{r,t}.
\end{aligned}
$$

The first line is an application of the Markov property; in the second step, we bring the $\mathcal{G}_r$–measurable expression inside the conditional expectation, and simplify the resulting integrand; in the third equality we simplify the qualification in time; in the first inequality we bound the exponential factor using the condition $x+\zeta_t\in I_t(t)$. The second inequality is obtained by using the Markov property at time r, and taking a supremum in x.

Returning to (4.15), we deduce, after taking suprema in r, that

$$
\Lambda_t \;\le\; \gamma\, t\, e^{(1+\overline{p})(\theta+\epsilon)t^{1/3}}\sup_{r\in[0,t]}h_{r,t}\cdot\sup_{r\in[0,t]}g_{r,t}
$$

where $\gamma := \int_0^\infty(1\wedge x)\Pi(dx)<\infty$, completing the proof. $\square$

To obtain an upper bound on Λ_t, we now estimate $\sup_{r\in[0,t]}h_{r,t}$ and $\sup_{r\in[0,t]}g_{r,t}$.

Step 4: Bounding $\sup_{r\in[0,t]}h_{r,t}$ *from above.* We remind the reader that

$$
h_{r,t} \;:=\; e^{(1+\overline{p})\left[\theta(t-r)t^{-2/3}+(\lambda+\epsilon)t^{1/3}\right]}\sup_{0\le x\le L_t}\mathbf{Q}\big(\zeta_s\in x+J_s(t)\ \ \forall s\in[0,t-r]\big)
$$

where $L_t := |I_r(t)| = (\lambda+\epsilon)t^{1/3}$.

Let $N,k\ge 1$ be integers, and set $t:=Nk$. We split the time interval $[0,t]$ in the supremum $\sup_{r\in[0,t]}h_{r,t}$ into intervals of the form $[(l-1)k,lk]$ for $1\le l\le N$. Then we replace all occurrences of $r\in[(l-1)k,lk]$ in the definition of $h_{r,t}$ with either $(l-1)k$ or lk as appropriate, to deduce that

$$\begin{aligned}
\sup_{0\le r\le t} h_{r,t} &= \max_{1\le l\le N} \sup_{(l-1)k\le r\le lk} e^{(1+\overline{p})\left[\theta(t-r)t^{-2/3}+(\lambda+\epsilon)t^{1/3}\right]} \\
&\qquad\cdot \sup_{0\le x\le L_t} \mathbf{Q}\left(\zeta_s\in x+J_s(t)\ \ \forall s\in[0,t-r]\right) \\
&\le \max_{1\le l\le N} e^{(1+\overline{p})\left[\theta(N-l+1)N^{-2/3}k^{1/3}+(\lambda+\epsilon)N^{1/3}k^{1/3}\right]} \\
&\qquad\cdot \sup_{0\le x\le L_{Nk}} \mathbf{Q}\left(\zeta_s\in x+J_s(Nk)\ \ \forall s\in[0,(N-l)k]\right) \\
&\le \max_{1\le l\le N} e^{(1+\overline{p})\left[\theta(N-l+1)N^{-2/3}k^{1/3}+(\lambda+\epsilon)N^{1/3}k^{1/3}\right]} \\
&\qquad\cdot \sup_{0\le x\le L_{Nk}} \mathbf{Q}\left(\zeta_i\in x+J_i(Nk)\ \ \forall 1\le i\le (N-l)k\right). \qquad (4.16)
\end{aligned}$$

In the first inequality, for instance, we note that $r\in[(l-1)k,lk] \Rightarrow t-r\ge (N-l)k$, so the qualification $\forall s\in[0,t-r]$ is more stringent than the qualification $\forall s\in[0,(N-l)k]$. In the second inequality we have done nothing but throw away qualification over non-integers.

We have now discretized time; the next step is to discretize space, partitioning $[0,L_{Nk}]$ into the intervals $[\frac{(m-1)L_{Nk}}{N},\frac{mL_{Nk}}{N}]$ for $1\le m\le N$. It will also be convenient to write the interval $J_i(Nk)$ as $[a_{i,N,k},b_{i,N,k}]$. This leads to the following inequality:

$$\begin{aligned}
&\sup_{0\le x\le L_{Nk}} \mathbf{Q}\left(\zeta_i\in x+J_i(Nk)\ \ \forall 1\le i\le (N-l)k\right) \ \le \\
&\max_{1\le m\le N} \mathbf{Q}\left(\zeta_i\in\left[a_{i,N,k}+\frac{(m-1)L_{Nk}}{N},\ b_{i,N,k}+\frac{mL_{Nk}}{N}\right],\ \forall 1\le i\le (N-l)k\right). \qquad (4.17)
\end{aligned}$$

Now we perform some rather ugly algebra. Let's write $c_{N,l} := \left(\frac{N}{N-l}\right)^{1/3}$. Then the probability indexed by m on the right-hand side of the previous display can be rewritten as follows:

$$\begin{aligned}
&\mathbf{Q}\Bigg(\theta\, c_{N,l}^{-2}\frac{i}{(N-l)k}-(\lambda+2\epsilon)c_{N,l}+\frac{m-1}{N}(\lambda+\epsilon)c_{N,l} \\
&\le \frac{\zeta_i}{(N-l)^{1/3}k^{1/3}} \ \le\ \theta\, c_{N,l}^{-2}\frac{i}{(N-l)k}+\epsilon c_{N,l}+\frac{m}{N}(\lambda+\epsilon)c_{N,l}\ \ \forall 1\le i\le (N-l)k\Bigg) \\
&\qquad =: Q(N,l,m,k).
\end{aligned}$$

The good news is we are now ready to apply Mogulskii's Theorem to these probabilities. The appropriate functions are

$$\begin{aligned}
g_1^{N,l,m}(t) &:= \theta\, c_{N,l}^{-2}\, t-(\lambda+2\epsilon)c_{N,l}+\frac{m-1}{N}(\lambda+\epsilon)c_{N,l} \quad \text{and} \\
g_2^{N,l,m}(t) &:= \theta\, c_{N,l}^{-2}\, t+\epsilon c_{N,l}+\frac{m}{N}(\lambda+\epsilon)c_{N,l}.
\end{aligned}$$

First note that

$$g_2^{N,l,m}(t)-g_1^{N,l,m}(t) \ = \ \left(\lambda+3\epsilon+\frac{1}{N}(\lambda+\epsilon)\right)c_{N,l} \ \le\ (\lambda+4\epsilon)c_{N,l}$$

whenever N is bigger than some $N_1(\epsilon, \lambda)$. Note that the m–dependence has been eradicated; all further estimates are therefore true uniformly in $1 \leq m \leq N$. We now apply Mogulskii's Theorem to arrive at the following conclusion:

For all $\epsilon, \lambda > 0$ there exists $N_1(\epsilon, \lambda) \in \mathbb{N}$ such that, whenever $N \geq N_1$, we have

$$\limsup_{k\to\infty} \frac{1}{(N-l)^{1/3}k^{1/3}} \log Q(N,l,m,k) \ \leq \ -\frac{\pi^2\sigma^2}{2(\lambda+4\epsilon)^2}\left(\frac{N-l}{N}\right)^{2/3},$$

for all $1 \leq m \leq N$, which we rewrite as

$$\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log Q(N,l,m,k) \ \leq \ -\frac{\pi^2\sigma^2}{2(\lambda+4\epsilon)^2}\frac{N-l}{N}.$$

Putting this estimate into (4.16), we deduce that, whenever $N \geq N_1$,

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log \sup_{0\leq r\leq Nk} h_{r,Nk} \\
&\quad \leq \ \max_{1\leq l\leq N}\left\{(1+\overline{p})\theta\frac{N-l+1}{N} + (1+\overline{p})(\lambda+\epsilon) - \frac{\pi^2\sigma^2}{2(\lambda+4\epsilon)^2}\frac{N-l}{N}\right\} \\
&\quad = \ (1+\overline{p})\theta\frac{N+1}{N} + (\lambda+\epsilon)(1+\overline{p}) - \frac{\pi^2\sigma^2}{2(\lambda+4\epsilon)^2} \\
&\qquad\qquad + \max_{1\leq l\leq N}\frac{l}{N}\left(\frac{\pi^2\sigma^2}{2(\lambda+4\epsilon)^2} - (1+\overline{p})\theta\right).
\end{aligned}$$

Henceforth we assume that

$$\lambda \ \geq \ \frac{\pi\sigma}{[2(1+\overline{p})\theta]^{1/2}} \ =: \ \lambda_0(\theta).$$

This inequality implies that the final maximum in the previous display is non-positive, which results in the following summary of what we have shown in this step:

For all $\epsilon, \theta > 0$, for all $N \in \mathbb{N}$, and for all $\lambda \geq \lambda_0(\theta)$, there exists $N_1 = N_1(\epsilon, \lambda) \in \mathbb{N}$ such that whenever $N \geq N_1$, the following inequality obtains:

$$\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log \sup_{0\leq r\leq Nk} h_{r,Nk} \ \leq \ (1+\overline{p})\theta\frac{N+1}{N} + (\lambda+\epsilon)(1+\overline{p}) - \frac{\pi^2\sigma^2}{2(\lambda+4\epsilon)^2}.$$

Step 5: Bounding $\sup_{r\in[0,t]} g_{r,t}$ *from above.* We remind the reader that

$$\begin{aligned}
g_{r,t} \ &:= \ \mathbf{Q}\left(\zeta_s \in I_s(t) \ \ \forall s\in[0,r]\right) \sup_{0\leq x\leq L_t} \mathbf{Q}\big(\zeta_s \in x + J_s(t) \ \ \forall s \in [0,t-r]\big) \\
&=: \ a_{r,t}\cdot b_{r,t}.
\end{aligned}$$

where $L_t := |I_r(t)| = (\lambda+\epsilon)t^{1/3}$.

