# Stable process in a cone

Andreas Kyprianou

based on joint work with Victor Rivero and Weerapat Satitkanitkul (a.k.a. Pite)

# Stable process

- For $d \geq 2$, let $X := (X_t : t \geq 0)$, with probabilities $\mathbb{P} = (\mathbb{P}_x, x \in \mathbb{R}^d)$, be a $d$-dimensional isotropic stable process of index $\alpha \in (0,2)$.
- Equivalently, this means $(X, \mathbb{P})$ is a $d$-dimensional Lévy process with characteristic exponent (up to a multiplicative constant)

$$\Psi(\theta) = |\theta|^\alpha, \qquad \theta \in \mathbb{R}^d.$$

- Equivalently $X$ is a Lévy process for which there is an $\alpha \in (0,2)$ and which satisfies:

$$\text{under } \mathbb{P}_x, \text{ the law of } (cX_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P}_{cx},$$

  for $c > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$.
- As a self-similar Markov process, $X$ can be represented by the Lamperti-Kiu transformation

$$X_t = \mathrm{e}^{\xi_{\varphi(t)}} \Theta_{\varphi(t)}, \qquad t \geq 0,$$

  where $(\xi, \Theta)$ is a Markov additive process on $\mathbb{R} \times \Omega$ with probabilities

$$\mathbf{P}_{\log|x|, \arg(x)}, \qquad x \in \mathbb{R}^d,$$

  and

$$\varphi(t) = \inf\{s > 0 : \int_0^s \mathrm{e}^{\alpha \xi_u} \mathrm{d}u > t\}.$$

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# Harmonic functions on the cone

- Lipchitz cone, $\Gamma = \{x \in \mathbb{R}^d : x \neq 0, \arg(x) \in \Omega\}$,
- Exit time from the cone i.e. $\kappa_\Gamma = \inf\{s > 0 : X_s \notin \Gamma\}$.
- Bañuelos and Bogdan (2004): There exists $M : \mathbb{R}^d \to \mathbb{R}$ such that
  - $M(x) = 0$ for all $x \notin \Gamma$.
  - $M$ is locally bounded on $\mathbb{R}^d$
  - There is a $\beta = \beta(\Gamma, \alpha) \in (0, \alpha)$, such that

    $$M(x) = |x|^\beta M(x/|x|) = |x|^\beta M(\arg(x)), \qquad x \neq 0.$$

  - Up to a multiplicative constant, $M$ is the unique such that

    $$M(x) = \mathbb{E}_x[M(X_{\tau_B})\mathbf{1}_{(\tau_B < \kappa_\Gamma)}], \qquad x \in \mathbb{R}^d,$$

    where $B$ is any open bounded domain and $\tau_B = \inf\{t > 0 : X_t \notin B\}$.
- Bañuelos and Bogdan (2004) and Bogdan, Palmowski, Wang (2018): We have

$$\lim_{a \to 0} \sup_{x \in \Gamma,\, |t^{-1/\alpha} x| \leq a} \frac{\mathbb{P}_x(\kappa_\Gamma > t)}{M(x) t^{-\beta/\alpha}} = C,$$

where $C > 0$ is a constant.

## Theorem

(i) *For any $t > 0$, and $x \in \Gamma$,*

$$\mathbb{P}_x^{\triangleleft}(A) := \lim_{s \to \infty} \mathbb{P}_x\left(A \,|\, \kappa_\Gamma > t + s\right), \qquad A \in \mathcal{F}_t,$$

*defines a family of conservative probabilities on the space of càdlàg paths such that*

$$\left.\frac{\mathrm{d}\mathbb{P}_x^{\triangleleft}}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} := \mathbf{1}_{(t<\kappa_\Gamma)} \frac{M(X_t)}{M(x)}, \qquad t \geq 0, \text{ and } x \in \Gamma. \tag{1}$$

*In particular, the right-hand side of (1) is a martingale.*
*(Note: this is nothing but an Esscher transform!)*

(ii) *Let $\mathbb{P}^{\triangleleft} := (\mathbb{P}_x^{\triangleleft}, x \in \Gamma)$. The process $(X, \mathbb{P}^{\triangleleft})$, is a self-similar Markov process.*

## Entrance law

Let $p_t^\Gamma(x,y)$, $x,y\in\Gamma$, $t\geq 0$, be the semigroup of $X$ killed on exiting the cone $\Gamma$.

### Theorem (Bogdan, Palmowski, Wang (2018))

*The following limit exits,*

$$n_t(y) := \lim_{\Gamma\ni x\to 0} \frac{p_t^\Gamma(x,y)}{\mathbb{P}_x(\kappa_\Gamma > t)t^{\beta/\alpha}}, \qquad x,y\in\Gamma, t>0, \tag{2}$$

*and* $(n_t(y)\mathrm{d}y, t>0)$*, serves as an entrance law to* $(\mathrm{X},\mathbb{P}^\Gamma)$*, in the sense that*

$$n_{t+s}(y) = \int_\Gamma n_t(x)p_s^\Gamma(x,y)\mathrm{d}x, \qquad y\in\Gamma, s,t\geq 0.$$

- Also easy to show that, in the sense of weak convergence,

$$\mathbb{P}_0^\triangleleft(X_t\in \mathrm{d}y) := \lim_{\Gamma\ni x\to 0}\frac{M(y)}{M(x)}\mathbb{P}_x(X_t\in\mathrm{d}y,\ t<\kappa_\Gamma) = CM(y)n_t(y)\mathrm{d}y.$$

- Can the process 'start from the apex of the cone' in a stronger sense?

