Censored stable processes

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Stable processes

Definition I

A Lévy process X is called α-stable if it satisfies the scaling property

$$\left(cX_{c^{-\alpha}t}\right)_{t\geq 0}\Big|_{\mathrm{P}_x} \stackrel{d}{=} X|_{\mathrm{P}_{cx}}, \quad c>0.$$

Necessarily $\alpha \in (0,2]$. [$\alpha = 2 \to$ BM, exclude this.]
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Definition II

Let α, ρ be admissible parameters, X the Lévy process with Lévy density

$$c_+ x^{-(\alpha+1)} \mathbb{1}_{(x>0)} + c_- |x|^{-(\alpha+1)} \mathbb{1}_{(x<0)}, \qquad x \in \mathbb{R},$$

no Gaussian part.

Stable processes

Two specific points:

- Assume X does not have one-sided jumps,
- When $\alpha = 1$, X is symmetric.

Problem statement

The problem

Let

$$\tau_{-1}^{1} = \inf\{t > 0 : X_t \in (-1, 1)\}$$

be the first hitting time of $(-1, 1)$.

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Theorem (B-G-R)

Let $x > 1$. Then, when $\alpha \in (0,1]$,

$$\begin{aligned} &\mathrm{P}_x(X_{\tau^1_{-1}} \in \mathrm{d}y,\ \tau^1_{-1} < \infty)/\mathrm{d}y \\ &\qquad = \frac{\sin(\pi\alpha/2)}{\pi}(x^2-1)^{\alpha/2}(1-y^2)^{-\alpha/2}(x-y)^{-1} \end{aligned}$$

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for $y \in (-1, 1)$.

Positive, self-similar Markov processes

α-pssMp

$[0,\infty)$-valued Markov process,
equipped with initial measures P_x, $x > 0$,
with 0 an absorbing state,
satisfying the scaling property

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Lamperti transform

$(X, \mathrm{P}_x)_{x>0}$ pssMp $\quad\leftrightarrow\quad$ $(\xi, \mathbb{P}_y)_{y\in\mathbb{R}}$ killed Lévy

$$X_t = \exp(\xi_{S(t)}),$$

S a random time-change

$$\xi_s = \log(X_{T(s)}),$$

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$$\left.\begin{array}{c} X \text{ never hits zero} \\ X \text{ hits zero continuously} \\ X \text{ hits zero by a jump} \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \xi \to \infty \text{ or } \xi \text{ oscillates} \\ \xi \to -\infty \\ \xi \text{ is killed} \end{array}\right.$$

Lamperti-stable processes

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Let X be a stable process, and define

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Then X^* is a pssMp, with Lamperti transform ξ^*.
ξ^* has Lévy density

$$c_+ \frac{e^x}{(e^x - 1)^{\alpha+1}} \mathbb{1}_{(x>0)} + c_- \frac{e^x}{(1 - e^x)^{\alpha+1}} \mathbb{1}_{(x<0)},$$

and is killed at rate $c_-/\alpha = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}$.

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- Let $A_t = \int_0^t \mathbb{1}_{(X_t>0)}\, \mathrm{d}t$.
- Let γ be the right-inverse of A, and put $\check{Y}_t := X_{\gamma(t)}$.
- Finally, make zero an absorbing state (needed in the case $\alpha \in (1,2)$): $Y_t = \check{Y}_t \mathbb{1}_{(t<T_0)}$.
  This is the censored stable process.

The Lamperti transform of Y and its structure

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Theorem

$\xi \overset{d}{=} \xi^{\mathsf{L}} + \xi^{\mathsf{C}}$ (independent sum), with

- *ξ^{L} equal in law to ξ^* with the killing removed,*
- *ξ^{C} a compound Poisson process with jump rate c_-/α.*

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By diagram.

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Proof.

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Tricky element – show Δ independent of ξ^{L}.
Lamperti: $\Delta \leftrightarrow \frac{X_\sigma}{X_{\tau-}}$. By Markov property, reduces to showing $\mathrm{P}_x\left(\frac{X_\sigma}{X_{\tau-}} \in \cdot\right)$ does not depend on x and this follows by scaling. $\square$

Wiener-Hopf factorisation

Recall: Wiener-Hopf factorisation

Let ξ be a Lévy process, $\mathbb{E}\left[e^{\mathrm{i}\theta\xi_1}\right] = e^{-\Psi(\theta)}$.
Then there exist κ, $\hat{\kappa}$, such that:

$$\Psi(\theta) = \kappa(-\mathrm{i}\theta)\hat{\kappa}(\mathrm{i}\theta),$$

κ and $\hat{\kappa}$ Laplace exponents of increasing, possibly killed Lévy processes (subordinators) H and $\hat{H}$:

$$\mathbb{E}\left[e^{-\lambda H_1}\right] = e^{-\kappa(\lambda)},\ \mathbb{E}\left[e^{-\lambda \hat{H}_1}\right] = e^{-\hat{\kappa}(\lambda)}, \qquad \lambda \geq 0.$$

- unique
- H and $\hat{H}$ related to maxima and minima of ξ: *ascending* and *descending ladder processes*.