Again we fix integers $N, k \geq 1$ and set $t := Nk$. For $r \in [(l-1)k, lk]$ (with $1 \leq l \leq N$), we have

$$\begin{aligned} b_{r,Nk} &\leq \sup_{0 \leq x \leq L_{Nk}} \mathbf{Q}\big(\zeta_s \in x + J_s(Nk) \ \forall s \in [0, (N-l)k]\big) \\ &\leq \sup_{0 \leq x \leq L_{Nk}} \mathbf{Q}\big(\zeta_i \in x + J_i(Nk) \ \forall 1 \leq i \leq (N-l)k\big). \end{aligned}$$

The final expression above is the same as the left-hand side of (4.17). Recycling the work from Step 4, we conclude that for $r \in [(l-1)k, lk]$, we have

$$\limsup_{k \to \infty} \frac{1}{N^{1/3} k^{1/3}} \log b_{r,Nk} \ \leq \ -\frac{\pi^2 \sigma^2}{2(\lambda + 4\epsilon)^2} \frac{N-l}{N}$$

whenever $N \geq N_1$. Now we turn to $a_{r,t}$. For $r \in [(l-1)k, lk]$ we have

$$\begin{aligned} a_{r,Nk} &\leq \mathbf{Q}\left(\zeta_i \in I_i(Nk) \ \forall 1 \leq i \leq (l-1)k\right) \\ &= \mathbf{Q}\Bigg(\theta k_{N,l}^{-2} \frac{i}{(l-1)k} - \lambda k_{N,l} \ \leq \ \frac{\zeta_i}{(l-1)^{1/3} k^{1/3}} \\ &\qquad\qquad \leq \ \theta k_{N,l}^{-2} \frac{i}{(l-1)k} + \epsilon k_{N,l} \ \forall 1 \leq i \leq (l-1)k\Bigg), \end{aligned}$$

where $k_{N,l} := \left(\frac{N}{l-1}\right)^{1/3}$.

We now use Mogulskii's Theorem to deduce that, for $r \in [(l-1)k, lk]$,

$$\limsup_{k \to \infty} \frac{1}{N^{1/3} k^{1/3}} \log a_{r,Nk} \ \leq \ -\frac{\pi^2 \sigma^2}{2(\lambda + \epsilon)^2} \frac{l-1}{N} \ \leq \ -\frac{\pi^2 \sigma^2}{2(\lambda + 4\epsilon)^2} \frac{l-1}{N}.$$

Now note that $\sup_{(l-1)k \leq r \leq lk} g_{r,Nk} \leq a_{(l-1)k,Nk} \cdot b_{lk,Nk}$ since $a_{r,t}$ decreases in r and $b_{r,t}$ increases in r. Putting together our estimates for $a_{r,t}$ and $b_{r,t}$, we deduce that for all $N \geq N_1$,

$$\begin{aligned} \limsup_{k \to \infty} \frac{1}{N^{1/3} k^{1/3}} \log \sup_{(l-1)k \leq r \leq lk} g_{r,Nk} &\leq -\frac{\pi^2 \sigma^2}{2(\lambda + 4\epsilon)^2} \frac{l-1}{N} - \frac{\pi^2 \sigma^2}{2(\lambda + 4\epsilon)^2} \frac{N-l}{N} \\ &= -\frac{\pi^2 \sigma^2}{2(\lambda + 4\epsilon)^2} \frac{N-1}{N}. \end{aligned}$$

The rightmost expression doesn't depend on l, so we arrive at the following summary of what we have shown in this step:

For all $\epsilon, \theta > 0$, for all $k, N \in \mathbb{N}$, and for all $\lambda > 0$, there exists $N_1 = N_1(\epsilon, \lambda) \in \mathbb{N}$ such that whenever $N \geq N_1$, the following inequality obtains:

$$\limsup_{k \to \infty} \frac{1}{N^{1/3} k^{1/3}} \log \sup_{0 \leq r \leq Nk} g_{r,Nk} \ \leq \ -\frac{\pi^2 \sigma^2}{2(\lambda + 4\epsilon)^2} \frac{N-1}{N}$$

whenever $N \geq N_1$.

We note that although θ and λ don't feature explicitly in this inequality, they do appear in the definition of $g_{r,Nk}$, via the definition of the intervals $J_r(Nk)$.

Step 6: Completing the argument. Now we return to (4.9). Using the estimates we obtained in the last two steps, we conclude that, whenever $N \geq N_1(\epsilon, \lambda)$,

$$\begin{aligned}\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log \Lambda_{Nk} \;&\leq\; (1+\overline{p})(\theta+\epsilon) + (1+\overline{p})\theta\frac{N+1}{N} + (\lambda+\epsilon)(1+\overline{p}) \\ &\qquad - \frac{\pi^2\sigma^2}{2(\lambda+4\epsilon)^2} - \frac{\pi^2\sigma^2}{2(\lambda+4\epsilon)^2}\frac{N-1}{N}. \qquad (4.18)\end{aligned}$$

In Step 2, we showed that

$$\liminf_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log E_{Nk} \;\geq\; (1+\overline{p})\theta - \frac{\pi^2\sigma^2}{2\lambda^2} \qquad (4.19)$$

and

$$\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log E_{Nk} \;\leq\; (1+\overline{p})(\theta+\epsilon) - \frac{\pi^2\sigma^2}{2(\lambda+\epsilon)^2}. \qquad (4.20)$$

Using the monotonicity of $t \mapsto \rho_\epsilon(\theta t^{-2/3}, t)$ together with our Paley-Zygmund statement, (4.7), we deduce that for all $N \geq N_1$,

$$\begin{aligned}&\liminf_{t\to\infty} \frac{1}{t^{1/3}} \;\log \rho_\epsilon(\theta t^{-2/3}, t) \\ =\;& \liminf_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log \rho_\epsilon(\theta(Nk)^{-2/3}, Nk) \\ \geq\;& 2\liminf_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \;\log E_{Nk} \\ &\quad - \max\left\{\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \;\log E_{Nk}\,,\, \limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \;\log \Lambda_{Nk}\right\}. \qquad (4.21)\end{aligned}$$

Using (4.20) we find that

$$\lim_{\epsilon\downarrow 0}\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \;\log E_{Nk} \;\leq\; (1+\overline{p})\theta - \frac{\pi^2\sigma^2}{2\lambda^2}. \qquad (4.22)$$

Using (4.18), we conclude that

$$\lim_{\epsilon\downarrow 0}\lim_{N\to\infty}\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log \Lambda_{Nk} \;\leq\; 2\left((1+\overline{p})\theta - \frac{\pi^2\sigma^2}{2\lambda^2}\right) + (1+\overline{p})\lambda. \qquad (4.23)$$

Remember that we are working under the hypothesis that

$$\lambda \;\geq\; \lambda_0 \;=\; \frac{\pi\sigma}{[2(1+\overline{p})\theta]^{1/2}}$$

which is equivalent to

$$(1+\overline{p})\theta - \frac{\pi^2\sigma^2}{2\lambda^2} \;\geq\; 0.$$

In consequence, the right-hand side of (4.23) is greater than the right-hand side of (4.22).

Now we return to (4.21). Using the fact that $(a, b) \mapsto \max(a, b)$ is continuous (allowing us to take the limits in N and ϵ inside the maximum), together with the calculations we've just made, and also (4.19), we conclude that

$$\liminf_{\epsilon \downarrow 0} \liminf_{t \to \infty} \frac{1}{t^{1/3}} \log \rho_\epsilon(\theta t^{-2/3}, t) \;\; \geq \;\; -(1 + \overline{p})\lambda.$$

The conclusion follows by setting $\lambda = \lambda_0$ and using (4.6). $\square$

4.2.3 Completing the proof

We are now in a position to prove the lower bound, (4.3). In fact, the only properties of $\rho(\epsilon)$ we will need are those contained in Proposition 4.7 and Proposition 4.8, and two further simple analytic properties. The probability content to follow solely concerns the Galton-Watson tree $\mathbb{G}$. To emphasize this fact, we state the final step in the following way. Let κ be a real valued function defined on $(0, a) \times [0, \infty)$, for some $a > 0$, with the further property that the limit $\kappa(\epsilon, \infty)$ exists for all $\epsilon \in (0, a)$. We will say that κ "satisfies Proposition 4.7" if Proposition 4.7 holds with occurrences of $\rho(\cdot, \cdot)$ replaced with κ, and occurrences of $\rho(\cdot)$ replaced with $\kappa(\cdot, \infty)$.

