Stable process in a cone

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based on joint work with Victor Rivero and Weerapat Satitkanitkul (a.k.a. Pite)

Stable process

- For $d \geq 2$, let $X := (X_t : t \geq 0)$, with probabilities $\mathbb{P} = (\mathbb{P}_x, x \in \mathbb{R}^d)$, be a d-dimensional isotropic stable process of index $\alpha \in (0,2)$.
- Equivalently, this means $(X, \mathbb{P})$ is a d-dimensional Lévy process with characteristic exponent (up to a multiplicative constant)

$$\Psi(\theta) = |\theta|^\alpha, \qquad \theta \in \mathbb{R}^d.$$

- Equivalently X is a Lévy process for which there is an $\alpha \in (0,2)$ and which satisfies:

$$\text{under } \mathbb{P}_x, \text{ the law of } (cX_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P}_{cx},$$

for $c > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$.

- As a self-similar Markov process, X can be represented by the Lamperti-Kiu transformation

$$X_t = e^{\xi_{\varphi(t)}} \Theta_{\varphi(t)}, \qquad t \geq 0,$$

where (ξ, Θ) is a Markov additive process on $\mathbb{R} \times \Omega$ with probabilities

$$\mathbf{P}_{\log |x|, \arg(x)}, \qquad x \in \mathbb{R}^d,$$

and

$$\varphi(t) = \inf\{s > 0 : \int_0^s e^{\alpha \xi_u} du > t\}.$$

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Harmonic functions on the cone

- Lipchitz cone, $\Gamma = \{x \in \mathbb{R}^d : x \neq 0, \arg(x) \in \Omega\}$,
- Exit time from the cone i.e. $\kappa_\Gamma = \inf\{s > 0 : X_s \notin \Gamma\}$.
- Bañuelos and Bogdan (2004): There exists $M : \mathbb{R}^d \to \mathbb{R}$ such that
  - $M(x) = 0$ for all $x \notin \Gamma$.
  - M is locally bounded on $\mathbb{R}^d$
  - There is a $\beta = \beta(\Gamma, \alpha) \in (0, \alpha)$, such that

$$M(x) = |x|^\beta M(x/|x|) = |x|^\beta M(\arg(x)), \qquad x \neq 0.$$

  - Up to a multiplicative constant, M is the unique such that

$$M(x) = \mathbb{E}_x[M(X_{\tau_B})\mathbf{1}_{(\tau_B < \kappa_\Gamma)}], \qquad x \in \mathbb{R}^d,$$

    where B is any open bounded domain and $\tau_B = \inf\{t > 0 : X_t \notin B\}$.
- Bañuelos and Bogdan (2004) and Bogdan, Palmowski, Wang (2018): We have

$$\lim_{a \to 0} \sup_{x \in \Gamma,\, |t^{-1/\alpha} x| \leq a} \frac{\mathbb{P}_x(\kappa_\Gamma > t)}{M(x) t^{-\beta/\alpha}} = C,$$

where $C > 0$ is a constant.

Theorem

(i) *For any* $t > 0$, *and* $x \in \Gamma$,

$$\mathbb{P}_x^{\triangleleft}(A) := \lim_{s \to \infty} \mathbb{P}_x\left(A \,|\kappa_\Gamma > t + s\right), \qquad A \in \mathcal{F}_t,$$

defines a family of conservative probabilities on the space of càdlàg paths such that

$$\left.\frac{\mathrm{d}\mathbb{P}_x^{\triangleleft}}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} := \mathbf{1}_{(t<\kappa_\Gamma)} \frac{M(X_t)}{M(x)}, \qquad t \geq 0, \text{ and } x \in \Gamma.$$

In particular, the right-hand side above is a martingale.
(Note: this is nothing but an Esscher transform for the underlying MAP!)

(ii) *Let* $\mathbb{P}^{\triangleleft} := \left(\mathbb{P}_x^{\triangleleft}, x \in \Gamma\right)$. *The process* $(X, \mathbb{P}^{\triangleleft})$, *is a self-similar Markov process.*

Entrance law

Let $p_t^\Gamma(x,y)$, $x,y\in\Gamma$, $t\geq 0$, be the semigroup of X killed on exiting the cone Γ.