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# Continuous entrance at the apex of the cone

### Theorem

*The limit* $\mathbb{P}_0^{\triangleleft} := \lim_{\Gamma \ni x \to 0} \mathbb{P}_x^{\triangleleft}$ *is well defined on the Skorokhod space, so that,* $(X, (\mathbb{P}_x^{\triangleleft}, x \in \Gamma \cup \{0\}))$ *is both Feller and self-similar which enters continuously at the origin, after which it never returns.*

## HEURISTIC OF PROOF

- **Step1:** construct the process condition to absorb continuously at the apex of the cone:

### Theorem

*For $A \in \mathcal{F}_t$, on the space of càdlàg paths with a cemetery state,*

$$\mathbb{P}^{\triangleright}_x(A,\, t < k^{\{0\}}) := \lim_{a \to 0} \mathbb{P}_x(A,\, t < \kappa_\Gamma \wedge \tau^{\oplus}_a | \tau^{\oplus}_a < \kappa_\Gamma),$$

*is well defined as a stochastic process which is continuously absorbed at the apex of $\Gamma$, where $k^{\{0\}} = \inf\{t > 0 : |X_t| = 0\}$ and $\tau^{\oplus}_a = \inf\{s > 0 : |X_s| < a\}$. Moreover, for $A \in \mathcal{F}_t$,*

$$\mathbb{P}^{\triangleright}_x(A,\, t < k^{\{0\}}) = \mathbb{E}_x \left[ \mathbf{1}_{(A,\, t<\kappa_\Gamma)} \frac{H(X_t)}{H(x)} \right], \qquad t \geq 0,$$

*where*

$$H(x) = |x|^{\alpha - d} M(x/|x|^2) = |x|^{\alpha - \beta - d} M(\arg(x)).$$

- **Step 2:** Check that $(X, \mathbb{P}^{\triangleright})$ is dual to $(X, \mathbb{P}^{\triangleleft})$ in the Hunt-Nagasawa sense that

$$X_{(t - k^{\{0\}})-}, \text{ for } t < k^{\{0\}} \text{ under } \mathbb{P}^{\triangleright}$$

has the same Markov transitions as $(X, \mathbb{P}^{\triangleleft})$

- **Step 3**: Control the convergence of $(X, \mathbb{P}^{\triangleleft})$ as $\Gamma \ni X_0 \to 0$ by controlling an appropriate functional e.g. $X_{\tau^{\ominus}_1}$, where $\tau^{\ominus}_1 = \inf\{t > 0 : |X_t| > 1\}$.

## Heuristic of proof

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# Where is the work?

- Step 1 needs
$$\lim_{\Gamma \ni a \to 0} \frac{\mathbb{P}_x(\tau_a^{\oplus} < \kappa_\Gamma)}{H(x) a^{d+\beta-\alpha}} = C \in (0, \infty),$$
where, $H(x) = |x|^{\alpha-\beta-d} M(\arg(x))$.
- Which in turn needs the stability (distributional convergence) as $a \to 0$ of the distribution of $(X_{\tau_a^{\oplus}}, \mathbb{P}^{\triangleleft})$.
- This is the same as the stability as $z \to -\infty$ of $(\xi_{\tau_z^-}, \Theta_{\tau_z^-})$ under $\mathbf{P}^{\triangleleft}$, where $\tau_z^- = \inf\{t > 0 : \xi_t < z\}$, where $((\xi, \Theta), \mathbf{P}^{\triangleleft})$ is the MAP representation of $(X, \mathbb{P}^{\triangleleft})$.

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## Where is the work?

- For Step 2:

**Theorem**

*Consider again the transformation of space via the sphere inversion $Kx = x/|x|^2$, $x \in \mathbb{R}^d$.*

(i) *The process $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}^{\triangleleft}_x$, $x \in \Gamma$, is equal in law to $(X_t, t < k^{\{0\}})$ under $\mathbb{P}^{\triangleright}_x$, $x \in \Gamma$, where*

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0. \tag{3}$$

*and $k^{\{0\}} = \inf\{t > 0 : X_t = 0\}$.*

(ii) *Under $\mathbb{P}^{\triangleleft}_0$, the time reversed process*

$$\overleftarrow{X}_t := X_{(k-t)-}, \qquad t \leq k,$$

*is a homogenous strong Markov process whose transitions agree with those of $(X, \mathbb{P}^{\triangleright}_x)$, $x \in \Gamma$, where $k$ is an L-time of $(X, \mathbb{P}^{\triangleleft}_x)$, $x \in \Gamma \cup \{0\}$.*

- Hence stability of $X_{\tau_a^{\oplus}}$ as $a \to 0$ translates to the the stability (distributional convergence) as $a \to \infty$ of the distribution of $X_{\tau_a^{\ominus}}$ where $\tau_a^{\ominus} = \inf\{t > 0 : |X_t| > a\}$. This is the same as the stability as $z \to +\infty$ of

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- So the main difficulty in the proof all boils down a more advanced form of the following theorem:

## Theorem

*Under $\mathbf{P}^{\triangleleft}$, the modulator process $\Theta^{+}$ of the descending ladder MAP of $(\xi, \Theta)$ has a stationary distribution. That is*

$$\pi^{\triangleleft,+}(\mathrm{d}\theta) := \lim_{t\to\infty} \mathbf{P}^{\triangleleft}_{x,\theta}(\Theta^{+}_{t} \in \mathrm{d}\theta), \qquad \theta \in \Omega, x \in \mathbb{R},$$

*exists as a non-degenerate distributional weak limit.*

Thank you!