Proposition 4.9. *Fix a function $\kappa : (0, a) \times [0, \infty) \to [0, 1]$, for some $a > 0$. Suppose that κ increases in the first co-ordinate, decreases in the second, and that* $\lim_{\epsilon \downarrow 0} \kappa(\epsilon, \infty) = 0$. *Suppose that κ satisfies Proposition* 4.7, *and that*

$$\liminf_{t \to \infty} \frac{1}{t^{1/3}} \log \kappa(\theta t^{-2/3}, t) \;\; \geq \;\; -\lambda \, \theta^{-1/2} \tag{4.24}$$

for some $\lambda > 0$ and all $\theta > 0$. Then

$$\liminf_{\epsilon \downarrow 0} \epsilon^{1/2} \log \kappa(\epsilon, \infty) \;\; \geq \;\; -\lambda.$$

This is essentially what the authors of [25] show on pages 18–19 of their paper. We emphasize that the proof to follow is identical to theirs; we make no claim to originality in this step. Since it is short and simple, we include it to keep this half of the proof self-contained. We remind the reader that M, K, γ and q are constants whose values are fixed by Proposition 4.7.

Proof. We start by fixing constants $\alpha \in (0, 1)$ and $b > \max\left\{\frac{M}{1-\alpha}, \frac{(3\lambda)^2}{\alpha(\log q)^2}\right\}$. Let $n > 1$, and let

$$\epsilon = \epsilon(n) := \frac{b}{n^{2/3}}, \qquad \text{and } \; L = L(n) := n - \lfloor n^{1/3} \rfloor.$$

Note, in particular, that for all sufficiently large n, the hypothesis $(n - L)M \leq (1 - \alpha)\epsilon L$ of Proposition 4.7 is met, since $b > \frac{M}{1-\alpha}$.

We are going now going to use the Galton-Watson tree discussed in Proposition 4.7. As α and b are fixed, ϵ is defined in terms of b and n, and L is defined in terms of n, we can write $\mathbb{G}_n$ for $\mathbb{G}(\epsilon, \alpha, n, L)$. Let p_n denote the extinction probability of $\mathbb{G}_n$, and write f_n for the generating function of G_n, which denotes the size of the first generation of $\mathbb{G}_n$. That is,

$$f_n(s) \;:=\; \mathbf{E}(s^{G_n}) \quad \text{for } s \in [0,1].$$

Elementary theory tells us that p_n is the smallest non-negative fixed point of f_n, so for all $r \in (0, p_n)$ we can write

$$p_n \;=\; f_n(0) + \int_0^{p_n - r} f_n'(s)\,ds + \int_{p_n - r}^{p_n} f_n'(s)\,ds.$$

We also know that f_n is an increasing, convex function. Consequently, f_n' increases, and $f_n'(p_n) \le 1$. This allows us to write

$$\int_0^{p_n - r} f_n'(s)\,ds \;\le\; (p_n - r) f_n'(p_n - r) \;\le\; f_n'(1 - r)$$

and

$$\int_{p_n - r}^{p_n} f_n'(s)\,ds \;\le\; r f_n'(p_n) \;\le\; r.$$

We deduce that

$$p_n \;\le\; f_n(0) + f_n'(1 - r) + r, \tag{4.25}$$

provided that $0 < r < p_n$. Next we use the inequality $1 - u \le e^{-u}$ for $u \ge 0$ to write

$$f_n'(1 - r) \;=\; \frac{1}{1 - r}\mathbf{E}(G_n (1 - r)^{G_n}) \;\le\; \frac{1}{1 - r}\mathbf{E}(G_n e^{-rG_n}).$$

Further insisting that $r < 1/2$, we arrive at the following conclusion (deduced from (4.25)):

$$1 - p_n \;\ge\; \mathbf{P}(G_n \ge 1) - 2\,\mathbf{E}(G_n e^{-rG_n}) - r$$

whenever $0 < r < p_n \wedge \frac{1}{2}$. Since κ satisfies Proposition 4.7, we know that $\mathbf{P}(G_n \ge 1) \ge \gamma\kappa(\alpha\epsilon, L) \ge \gamma\kappa(\alpha\epsilon, n)$; the second inequality holds because κ decreases in its second argument. Since the function $u \mapsto u e^{-ru}$ decreases on $[\frac{1}{r}, \infty)$, we can write

$$\begin{aligned}\mathbf{E}(G_n e^{-rG_n}) \;&\le\; \mathbf{E}(G_n e^{-rG_n};\; 1 \le G_n \le r^{-2}) + r^{-2} e^{-\frac{1}{r}} \\ &\le\; r^{-2}\,\mathbf{P}(1 \le G_n \le r^{-2}) + r^{-2} e^{-\frac{1}{r}}.\end{aligned}$$

Elementary calculations tell us that $x \mapsto x^3\, e^{-x}$ decreases on $[3, \infty)$. Consequently, $\sup_{\{x \ge 16\}} x^3\, e^{-x} \;=\; (16)^3\, e^{-16} \;<\; \frac{1}{2}$, so that $\sup_{\{0 < r \le \frac{1}{16}\}} r^{-3} e^{-\frac{1}{r}} \;<\; \frac{1}{2}$. Altogether, we deduce that

$$1 - p_n \;\ge\; \gamma\kappa(\alpha\epsilon, n) - 2r^{-2}\,\mathbf{P}(1 \le G_n \le r^{-2}) - 2r \tag{4.26}$$

whenever $0 < r < p_n \wedge \frac{1}{16}$.

Now we claim that whenever (a_n) is a non-negative sequence decreasing to zero, $\lim_n \kappa(a_n, n)$ goes to zero too. Indeed, fix $\epsilon > 0$. Then for all large n, $\kappa(a_n, n) \le$

$\kappa(\epsilon, n)$ since κ decreases in its first argument. We deduce that $\limsup_{n\to\infty} \kappa(a_n, n) \leq \kappa(\epsilon, \infty)$. Sending ϵ to zero gives the required conclusion. On the other hand, we are assuming that $\kappa(\epsilon_n, \infty) \geq 1 - p_n$ for all $n \in \mathbb{N}$. Since $\lim_n \kappa(\epsilon_n, \infty) = 0$, we deduce that $\lim_n p_n = 1$. Putting these two statements together, we deduce that $\frac{\gamma}{8}\kappa(\alpha\epsilon_n, n) < p_n \wedge \frac{1}{16}$ for all sufficiently large n, since the left-hand side goes to 0 and the right-hand side goes to $1/16$. We can therefore use the values $r = r_n := \frac{\gamma}{8}\kappa(\alpha\epsilon_n, n)$ in the work above.

Now we claim that $K\, q^{L-n} \leq r_n^3$ whenever n is sufficiently large. This is equivalent to showing that $q^{n-L}\, \kappa(\alpha b n^{-2/3}, n)^3 \geq K\,(8/\gamma)^3$ whenever n is large. In fact, the left-hand side of this inequality goes to infinity in n, because

$$\liminf_{n\to\infty} \frac{1}{n^{1/3}} \log \kappa(\alpha b n^{-2/3}, n) \;>\; -\frac{1}{3}\log q.$$

To see why this inequality holds, note that the left-hand side exceeds $-\lambda(\alpha b)^{-1/2}$ by our hypothesis (4.24). Our assumption that $b > \frac{(3\lambda)^2}{\alpha(\log q)^2}$ does the rest.

Since $r_n \to 0$ in n, we deduce from the result of the last paragraph that $r_n^2 \geq q^{L-n}$ for all large n. We can therefore write

$$\mathbf{P}(1 \leq G_n \leq r_n^{-2}) \;\leq\; \mathbf{P}(1 \leq G_n \leq q^{n-L}) \;\leq\; \frac{K}{q^{n-L}} \;\leq\; r_n^3$$

for all sufficiently large n. The second inequality uses point 3 of Proposition 4.7. Going back to (4.26), we conclude that for all sufficiently large n,

$$1 - p_n \;\geq\; \gamma\kappa(\alpha\epsilon_n, n) - 2r_n - 2r_n \;\geq\; \frac{\gamma}{2}\kappa(\alpha\epsilon_n, n).$$

Recalling that $\kappa(\epsilon_n, \infty) \geq 1 - p_n$, we deduce that

$$\kappa(bn^{-2/3}, \infty) \;\geq\; \frac{\gamma}{2}\kappa(\alpha b n^{-2/3}, n)$$

for all sufficiently large n. Using (4.24), we deduce that

$$\liminf_{n\to\infty} \frac{1}{n^{1/3}} \log \kappa(bn^{-2/3}, \infty) \;\geq\; -\lambda(\alpha b)^{-1/2}.$$

Since $\epsilon \mapsto \kappa(\epsilon, \infty)$ decreases as ϵ does, we conclude that

$$\liminf_{\epsilon\to 0} \epsilon^{1/2} \log \kappa(\epsilon, \infty) \;\geq\; -\lambda\alpha^{-1/2}.$$

The argument is concluded by sending $\alpha \to 1$. $\square$

To complete the proof of Theorem 1.15, it remains to note that the function $(\epsilon, t) \mapsto \rho(\epsilon, t)$ satisfies the additional hypotheses of Proposition 4.9. Indeed, survival becomes increasingly difficult as ϵ decreases (with t fixed), and as t increases (with ϵ fixed). This means that ρ increases in its first argument and decreases in its second. The fact that $\lim_{\epsilon\downarrow 0} \rho(\epsilon, \infty) = 0$ is a simple consequence of the upper bound proved in §4.1. Finally, assumption (4.24) is satisfied (for the appropriate value of λ) by Proposition 4.8. $\square$

CHAPTER 5

SURVIVAL OF CRITICALLY KILLED FRAGMENTATION PROCESSES

In this chapter we will prove Theorem 1.16, which concerns the asymptotic behaviour of the probability that a $c_{\overline{p}}$–killed fragmentation process survives until large times.