Theorem (Bogdan, Palmowski, Wang (2018))

The following limit exits,

$$n_t(y) := \lim_{\Gamma\ni x\to 0}\frac{p_t^\Gamma(x,y)}{\mathbb{P}_x(\kappa_\Gamma > t)t^{\beta/\alpha}}, \qquad x,y\in\Gamma, t>0, \tag{1}$$

and $(n_t(y)\mathrm{d}y, t>0)$, *serves as an entrance law to* $(X,\mathbb{P}^\Gamma)$, *in the sense that*

$$n_{t+s}(y) = \int_\Gamma n_t(x)p_s^\Gamma(x,y)\mathrm{d}x, \qquad y\in\Gamma, s,t\geq 0.$$

- Also easy to show that, in the sense of weak convergence,

$$\mathbb{P}_0^\triangleleft(X_t\in \mathrm{d}y) := \lim_{\Gamma\ni x\to 0}\frac{M(y)}{M(x)}\mathbb{P}_x(X_t\in\mathrm{d}y,\ t<\kappa_\Gamma) = CM(y)n_t(y)\mathrm{d}y.$$

- Can the process 'start from the apex of the cone' in a stronger sense?

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Continuous entrance at the apex of the cone

Theorem

The limit $\mathbb{P}_0^{\triangleleft} := \lim_{\Gamma \ni x \to 0} \mathbb{P}_x^{\triangleleft}$ *is well defined on the Skorokhod space, so that,* $(X, (\mathbb{P}_x^{\triangleleft}, x \in \Gamma \cup \{0\}))$ *is both Feller and self-similar which enters continuously at the origin, after which it never returns.*

Heuristic of proof

- **Step1:** construct the process condition to absorb continuously at the apex of the cone:

Theorem

For $A \in \mathcal{F}_t$, on the space of càdlàg paths with a cemetery state,

$$\mathbb{P}^{\triangleright}_x(A,\, t < \mathtt{k}^{\{0\}}) := \lim_{a \to 0} \mathbb{P}_x(A,\, t < \kappa_\Gamma \wedge \tau_a^{\oplus} | \tau_a^{\oplus} < \kappa_\Gamma),$$

is well defined as a stochastic process which is continuously absorbed at the apex of Γ, where $\mathtt{k}^{\{0\}} = \inf\{t > 0 : |X_t| = 0\}$ and $\tau_a^{\oplus} = \inf\{s > 0 : |X_s| < a\}$. Moreover, for $A \in \mathcal{F}_t$,

$$\mathbb{P}^{\triangleright}_x(A,\, t < \mathtt{k}^{\{0\}}) = \mathbb{E}_x\left[\mathbf{1}_{(A,\, t < \kappa_\Gamma)} \frac{H(X_t)}{H(x)}\right], \qquad t \geq 0,$$

where

$$H(x) = |x|^{\alpha - d} M(x/|x|^2) = |x|^{\alpha - \beta - d} M(\arg(x)).$$

- **Step 2:** Check that $(X, \mathbb{P}^{\triangleright})$ is dual to $(X, \mathbb{P}^{\triangleleft})$ in the Hunt-Nagasawa sense that

$$X_{(t - \mathtt{k}^{\{0\}})-}, \text{ for } t < \mathtt{k}^{\{0\}} \text{ under } \mathbb{P}^{\triangleright}$$

  has the same Markov transitions as $(X, \mathbb{P}^{\triangleleft})$ (this gives us continuous entrance at 0 of $\mathbb{P}^{\triangleleft}$).
- **Step 3**: Control the convergence of $(X, \mathbb{P}^{\triangleleft})$ as $\Gamma \ni X_0 \to 0$ by controlling an appropriate functional e.g. $X_{\tau_1^{\ominus}}$, where $\tau_1^{\ominus} = \inf\{t > 0 : |X_t| > 1\}$.