Let's state Theorem 1.16 again:

Theorem. *The probability $\kappa(t)$ that the critically killed fragmentation process survives until time t satisfies the following asymptotic identity:*

$$\lim_{t\to\infty} \frac{1}{t^{1/3}} \log \kappa(t) \;=\; -\left(\frac{3\pi^2(1+\overline{p})^2|\Phi''(\overline{p})|}{2}\right)^{1/3}.$$

The structure of this chapter has much in common with that of Chapter 4. Again our proof will be based on a paper which addresses the corresponding branching random walk question, in this case Aïdékon and Jaffuel [2]. We will use the work in this paper to prove the upper bound using Lemma 2.1, and will prove the lower bound from scratch using the second moment method in combination with Mogulskii's Theorem, Lemma 2.14.

The proof of the lower bound in this chapter has a few qualitative differences to the proof of the lower bound of Theorem 1.15. The reason for this essentially derives from the fact that the intervals we will use in when applying Mogulskii's Theorem in this chapter do not have constant width. In [2], the authors deal with this issue by proving a series of sophisticated corollaries of Mogulskii's Theorem; the reader is referred to section 4 of that paper. We prefer to avoid using these corollaries, and instead adopt a bare-hands approach. The resulting calculations are consequently quite messy, but rely only on our simple version of Mogulskii's Theorem.

5.1 Proof of the upper bound

As in the proof of the upper bound in Chapter 4, we can use Lemma 2.1 (taking $g = 0$ there) to arrive at the following conclusion:

For any array $(b_i^{(n)} : \; 1 \le i \le n)_{n\in\mathbb{N}}$ satisfying $b_i^{(n)} \le b_j^{(n)}$ whenever $i > j$, and any $n \in \mathbb{N}$, we have

$$\kappa(n) \;\le\; I_n^{(n)} \;+\; \sum_{j=0}^{n-1} e^{-b_{j+1}^{(n)}(1+\overline{p})} I_j^{(n)} \tag{5.1}$$

where $I_0^{(n)} := 1$ for all $n \in \mathbb{N}$, and

$$\begin{aligned} S_j &:= -\zeta_j, \quad \forall j \in \mathbb{N}, \\ I_j^{(n)} \;&:=\; \mathbf{Q}\big(0 \le S_i \le b_i^{(n)} \quad \forall i \le j\big), \quad 1 \le j \le n. \end{aligned}$$

For any centred random walk Y with finite variance σ_Y^2, any $\nu > 0$, and any array $(b_i^{(n)} : \; 1 \le i \le n)_{n\in\mathbb{N}}$, we define

$$F^Y(n, b^{(n)}, \nu) \;:=\; I_{Y,n}^{(n)} \;+\; \sum_{j=0}^{n-1} e^{-\nu\, b_{j+1}^{(n)}} I_{Y,j}^{(n)}$$

where $I_{Y,0}^{(n)} := 1$ for all $n \in \mathbb{N}$, and where

$$I_{Y,j}^{(n)} \;:=\; \mathbf{Q}\big(0 \le Y_i \le b_i^{(n)} \quad \forall i \le j\big), \quad 1 \le j \le n.$$

The right-hand side of (5.1) can then be written as $F^S(n, b^{(n)}, 1+\overline{p})$.

Now let us fix $b_j^{(n)} := d(n-j)^{1/3}$, where $d := \left(\frac{3\pi^2\sigma_Y^2}{2(1+\overline{p})}\right)^{1/3}$. The authors of [2] show on page 1931 of their paper that for any $\nu > 0$,

$$\limsup_{\mathbb{N}\ni n\to\infty} \frac{1}{n^{1/3}} \log F^Y(n, b^{(n)}, \nu) \;\le\; -\left(\frac{3\pi^2\nu^2\sigma_Y^2}{2}\right)^{1/3}.$$

For our embedded random walk S, $\sigma_S^2 = -\Phi''(\overline{p})$. Using the monotonicity of $t \mapsto \kappa(t)$, and taking $\nu = 1 + \overline{p}$, we deduce that

$$\limsup_{t\to\infty} \frac{1}{t^{1/3}} \log \kappa(t) \;\le\; -\left(\frac{3\pi^2(1+\overline{p})^2|\Phi''(\overline{p})|}{2}\right)^{1/3},$$

completing the proof of the upper bound. □

5.2 Proof of the lower bound

In this section we will prove that

$$\liminf_{t\to\infty} \frac{1}{t^{1/3}} \log \kappa(t) \;\ge\; -\left(\frac{3\pi^2(1+\overline{p})^2|\Phi''(\overline{p})|}{2}\right)^{1/3}.$$

We split the proof into several steps, which replicate those used in the proof of Proposition 4.8.

Step 1: Preliminaries. This step is formally identical to the first step in the proof of Proposition 4.8, with only the definitions of the intervals $I_s(t)$ differing. We will repeat it anyway, to fix notation.

First note that

$$\kappa(t) \quad = \quad \mathbf{P}(\exists x \in (0,1): \;\; \zeta_s^x \leq 0 \;\; \forall s \in [0,t]\,).$$

First we move the killing barrier $s \mapsto 0$ on $[0,t]$ up by the amount $\epsilon t^{1/3}$: by Corollary 2.11 and the fragmentation property, there exist $\eta, p > 0$ such that, for all $t \geq 0$ and $\epsilon > 0$, we can write

$$\kappa(t) \;\; \geq \;\; p^{\alpha} \;\cdot\; \mathbf{P}\big(\exists x \in (0,1): \;\; \zeta_s^x \leq \epsilon t^{1/3} \;\; \forall s \in [0,t]\big), \tag{5.2}$$

where $\alpha = \alpha(t,\epsilon,\eta) := \lceil \frac{\epsilon t^{1/3}}{\eta} \rceil$. Since

$$\lim_{t\to\infty} \frac{1}{t^{1/3}} \log p^{\alpha} \quad = \quad o_{\epsilon}(1),$$

we can focus on the second factor in (5.2) from now on. We will estimate this factor by using the Paley-Zygmund inequality. To this end, we define, for $t \geq 0$, the random variable

$$Z_t \;\; := \;\; \sum_{[x]_t} \mathbf{1}\{\zeta_s^x \in I_s(t) \;\; \forall s \in [0,t]\}$$

where this time

$$I_s(t) \;\; := \;\; [-d(t-s)^{1/3}, \epsilon t^{1/3}] \quad \text{with} \quad d \;\; := \;\; \left(\frac{3\pi^2\sigma^2}{2(1+\overline{p})}\right)^{1/3}.$$

Using the Paley-Zygmund inequality, we have

$$\mathbf{P}(\exists x \in (0,1): \;\; \zeta_s^x \leq \epsilon t^{1/3} \;\; \forall s \in [0,t]) \;\; \geq \;\; \mathbf{P}(Z_t > 0) \;\; \geq \;\; \frac{(\mathbf{E}Z_t)^2}{\mathbf{E}Z_t^2},$$

and we now proceed to write the first and second moments of Z_t in terms of the spine $(\zeta, \mathbf{Q})$.

Using the Many-to-One Lemma, we have

$$E_t \;\; := \;\; \mathbf{E}Z_t \;\; = \;\; \mathbf{Q}(e^{(1+\overline{p})\zeta_t}\mathbf{1}_{(\zeta_s \in I_s(t) \;\; \forall s \in [0,t])}) \;\; = \;\; \mathbf{Q}(e^{(1+\overline{p})\zeta_t}\mathbf{1}_{A_{[0,t]}})$$

where for $0 \leq v \leq w \leq t$ we write $A_{[v,w]}$ for the event that $\zeta_s \in I_s(t)$ for all $v \leq s \leq w$.

By Corollary 2.7,

$$\mathbf{E}(Z_t^2) \;\; = \;\; E_t + \Lambda_t,$$

where

$$\Lambda_t \;\; := \;\; \int_0^t dr \;\cdot\; \mathbf{Q}\left(e^{(1+\overline{p})\zeta_r}\mathbf{1}_{A_{[0,r]}} \int_{\mathcal{U}} G(\zeta_r, u)\nu(du)\right),$$

with

$$G(\alpha,u) \;:=\; \sum_{a_u\neq b_u} F(\alpha-\log|a_u|)F(\alpha-\log|b_u|),$$

and

$$F(\alpha) \;:=\; \mathbf{Q}\, e^{\zeta_{t-r}(\overline{p}+1)}\mathbf{1}_{\left(\alpha+\zeta_s\in I_{s+r}(t)\ \ \forall s\in[0,t-r]\right)}.$$

For future reference, we rewrite our application of the Paley-Zygmund inequality in these terms: for all $t\geq 0$, we have

$$P_t \;:=\; \mathbf{P}(\exists x\in(0,1):\ \ \zeta_s^x\leq \epsilon t^{1/3}\ \ \forall s\in[0,t]) \;\geq\; \frac{E_t^2}{E_t+\Lambda_t}. \tag{5.3}$$

We now proceed to find upper and lower bounds on E_t, and an upper bound on Λ_t.

Step 2: Bounding E_t from above and below. We start by using the bounds $0\leq\zeta_t\leq \epsilon t^{1/3}$ on $A_{[0,t]}$ to write

$$\mathbf{Q}\left(A_{[0,t]}\right) \;\leq\; E_t \;\leq\; e^{\epsilon(1+\overline{p})t^{1/3}}\,\mathbf{Q}\left(A_{[0,t]}\right).$$

Now,

$$\mathbf{Q}\left(A_{[0,t]}\right) \;=\; \mathbf{Q}\left(-d\left(1-\frac{s}{t}\right)^{1/3} \;\leq\; \frac{\zeta_s}{t^{1/3}} \;\leq\; \epsilon \quad \forall s\in[0,t]\right),$$

so that, by Lemma 2.14, we deduce that

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t^{1/3}}\log\mathbf{Q}\left(A_{[0,t]}\right) &= -\frac{\pi^2\sigma^2}{2}\int_0^1\left(\epsilon+d(1-s)^{1/3}\right)^{-2}ds\\ &=: -\frac{3\pi^2\sigma^2}{2d^2}+\beta_1(\epsilon).\end{aligned}$$

By the dominated convergence theorem, $\beta_1(\epsilon)=o(1)$ as $\epsilon\downarrow 0$, since the integral in the previous display approaches $3d^{-2}$ as $\epsilon\downarrow 0$. Returning to the first display of this step, we deduce that

$$\liminf_{t\to\infty}\frac{1}{t^{1/3}}\log E_t \;\geq\; -\frac{3\pi^2\sigma^2}{2d^2}+\beta_1(\epsilon)$$

and that

$$\limsup_{t\to\infty}\frac{1}{t^{1/3}}\log E_t \;\leq\; -\frac{3\pi^2\sigma^2}{2d^2}+\beta_1(\epsilon)+(1+\overline{p})\epsilon \;=:\; -\frac{3\pi^2\sigma^2}{2d^2}+\beta_2(\epsilon).$$

Step 3: A preliminary upper bound on Λ_t. Let us widen the intervals $I_s(t)$ by $\epsilon t^{1/3}$ on either side, by defining

$$J_s(t) \;:=\; [-d(t-s)^{1/3}-\epsilon t^{1/3},\ \ 2\epsilon t^{1/3}].$$

We also define

$$L_{s,t} \;:=\; |I_s(t)| \;=\; d(t-s)^{1/3}+\epsilon t^{1/3}$$

and let $\gamma:=\int_0^\infty(1\wedge x)\Pi(dx)<\infty$.

We claim that

$$\Lambda_t \;\leq\; \gamma\, t\, e^{2\epsilon(1+\overline{p})t^{1/3}} \sup_{r\in[0,t]} h_{r,t} \cdot \sup_{r\in[0,t]} g_{r,t} \tag{5.4}$$

where

$$h_{r,t} \;:=\; e^{d(1+\overline{p})(t-r)^{1/3}} \sup_{0\leq x\leq L_{r,t}} \mathbf{Q}\left(\zeta_s \in x + J_{s+r}(t) \;\; \forall s\in[0,t-r]\right)$$

and

$$g_{r,t} \;:=\; \mathbf{Q}\left(\zeta_s \in I_s(t) \;\; \forall s\in[0,r]\right) \sup_{0\leq x\leq L_{r,t}} \mathbf{Q}\big(\zeta_s \in x + J_{s+r}(t) \;\; \forall s\in[0,t-r]\big).$$

The proof of (5.4) is formally the same as the one contained in Step 2 of the proof of Proposition 4.8. The only differences arise when estimating terms of the form $e^{(1+\overline{p})\zeta_s}$ on the event $A_{[0,t]}$, as we are using different intervals $I_s(t)$ in this section. For this reason, we feel it's safe to omit the details, and proceed to bounding $\sup_{r\in[0,t]} h_{r,t}$ and $\sup_{r\in[0,t]} g_{r,t}$ from above.

Step 4: Bounding $\sup_{r\in[0,t]} h_{r,t}$ *from above.* We start by fixing integers $N, k \geq 1$. We then set $t = Nk$, and split the time interval $[0,t]$ into intervals of the form $[(l-1)k, lk]$ for $1\leq l\leq N$. For $r\in[(l-1)k, lk]$, we have $(N-l)k \leq t-r \leq (N-1+1)k$. We also note that $s\mapsto L_{s,t}$ is a decreasing map and that $J_v(t)\subset J_u(t)$ whenever $0\leq u\leq v\leq t$. These observations allow us to make the following estimate:

$$\begin{aligned}
\sup_{r\in[0,t]} h_{r,t} \;&=\; \max_{1\leq l\leq N} \sup_{(l-1)k\leq r\leq lk} e^{d(1+\overline{p})(t-r)^{1/3}} \\
&\qquad\qquad \cdot \sup_{0\leq x\leq L_{r,t}} \mathbf{Q}\left(\zeta_s \in x + J_{s+r}(t) \;\; \forall s\in[0,t-r]\right) \\
&\leq\; \max_{1\leq l\leq N} e^{d(1+\overline{p})(N-l+1)^{1/3}k^{1/3}} \\
&\qquad\qquad \cdot \sup_{0\leq x\leq L_{(l-1)k,Nk}} \mathbf{Q}\left(\zeta_s \in x + J_{s+(l-1)k}(Nk) \;\; \forall s\in[0,(N-l)k]\right).
\end{aligned}$$

Discretizing in s, we deduce that

$$\sup_{r\in[0,t]} h_{r,t} \;\leq\; \max_{1\leq l\leq N} e^{d(1+\overline{p})(N-l+1)^{1/3}k^{1/3}}\, H_{l,N,k} \tag{5.5}$$

where

$$H_{l,N,k} \;:=\; \sup_{0\leq x\leq L_{(l-1)k,Nk}} \mathbf{Q}\left(\zeta_i \in x + J_{i+(l-1)k}(Nk) \;\; \forall 1\leq i\leq (N-l)k\right).$$

Let's write $L_{l,N,k}$ for $L_{(l-1)k,Nk}$. The next step is to split the x-values in the supremum over $x\in[0,L_{l,N,k}]$ in the definition of $H_{l,N,k}$ into intervals of the form

$$\left[\frac{(m-1)L_{l,N,k}}{N},\; \frac{mL_{l,N,k}}{N}\right], \qquad 1\leq m\leq N.$$

Write $J_{i+(l-1)k}(Nk)$ as $[a_{i,l,k}, b_{i,l,k}]$. Then $H_{l,N,k}$ is bounded above by

$$\max_{1\leq m\leq N} \mathbf{Q}\left(\zeta_i \in \left[a_{i,l,k} + \frac{(m-1)L_{l,N,k}}{N},\;\; b_{i,l,k} + \frac{mL_{l,N,k}}{N}\right],\;\; \forall 1\leq i\leq (N-l)k\right).$$

The probability indexed by m in the previous display can be rewritten as

$$\mathbf{Q}\left(-d\left[\frac{N-l+1}{N-l}-\frac{i}{(N-l)k}\right]^{1/3}-\epsilon\left[\frac{N}{N-l}\right]^{1/3}+\frac{m-1}{N}\alpha_{N,l,\epsilon}\right.$$
$$\left.\leq\ \frac{\zeta_i}{(N-l)^{1/3}k^{1/3}}\ \leq\ 2\epsilon\left[\frac{N}{N-l}\right]^{1/3}+\frac{m}{N}\alpha_{N,l,\epsilon},\ \forall 1\leq i\leq(N-l)k\right),\quad(5.6)$$

where

$$\alpha_{N,l,\epsilon}\ :=\ \epsilon\left[\frac{N}{N-l}\right]^{1/3}+d\left[\frac{N-l+1}{N-l}\right]^{1/3}.$$

We want to apply Mogulskii's Theorem (sending k to infinity) to estimate these probabilities, for each pair $1\leq m,l\leq N$. For minor technical reasons, we treat pairs with $1\leq l\leq N-N^{1/3}$ and pairs with $N-N^{1/3}<l\leq N$ separately.