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Where is the work?

- Step 1 needs
$$\lim_{\Gamma \ni a \to 0} \frac{\mathbb{P}_x(\tau_a^{\oplus} < \kappa_\Gamma)}{H(x) a^{d+\beta-\alpha}} = C \in (0, \infty),$$
where, $H(x) = |x|^{\alpha-\beta-d} M(\arg(x))$.
- Which in turn needs the stability (distributional convergence) as $a \to 0$ of the distribution of $(X_{\tau_a^{\oplus}}, \mathbb{P}^{\triangleleft})$.
- This is the same as the stability as $z \to -\infty$ of $(\xi_{\tau_z^-}, \Theta_{\tau_z^-})$ under $\mathbf{P}^{\triangleleft}$, where $\tau_z^- = \inf\{t > 0 : \xi_t < z\}$, where $((\xi, \Theta), \mathbf{P}^{\triangleleft})$ is the MAP representation of $(X, \mathbb{P}^{\triangleleft})$.

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- For Step 2:

Theorem

Consider again the transformation of space via the sphere inversion $Kx = x/|x|^2$, $x \in \mathbb{R}^d$.

(i) *The process $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}_x^{\triangleleft}$, $x \in \Gamma$, is equal in law to $(X_t, t < \mathtt{k}^{\{0\}})$ under $\mathbb{P}_x^{\triangleright}$, $x \in \Gamma$, where*

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0. \tag{2}$$

and $\mathtt{k}^{\{0\}} = \inf\{t > 0 : X_t = 0\}$.

(ii) *Under $\mathbb{P}_0^{\triangleleft}$, the time reversed process*

$$\overleftarrow{X}_t := X_{(\mathtt{k}-t)-}, \qquad t \leq \mathtt{k},$$

is a homogenous strong Markov process whose transitions agree with those of $(X, \mathbb{P}_x^{\triangleright})$, $x \in \Gamma$, where $\mathtt{k}$ is an L-time of $(X, \mathbb{P}_x^{\triangleleft})$, $x \in \Gamma \cup \{0\}$.

- Hence stability of $X_{\tau_a^{\oplus}}$ as $a \to 0$ translates to the the stability (distributional convergence) as $a \to \infty$ of the distribution of $X_{\tau_a^{\ominus}}$ where $\tau_a^{\ominus} = \inf\{t > 0 : |X_t| > a\}$. This is the same as the stability as $z \to +\infty$ of

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Where is the work?

- So the main difficulty in the proof all boils down showing the stability of

$$|X_{\tau_a^\ominus}| = \exp(\xi_{\tau^+_{\log a}}) \qquad \text{and} \qquad \arg(X_{\tau_a^\ominus}) = \Theta_{\tau^+_{\log a}}$$

as $a \to \infty$, where

$$\tau_a^\ominus = \inf\{t > 0 : |X_t| > a\} \qquad \text{and} \qquad \tau^+_{\log a} = \inf\{t > 0 : \xi_t > \log a\}.$$

MAPs are like Lévy processes

- The radial first passage occurs in the range of

$$e^{\xi_t}\Theta_t, \qquad \text{for } t \text{ such that } \overline{\xi}_t - \xi_t = 0$$

where $\overline{\xi}_t = \sup_{u\le t}\xi_u$.

- (Just like the story of the Wiener-Hopf factorisation for Lévy processes) there is a MAP (H^+_t, Θ^+_t), $t \ge 0$, such that H^+_t has non-decreasing paths (a MAP subordinator) such that

$$\texttt{range}(e^{H^+_t}\Theta^+_t : t \ge 0) = \texttt{range}(e^{\xi_t}\Theta_t : \overline{\xi}_t - \xi_t = 0, t \ge 0)$$

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We need some MAP excursion theory

- Distributional convergence of

$$X_{\tau_a^{\ominus}}/a \qquad a \to \infty$$

agrees with that of

$$\mathrm{e}^{H^+_{T_b} - b}\Theta^+_{T_b} \qquad b = \log a \to \infty,$$

where

$$T_b = \inf\{t > 0 : H^+_t > b\}$$

- We are almost back to renewal theory, were it not for the MAP nature of (H^+, Θ^+).
- BIG PROBLEM: We have very little understanding of how these two processes (the new radial maxima and the angular positioning at new radial maxima) are corollated!