Let's start with the more difficult case, fixing $1\leq m\leq l$ and $1\leq l\leq N-N^{1/3}$. The appropriate functions for applying Mogulskii's Theorem, Lemma 2.14, to the probabilities in (5.6) are

$$g_1^{m,l}(t)\ :=\ -d\left[\frac{N-l+1}{N-l}-t\right]^{1/3}-\epsilon\left[\frac{N}{N-l}\right]^{1/3}+\frac{m-1}{N}\alpha_{N,l,\epsilon}$$

and

$$g_2^{m,l}(t)\ :=\ 2\epsilon\left[\frac{N}{N-l}\right]^{1/3}+\frac{m}{N}\alpha_{N,l,\epsilon}$$

for $t\in[0,1]$. We have

$$g_2^{m,l}(t)-g_1^{m,l}(t)\ =\ 3\epsilon\left[\frac{N}{N-l}\right]^{1/3}+d\left[\frac{N-l+1}{N-l}-t\right]^{1/3}+\frac{\alpha_{N,l,\epsilon}}{N}\ =:\ G_{N,l}(t).$$

Note that the m–dependence has disappeared; all the following estimates are therefore true uniformly in m. Applying Lemma 2.14 to our work above, we find that

$$\limsup_{k\to\infty}\frac{1}{(N-l)^{1/3}k^{1/3}}\log H_{l,N,k}\ \leq\ -\frac{\pi^2\sigma^2}{2}\int_0^1 G_{N,l}^{-2}(t)dt.\quad(5.7)$$

We want to bound this expression above, which is equivalent to bounding $G_{N,l}$ above; this is where we use our restriction $l\leq N-N^{1/3}$. For all such l,

$$\frac{N-l+1}{N-l}\ =\ 1+\frac{1}{N-l}\ \leq\ 1+\frac{1}{N^{1/3}}.$$

Uniformly in $1\leq l\leq N-N^{1/3}$, we can therefore use the bound

$$\frac{N-l+1}{N-l}\ \leq\ 1+\epsilon$$

for all sufficiently large N. We remark, in particular, that it will therefore be important to take N to infinity before ϵ is sent to zero when concluding the argument. The term $\frac{\alpha_{N,l,\epsilon}}{N}$ is $O(N^{-2/3})$, uniformly in $1\leq l<N$, so this term can be bounded

above by ϵ for all large N. These observations lead to the following bound, valid for all large N and all $1 \le l \le N - N^{1/3}$:

$$\begin{aligned} G_{N,l}(t) &\le 3\epsilon \left[\frac{N}{N-l}\right]^{1/3} + d(1+\epsilon-t)^{1/3} + \epsilon \\ &\le 4\epsilon \left[\frac{N}{N-l}\right]^{1/3} + d(1+\epsilon-t)^{1/3}. \end{aligned}$$

Returning to (5.7), we conclude that for all large N (and uniformly in l) we have

$$\begin{aligned} &\limsup_{k\to\infty} \frac{1}{(Nk)^{1/3}} \log H_{l,N,k} \\ &\qquad \le -\frac{\pi^2\sigma^2}{2}\left(\frac{N-l}{N}\right)^{1/3} \int_0^1 \frac{dt}{\left[d(1+\epsilon-t)^{1/3} + 4\epsilon\left[\frac{N}{N-l}\right]^{1/3}\right]^2} . \end{aligned} \tag{5.8}$$

Returning to (5.5), with the restriction $1 \le l \le N - N^{1/3}$, we conclude that for all large enough N,

$$\limsup_{k\to\infty} \frac{1}{(Nk)^{1/3}} \log \max_{1\le l\le N-N^{1/3}} e^{d(1+\overline{p})(N-l+1)^{1/3}k^{1/3}} H_{l,N,k}$$

is bounded above by the maximum over $1 \le l \le N - N^{1/3}$ of

$$d(1+\overline{p})\left(\frac{N-l+1}{N}\right)^{1/3} - \frac{\pi^2\sigma^2}{2d^2}\left(\frac{N-l}{N}\right)^{1/3} \int_0^1 \frac{dt}{\left[(1+\epsilon-t)^{1/3} + \epsilon'\left[\frac{N}{N-l}\right]^{1/3}\right]^2} \tag{5.9}$$

where $\epsilon' := 4\epsilon/d$. We want to replace the term $\left(\frac{N-l+1}{N}\right)^{1/3}$ with $\left(\frac{N-l}{N}\right)^{1/3}$. But this can easily be done at ϵ–expense, uniformly in l, because

$$\max_{1\le l\le N-N^{1/3}} \left\{\left(\frac{N-l+1}{N}\right)^{1/3} - \left(\frac{N-l}{N}\right)^{1/3}\right\} \le \sup_{x\in[N^{-1},1]} \left\{x^{1/3} - \left(x - N^{-1}\right)^{1/3}\right\}$$

which goes to zero in N by the uniform continuity of the map $x \mapsto x^{1/3}$ on $[0,1]$. As a result, we can bound (5.9) above by

$$\epsilon + d(1+\overline{p})\left(\frac{N-l}{N}\right)^{1/3} - \frac{\pi^2\sigma^2}{2d^2}\left(\frac{N-l}{N}\right)^{1/3} \int_0^1 \frac{dt}{\left[(1+\epsilon-t)^{1/3} + \epsilon'\left[\frac{N}{N-l}\right]^{1/3}\right]^2} \tag{5.10}$$

for all sufficiently large N. Now note that

$$(1+\overline{p})d = (1+\overline{p})\left(\frac{3\pi^2\sigma^2}{2(1+\overline{p})}\right)^{1/3} = \left(\frac{3\pi^2\sigma^2(1+\overline{p})^2}{2}\right)^{1/3}$$

and that

$$\frac{3\pi^2\sigma^2}{2d^2} = \frac{3\pi^2\sigma^2}{2}\left(\frac{2^2(1+\overline{p})^2}{3^2\pi^4\sigma^4}\right)^{1/3} = \left(\frac{3\pi^2\sigma^2(1+\overline{p})^2}{2}\right)^{1/3},$$

so that in fact $\alpha := (1+\overline{p})d = \frac{3\pi^2\sigma^2}{2d^2}$. Our upper bound now reads as the maximum over $1 \le l \le N - N^{1/3}$ of

$$\epsilon + \alpha \left\{ \left(\frac{N-l}{N}\right)^{1/3} - \frac{1}{3}\left(\frac{N-l}{N}\right)^{1/3} \int_0^1 \frac{dt}{\left[(1+\epsilon-t)^{1/3} + \epsilon' \left[\frac{N}{N-l}\right]^{1/3}\right]^2} \right\}.$$

Now we need to study the expression in the curly brackets, which is bounded above by

$$\sup_{x\in[0,1]} x(1 - I_\epsilon(x)),$$

where

$$\begin{aligned} I_\epsilon(x) \;&:=\; \frac{1}{3}\int_0^1 \frac{dt}{\left[(1+\epsilon-t)^{1/3} + \frac{\epsilon'}{x}\right]^2} \\ &=\; x^2 \int_{\epsilon^{1/3}}^{(1+\epsilon)^{1/3}} \frac{t^2}{(xt+\epsilon')^2}\,dt. \end{aligned}$$

We state the next part of the proof as a lemma:

Lemma 5.1. *The function $x \mapsto x(1 - I_\epsilon(x))$ is increasing on $[0,\infty)$.*

Proof. Define $f(x) := x(1 - I_\epsilon(x))$. Then $f'(x) = 1 - I_\epsilon(x) - xI_\epsilon'(x)$. Elementary calculations show that $f'(x) \ge 0$ if and only if

$$\int_{\epsilon^{1/3}}^{(1+\epsilon)^{1/3}} \frac{x^3t + 3\epsilon' x^2}{(xt+\epsilon')^3} t^2 dt \;\le\; 1,$$

for all $x \in [0,\infty)$. Now we note that

$$\frac{\partial}{\partial x}\frac{x^3t + 3\epsilon' x^2}{(xt+\epsilon')^3} \;=\; \frac{6(\epsilon')^2 x}{(xt+\epsilon')^4}$$

which is positive for $x, t \ge 0$. In other words, for fixed t, the integrand of the previous display is increasing in x, and so we obtain an upper bound for the integral by sending $x \to \infty$ there. But then the integral equals $(1+\epsilon)^{1/3} - \epsilon^{1/3}$ which is strictly less than one by the concavity of the function $x \mapsto x^{1/3}$. $\square$

Using the previous lemma, we deduce that $\sup_{x\in[0,1]} f(x) = f(1)$. Consequently, the maximum of (5.10) over $1 \le l \le N - N^{1/3}$ is bounded above, for all large N, by

$$\epsilon + \alpha \left\{ 1 - \frac{1}{3}\int_0^1 \frac{dt}{\left[(1+\epsilon-t)^{1/3} + \epsilon'\right]^2} \right\}. \tag{5.11}$$

To be precise, we have shown so far that: for all $\epsilon > 0$ there exists $N_0(\epsilon) \in \mathbb{N}$ such that for all $N \ge N_0$ the following expression is bounded above by (5.11):

$$\limsup_{k\to\infty} \frac{1}{(Nk)^{1/3}} \log \max_{1\le l\le N-N^{1/3}} e^{d(1+\overline{p})(N-l+1)^{1/3}k^{1/3}} H_{l,N,k}\,.$$

It is very easy to deal with the values $N - N^{1/3} < l \leq N$. Simply note that, since all the $H_{N,l,k}$ are bounded above by 1, we have

$$\max_{N-N^{1/3}<l\leq N} e^{d(1+\overline{p})(N-l+1)^{1/3}k^{1/3}} H_{l,N,k} \;\leq\; e^{d(1+\overline{p})(N^{1/3}+1)^{1/3}k^{1/3}} \;\leq\; e^{cN^{1/9}k^{1/3}}$$

for all $N \geq 1$, where $c := 2^{1/3}d(1+\overline{p})$. We conclude that for all $N \geq 1$ we have

$$\limsup_{k\to\infty} \frac{1}{(Nk)^{1/3}} \log \max_{N-N^{1/3}<l\leq N} e^{d(1+\overline{p})(N-l+1)^{1/3}k^{1/3}} H_{l,N,k} \;\leq\; cN^{-2/9}.$$

Returning to (5.5), we conclude that for all $N \geq N_0(\epsilon)$, we have

$$\begin{aligned}
\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log \sup_{0\leq r\leq Nk} h_{r,Nk} & \\
&\leq\; \epsilon + cN^{-2/9} + \alpha \left\{ 1 - \frac{1}{3}\int_0^1 \frac{dt}{[(1+\epsilon-t)^{1/3}+\epsilon']^2} \right\} \\
&=:\; \lambda(N,\epsilon).
\end{aligned}$$

We remark that upon letting $N \to \infty$ and then $\epsilon \to 0$, $\lambda(N,\epsilon)$ goes to zero; we'll use this at the end of the argument.

Step 5: Bounding $\sup_{r\in[0,t]} g_{r,t}$ *from above.* We recall that

$$\begin{aligned}
g_{r,t} \;&:=\; \mathbf{Q}\left(\zeta_s \in I_s(t)\ \ \forall s \in [0,r]\right) \sup_{0\leq x\leq L_{r,t}} \mathbf{Q}\big(\zeta_s \in x + J_{s+r}(t)\ \ \forall s\in[0,t-r]\big) \\
&=:\; a_{r,t}\cdot b_{r,t}
\end{aligned}$$

where

$$J_s(t) \;:=\; [-d(t-s)^{1/3} - \epsilon t^{1/3},\ \ 2\epsilon t^{1/3}]$$

and

$$L_{s,t} \;:=\; |I_s(t)| \;=\; d(t-s)^{1/3} + \epsilon t^{1/3}.$$

As before, we set $t = Nk$ for integers $N, k \geq 1$. We have already shown (see (5.8)) that there exists $N_0(\epsilon) \in \mathbb{N}$ such that for all $N \geq N_0$ and $1 \leq l \leq N - N^{1/3}$ we have

$$\begin{aligned}
\limsup_{k\to\infty} \frac{1}{(Nk)^{1/3}} \log \sup_{(l-1)k\leq r\leq lk} b_{r,Nk} & \\
&\leq\; -\frac{\pi^2\sigma^2}{2d^2}\left(\frac{N-l}{N}\right)^{1/3} \int_0^1 \frac{dt}{\left[(1+\epsilon-t)^{1/3} + \epsilon'\left[\frac{N}{N-l}\right]^{1/3}\right]^2}.
\end{aligned}$$

Now let's use Mogulskii's Theorem to treat the factor $a_{r,t}$ in a similar way. For $r \in [(l-1)k, lk]$, $a_{r,Nk}$ is bounded above by

$$\begin{aligned}
A_{N,l,k} \;:=\; \mathbf{Q}\Bigg(-d\left[\frac{N}{l-1} - \frac{i}{(l-1)k}\right]^{1/3} \;&\leq\; \frac{\zeta_i}{(l-1)^{1/3}k^{1/3}} \\
&\leq\; \epsilon\left[\frac{N}{l-1}\right]^{1/3},\ \ \forall 1\leq i\leq (l-1)k\Bigg).
\end{aligned}$$

By Mogulskii's Theorem we have

$$\lim_{k\to\infty}\frac{1}{(l-1)^{1/3}k^{1/3}}\log A_{N,l,k} \;=\; -\frac{\pi^2\sigma^2}{2d^2}\int_0^1 \frac{dt}{\left[\left(\frac{N}{l-1}-t\right)^{1/3}+\epsilon''\left[\frac{N}{l-1}\right]^{1/3}\right]^2}$$

where $\epsilon'' := \epsilon/d$, so that

$$\lim_{k\to\infty}\frac{1}{N^{1/3}k^{1/3}}\log A_{N,l,k} \;=\; -\frac{\pi^2\sigma^2}{2d^2}\left(\frac{l-1}{N}\right)^{1/3}\int_0^1 \frac{dt}{\left[\left(\frac{N}{l-1}-t\right)^{1/3}+\epsilon''\left[\frac{N}{l-1}\right]^{1/3}\right]^2}.$$

Now we combine our estimates for $a_{r,t}$ and $b_{r,t}$ to deduce that, for $N \geq N_0(\epsilon)$ and $1 \leq l \leq N - N^{1/3}$, we have

$$\begin{aligned}
-\frac{2d^2}{3\pi^2\sigma^2}\limsup_{k\to\infty}\frac{1}{N^{1/3}k^{1/3}}\log\sup_{(l-1)k\leq r\leq lk} g_{r,Nk}& \\
\geq \left(\frac{N-l}{N}\right)^{1/3} I_{N,l,\epsilon} + \left(\frac{l-1}{N}\right)^{1/3} J_{N,l,\epsilon}& \qquad (5.12)
\end{aligned}$$

where

$$I_{N,l,\epsilon} \;:=\; \frac{1}{3}\int_0^1 \frac{dt}{\left[(1+\epsilon-t)^{1/3}+\epsilon'\left[\frac{N}{N-l}\right]^{1/3}\right]^2}$$

and

$$J_{N,l,\epsilon} \;:=\; \frac{1}{3}\int_0^1 \frac{dt}{\left[\left(\frac{N}{l-1}-t\right)^{1/3}+\epsilon''\left[\frac{N}{l-1}\right]^{1/3}\right]^2}.$$

We now work on estimating the right-hand side of (5.12) from below. The right-hand side of (5.12) trivially equals

$$\begin{aligned}
&\left(\frac{N-l}{N}\right)^{1/3} I_{N,l,0} \;+\; \left(\frac{N-l}{N}\right)^{1/3}(I_{N,l,\epsilon}-I_{N,l,0}) \\
&\qquad + \left(\frac{l-1}{N}\right)^{1/3} J_{N,l,0} \;+\; \left(\frac{l-1}{N}\right)^{1/3}(J_{N,l,\epsilon}-J_{N,l,0}).
\end{aligned}$$

Now we make the explicit evaluations $I_{N,l,0} = 1$ and $J_{N,l,0} = \left(\frac{N}{l-1}\right)^{1/3} - \left(\frac{N-l+1}{l-1}\right)^{1/3}$, which we substitute into the previous display; after rearrangement, we obtain

$$\begin{aligned}
&1 - \left\{\left(\frac{N-l+1}{N}\right)^{1/3} - \left(\frac{N-l}{N}\right)^{1/3}\right\} \\
&+ \left(\frac{N-l}{N}\right)^{1/3}(I_{N,l,\epsilon}-1) \\
&+ \left(\frac{l-1}{N}\right)^{1/3}\left\{J_{N,l,\epsilon} - \left[\left(\frac{N}{l-1}\right)^{1/3} - \left(\frac{N-l+1}{l-1}\right)^{1/3}\right]\right\}. \qquad (5.13)
\end{aligned}$$

Note that

$$\begin{aligned}
\left(\frac{N-1+1}{N}\right)^{1/3} - \left(\frac{N-l}{N}\right)^{1/3} &= \left(1-\frac{l}{N}+\frac{1}{N}\right)^{1/3} - \left(1-\frac{l}{N}\right)^{1/3} \\
&\leq \sup_{x\in[0,1]} \left\{ \left(x+\frac{1}{N}\right)^{1/3} - x^{1/3} \right\} \\
&= N^{-1/3},
\end{aligned}$$

where the second equality follows from the concavity of the function $x \mapsto x^{1/3}$. So, uniformly in l, we obtain the following lower bound on the first line of (5.13):

$$1 - \left\{ \left(\frac{N-1+1}{N}\right)^{1/3} - \left(\frac{N-l}{N}\right)^{1/3} \right\} \geq 1 - N^{-1/3}.$$

For the second line of (5.13), we can recycle some of the work we did in Step 4. Indeed, in the notation used there, the second line of (5.13) is bounded below by

$$\inf_{x\in[0,1]} x(I_\epsilon(x)-1) = -\sup_{x\in[0,1]} x(1-I_\epsilon(x)).$$

But in Step 4 we showed that $x \mapsto x(1-I_\epsilon(x))$ increases, so

$$\sup_{x\in[0,1]} x(1-I_\epsilon(x)) = 1 - I_\epsilon(1).$$

It remains to deal with the third line of (5.13), which has the lower bound

$$\inf_{x\in[0,1]} x^{1/3} \left\{ \frac{1}{3}\int_0^1 \frac{dt}{\left[(x^{-1}-t)^{1/3} + \epsilon'' x^{-1/3}\right]^2} - \left(x^{-1/3} - (x^{-1}-1)^{1/3}\right) \right\},$$

which equals

$$\inf_{x\in[0,1]} \left\{ \frac{1}{3}\int_0^1 \frac{x}{\left[(1-xt)^{1/3} + \epsilon''\right]^2} dt - \left(1 - (1-x)^{1/3}\right) \right\}.$$

We state the next part of the proof as a lemma:

Lemma 5.2. *The function*

$$F: \; x \mapsto \frac{1}{3}\int_0^1 \frac{x}{\left[(1-xt)^{1/3} + c\right]^2} dt - \left(1 - (1-x)^{1/3}\right)$$

decreases on $[0,1]$ *for all* $c \geq 0$.