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Markov additive renewal theory to the rescue

- Classical work of Gerold Alsmeyer (and others before him) hold the key to the convergence of $(H^+_{T_b} - b, \Theta^+_{T_b})$ as $b \to \infty$
- Formally speaking, we can write

$$\mathbf{E}^{\triangleleft}_{0,\phi}[f(H^+_{T_b} - b, \Theta^+_{T_b})] = \int_0^b \int_\Omega U_\phi(\mathrm{d}z, \mathrm{d}\theta)\mathbb{N}^{\triangleleft}_\theta\Big(f(\epsilon(\zeta) - (b-z), \Theta^\epsilon(\zeta)); \epsilon(\zeta) > b - z\Big)$$

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for $z \geq 0$ and $\theta \in \Omega$, and $(\mathbb{N}^{\triangleleft}_\theta, \theta \in \Omega)$ is a family of excursion measures on the canonical space of MAP excursions $((\epsilon(t), \Theta^\epsilon(t)), t \leq \zeta)$.

- Just as with classical renewal theory we can take limits of the above convolution and expect

$$\lim_{b\to\infty} \mathbf{E}^{\triangleleft}_{0,\phi}[f(H^+_{T_b} - b, \Theta^+_{T_b})] = \int_0^\infty \int_\Omega \pi(\mathrm{d}\theta)\mathrm{d}r\mathbb{N}^{\triangleleft}_\theta\Big(f(\epsilon(\zeta) - r, \Theta^\epsilon(\zeta)); \epsilon(\zeta) > r\Big),$$

PROVIDING: $\exists\pi(\mathrm{d}\theta) := \lim_{t\to\infty} \mathbf{P}^{\triangleleft}_{0,\phi}(\Theta^+_t \in \mathrm{d}\theta)$

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Making it an easier question

- Under $\mathbb{P}^{\triangleleft}_x$, define the following sequence of stopping times,

$$T_n := \inf\{t > T_{n-1} : |X_t| > \mathrm{e}|X_{T_{n-1}}|\}, \qquad n \geq 1,$$

with $T_0 = 0$, and

$$S_n = \sum_{k=1}^{n} A_k \qquad A_n = \log \frac{|X_{T_n}|}{|X_{T_{n-1}}|} \quad \text{and} \quad \Xi_n = \arg(X_{T_n}), \qquad n \geq 1.$$

- Note in particular that

$$X_{T_n} = |x|\mathrm{e}^{S_n}\Xi_n, \qquad n \geq 1.$$

and that $((S_n, \Xi_n), n \geq 0)$, is a Markov additive renewal process.

- Defining $V_\phi(\mathrm{d}r, \mathrm{d}\theta) := \sum_{n=0}^{\infty} \mathbb{P}^{\triangleleft}_\phi(S_n \in \mathrm{d}r, \Xi_n \in \mathrm{d}\theta)$, $r \in \mathbb{R}, \phi \in \Omega$,

$$\mathbb{E}^{\triangleleft}_x\left[f(X_{\tau_1^\ominus})\right] = \int_0^{-\log|x|}\int_\Omega V_{\arg(x)}(\mathrm{d}r, \mathrm{d}\theta) G(-\log|x| - r, \theta),$$

where, for $\phi \in \Omega$ and $y \geq 0$, $G(y,\theta) := \mathbb{E}^{\triangleleft}_{\mathrm{e}^{-y}\theta}\left[f(X_{\tau_1^\ominus})\mathbf{1}_{(\tau_1^\ominus \leq \tau^\ominus_{\mathrm{e}^{1-y}})}\right]$.

Ignore the purple crap again.