Proof. The derivative of F evalulated at $x \in [0,1]$ equals

$$\frac{1}{3}\int_0^1 \frac{(1-xt)^{1/3} + c + \frac{2xt}{3}(1-xt)^{-2/3}}{\left[(1-xt)^{1/3} + c\right]^3} dt - \frac{1}{3}(1-x)^{-2/3}$$

which is less than or equal to zero precisely when

$$(1-x)^{2/3}\int_0^1 \frac{(1-xt)^{1/3}+c+\frac{2xt}{3}(1-xt)^{-2/3}}{\left[(1-xt)^{1/3}+c\right]^3}dt \;\leq\; 1.$$

First we make the change of variable $s := 1-xt$, to transform this inequality into

$$\frac{(1-x)^{2/3}}{x}\int_{1-x}^1 \frac{s^{1/3}+c+\frac{2}{3}(1-s)s^{-2/3}}{\left[s^{1/3}+c\right]^3}ds \;\leq\; 1.$$

Then we make the change of variable $w := s^{1/3}$, which results in the equivalent inequality

$$\frac{(1-x)^{2/3}}{x}\int_{(1-x)^{1/3}}^1 \frac{3w^2(w+c)+2(1-w^3)}{(w+c)^3}dw \;\leq\; 1. \tag{5.14}$$

Call the integrand $B(w,c)$. We have

$$\frac{\partial}{\partial c}B(w,c) \;=\; -6\frac{1+cw^2}{(w+c)^4} \;\leq\; 0.$$

The left-hand side of (5.14) is therefore bounded above by

$$\frac{(1-x)^{2/3}}{x}\int_{(1-x)^{1/3}}^1 B(w,0)\;dw \;=\; \frac{(1-x)^{2/3}}{x}\int_{(1-x)^{1/3}}^1 1+\frac{2}{w^3}\;dw \;=\; 1.$$

as required. □

In consequence,

$$\inf_{x\in[0,1]} F(x) \;=\; F(1) \;=\; \frac{1}{3}\int_0^1 \frac{dt}{\left[(1-t)^{1/3}+\epsilon''\right]^2} - 1.$$

Let us summarize what we have shown so far: there exists $N_0(\epsilon)\in\mathbb{N}$ such that whenever $N\geq N_0$ we have

$$\begin{aligned}
&-\frac{2d^2}{3\pi^2\sigma^2}\limsup_{k\to\infty}\frac{1}{N^{1/3}k^{1/3}}\log\sup_{0\leq r\leq(N-N^{1/3})k} g_{r,Nk}\\
&\geq\; 1-N^{-1/3}-\left[1-\frac{1}{3}\int_0^1\frac{dt}{\left[(1+\epsilon-t)^{1/3}+\epsilon'\right]^2}\right]+\left[\frac{1}{3}\int_0^1\frac{dt}{\left[(1-t)^{1/3}+\epsilon''\right]^2}-1\right]\\
&=:\; \lambda_1(N,\epsilon).
\end{aligned} \tag{5.15}$$

It remains to estimate $g_{r,t}$ for values $r\in[(N-N^{1/3})k, Nk]$, which is easy. We start with the trivial inequalities

$$g_{r,Nk} \;\leq\; \mathbf{Q}\left(\zeta_s\in I_s(t)\;\;\forall s\in[0,r]\right) \;\leq\; \mathbf{Q}\left(\zeta_i\in I_i(Nk)\;\;\forall 1\leq i\leq(N-N^{1/3})k\right).$$

The probability on the right-hand side is equal to

$$\mathbf{Q}\Bigg(-d\left[\frac{N}{N-N^{1/3}} - \frac{i}{(N-N^{1/3})k}\right]^{1/3} \\ \leq \frac{\zeta_i}{(N-N^{1/3})^{1/3}k^{1/3}} \leq \epsilon\left[\frac{N}{N-N^{1/3}}\right]^{1/3}, \ \forall 1 \leq i \leq (N-N^{1/3})k\Bigg)$$

which in turn is bounded above by

$$\mathbf{Q}\Bigg(-d\left[1+\epsilon - \frac{i}{(N-N^{1/3})k}\right]^{1/3} \\ \leq \frac{\zeta_i}{(N-N^{1/3})^{1/3}k^{1/3}} \leq 2\epsilon, \ \forall 1 \leq i \leq (N-N^{1/3})k\Bigg)$$

for all $N \geq N_1(\epsilon)$, for some $N_1(\epsilon) \in \mathbb{N}$. Using Mogulskii's Theorem, we conclude that

$$\begin{aligned} \limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log \sup_{(N-N^{1/3})k \leq r \leq Nk} g_{r,Nk} \\ &\leq -\frac{3\pi^2\sigma^2}{2d^2} \cdot \frac{(N-N^{1/3})^{1/3}}{N^{1/3}} \cdot \frac{1}{3}\int_0^1 \frac{dt}{\left[(1+\epsilon-t)^{1/3}+\epsilon'\right]^2} \\ &=: -\frac{3\pi^2\sigma^2}{2d^2}\lambda_2(N,\epsilon) \end{aligned} \tag{5.16}$$

where, as before, $\epsilon' := 4\epsilon/d$.

Now we combine our estimates (5.15) and (5.16). Writing $N_2(\epsilon) = N_0 \vee N_1$, we conclude that whenever $N \geq N_2$, we have

$$\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log \sup_{0\leq r\leq Nk} g_{r,Nk} \leq -\frac{3\pi^2\sigma^2}{2d^2}\big(\lambda_1(N,\epsilon) \wedge \lambda_2(N,\epsilon)\big).$$

We now have all the ingredients required to complete the proof.

Step 6: Completing the argument. Combining Steps 3, 4 and 5 we deduce that for any $\epsilon > 0$ and any $N \geq N_2(\epsilon)$ we have

$$\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log \Lambda_{Nk} \leq 2(1+\overline{p})\epsilon + \lambda(N,\epsilon) - \frac{3\pi^2\sigma^2}{2d^2}\big(\lambda_1(N,\epsilon) \wedge \lambda_2(N,\epsilon)\big).$$

Our statement of the Paley-Zygmund inequality, (5.3), implies that

$$\liminf_{t\to\infty} \frac{1}{t^{1/3}} \log P_t \geq 2\liminf_{t\to\infty} \frac{1}{t^{1/3}} \log E_t \\ - \max\left\{\limsup_{t\to\infty} \frac{1}{t^{1/3}} \log E_t\,, \ \limsup_{t\to\infty} \frac{1}{t^{1/3}} \log \Lambda_t\right\}.$$

Applying this inequality along the sequence $(Nk : k \in \mathbb{N})$, and using the first display of this step along with the results of Step 2, we deduce that

$$\liminf_{k\to\infty} \frac{1}{(Nk)^{1/3}} \log P_{Nk}$$

is bounded from below by

$$2\left(-\frac{3\pi^2\sigma^2}{2d^2} + \beta_1(\epsilon)\right)$$
$$- \max\left\{-\frac{3\pi^2\sigma^2}{2d^2} + \beta_2(\epsilon)\,,\;\; 2(1+\overline{p})\epsilon + \lambda(N,\epsilon) - \frac{3\pi^2\sigma^2}{2d^2}\big(\lambda_1(N,\epsilon) \wedge \lambda_2(\epsilon)\big)\right\}$$

for any $\epsilon > 0$ and $N \geq N_2(\epsilon)$. Now we note that $\beta_1(\epsilon)$ and $\beta_2(\epsilon)$ are $o(1)$ as $\epsilon \downarrow 0$; that $\lim_{\epsilon\downarrow 0} \lim_{N\to\infty} \lambda(N,\epsilon) = 0$; and that $\lim_{\epsilon\downarrow 0} \lim_{N\to\infty} \lambda_i(N,\epsilon) = 1$ for $i = 1$ and 2. Consequently, the expression in the previous display goes to $-\frac{3\pi^2\sigma^2}{2d^2}$ upon sending $N \to \infty$ and then $\epsilon \downarrow 0$.

On the other hand, by the monotonicity of the function $t \mapsto P_t$,

$$\liminf_{t\to\infty} \frac{1}{t^{1/3}} \log P_t \;=\; \liminf_{k\to\infty} \frac{1}{(Nk)^{1/3}} \log P_{Nk} \qquad \text{for any } \; N \geq 1.$$

We deduce that

$$\liminf_{\epsilon\downarrow 0} \liminf_{t\to\infty} \frac{1}{t^{1/3}} \log P_t \;\geq\; -\frac{3\pi^2\sigma^2}{2d^2}.$$

The lower bound then follows from (5.2), and the fact that

$$\frac{3\pi^2\sigma^2}{2d^2} \;=\; \left(\frac{3\pi^2(1+\overline{p})^2|\Phi''(\overline{p})|}{2}\right)^{1/3},$$

completing the proof of Theorem 1.16. □

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