- Alsmeyer's Markov additive renewal Theorem now only means we need to find $\upsilon(\mathrm{d}\theta) = \lim_{n\to\infty} \mathbf{P}^{\triangleleft}_{0,\phi}(\Xi_n \in \mathrm{d}\theta)$.

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$$T_n := \inf\{t > T_{n-1} : |X_t| > \mathrm{e}|X_{T_{n-1}}|\}, \qquad n \geq 1,$$

with $T_0 = 0$, and

$$S_n = \sum_{k=1}^{n} A_k \qquad A_n = \log \frac{|X_{T_n}|}{|X_{T_{n-1}}|} \quad \text{and} \quad \Xi_n = \arg(X_{T_n}), \qquad n \geq 1.$$

- Note in particular that

$$X_{T_n} = |x|\mathrm{e}^{S_n}\Xi_n, \qquad n \geq 1.$$

and that $((S_n, \Xi_n), n \geq 0)$, is a Markov additive renewal process.

- Defining $V_\phi(\mathrm{d}r, \mathrm{d}\theta) := \sum_{n=0}^{\infty} \mathbb{P}^{\triangleleft}_\phi(S_n \in \mathrm{d}r, \Xi_n \in \mathrm{d}\theta)$, $r \in \mathbb{R}, \phi \in \Omega$,

$$\mathbb{E}^{\triangleleft}_x\left[f(X_{\tau_1^{\ominus}})\right] = \int_0^{-\log|x|} \int_\Omega V_{\arg(x)}(\mathrm{d}r, \mathrm{d}\theta) G(-\log|x| - r, \theta),$$

where, for $\phi \in \Omega$ and $y \geq 0$, $G(y, \theta) := \mathbb{E}^{\triangleleft}_{\mathrm{e}^{-y}\theta}\left[f(X_{\tau_1^{\ominus}})\mathbf{1}_{(\tau_1^{\ominus} \leq \tau^{\ominus}_{\mathrm{e}^{1-y}})}\right]$.

Ignore the purple crap again.

- Alsmeyer's Markov additive renewal Theorem now only means we need to find $\upsilon(\mathrm{d}\theta) = \lim_{n\to\infty} \mathbf{P}^{\triangleleft}_{0,\phi}(\Xi_n \in \mathrm{d}\theta)$.

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WE NEED TO FIND υ

- This only appears to have transferred the problem of the existence of π to the existence of υ
- We are going to use the theory of 'Harris recurrence' to check the condition that there exists a probability measure, $\rho(\cdot)$ on $\mathcal{B}(\Omega)$ (Borel sets in Ω) such that, for some $\lambda > 0$,

$$\mathbb{P}^{\triangleleft}_{\theta}(\Xi_1 \in E) \geq \lambda\rho(E), \text{ for all } \theta \in \Omega, E \in \mathcal{B}(\Omega),$$

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- The function

$$g(x;E) := \mathbb{E}_x\left[M(X_{\tau_1^\ominus})\mathbf{1}_{(\arg(X_{\tau_1^\ominus})\in E,\, \tau_1^\ominus<\kappa_\Gamma)}\right],$$

for $x \in \Gamma$ such that $|x| < 1$ is a regular harmonic function and note that, by scaling

$$g(\theta/\mathrm{e};E) = \mathbb{E}_\theta\left[M(X_{T_1})\mathbf{1}_{(\Xi_1\in E,\, T_1<\kappa_\Gamma)}\right], \qquad \theta \in \Omega.$$

- The function $M(x)$ is similarly regular harmonic.
- Hence, fix θ_0 with $|\theta_0| = 1$ so that $M(\theta_0/\mathrm{e}) = 1$ and then thanks to Bogdan's Harnack inequality we have, for $x \in \Gamma$ such that $|x| < 1/2$,

$$C^{-1}M(x) \leq \frac{g(x;E)}{g(\theta_0/\mathrm{e};E)} \leq CM(x)$$

for a universal constant C which does not depend on E, x or x_0.

- Rearranging gives us for $x = \theta/\mathrm{e}$

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Thank you!