Wiener-Hopf factorisation for ξ: $\alpha \in (0,1]$

WHF for $\alpha \in (0,1]$

$$\kappa(\lambda) = \frac{\Gamma(\alpha\rho + \lambda)}{\Gamma(\lambda)}, \qquad \hat{\kappa}(\lambda) = \frac{\Gamma(1 - \alpha\rho + \lambda)}{\Gamma(1 - \alpha + \lambda)}, \qquad \lambda \geq 0.$$

H: Lamperti-stable subordinator with parameters $(\alpha\rho, 1)$, i.e. pure jump subordinator with Lévy density $e^x/(e^x - 1)^{\alpha\rho}$
$\hat{H}$: (killed) Lamperti-stable subordinator with parameters $(\alpha\hat{\rho}, \alpha)$.

Lamperti-stable subordinators are nice! We can calculate:

- The Lévy measure of ξ,
- The Lévy measures of H and $\hat{H}$,
- The renewal measures, $\mathbb{E} \int_0^\infty \mathbb{1}_{(H_t \in \cdot)} \, \mathrm{d}t$ and $\mathbb{E} \int_0^\infty \mathbb{1}_{(\hat{H}_t \in \cdot)} \, \mathrm{d}t$.

Wiener-Hopf factorisation for ξ: $\alpha \in (1,2)$

WHF for $\alpha \in (1,2)$

$$\kappa(\lambda) = (\alpha - 1 + \lambda)\frac{\Gamma(\alpha\rho + \lambda)}{\Gamma(1+\lambda)}, \qquad \hat{\kappa}(\lambda) = \lambda\frac{\Gamma(1-\alpha\rho+\lambda)}{\Gamma(2-\alpha+\lambda)},$$

for $\lambda \geq 0$.

- $\kappa(\lambda) = \frac{\lambda}{\mathcal{T}_{\alpha-1}\psi(\lambda)}$, with ψ LSS$(1-\alpha\rho, \alpha\hat{\rho})$.
- $\hat{\kappa}(\lambda) = \frac{\lambda}{\phi(\lambda)}$, with ϕ LSS$(1-\alpha\hat{\rho}, \alpha\rho)$.

Not as nice, but we can still calculate Lévy measures and renewal measures.

Results

Recall: the problem

Let X be a stable process and $x > 1$.

$$\mathrm{P}_x\left(X_{\tau_{-1}^1} \in \mathrm{d}y,\ \tau_{-1}^1 < \infty\right) = \text{what?}$$

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As stable processes are self-similar and have stationary and independent increments, we can shift-and scale and reduce the probability of interest to:

$$\mathrm{P}_1\left(X_{\tau_0^b} \in \mathrm{d}z, \tau_0^b < \infty\right), \qquad 0 < b < 1.$$

where $\tau_0^b = \inf\{t > 0 : X_t \in (0, b)\}$.

Results

Key fact 1: $\mathrm{P}_1\left(X_{\tau_0^b} \in \mathrm{d}z, \tau_0^b < \infty\right) = \mathrm{P}_1\left(Y_{\eta_0^b} \in \mathrm{d}z, \eta_0^b < \infty\right)$
where $\eta_0^b = \inf\{t > 0 : Y_t \in [0, b)\}$.

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Recall: Lamperti transform

$$Y_t = \exp(\xi_{S(t)}), \quad \text{and} \quad \xi_s = \log Y_{T(s)},$$

where S, T are random, mutually inverse time-changes.

Key fact 2: $(0, b)$ for Y corresponds to $(-\infty, \log b)$ for ξ and η_0^b corresponds to $S_a^- = \inf\{s > 0 : \xi_s < \log b\}$. Then,

$$Y_{\eta_0^b} = \exp\big(\xi_{S^-_{\log b}}\big).$$

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Results

So now we are looking for $\mathbb{P}(\xi_{S_a^-} \in \mathrm{d}w,\ S_a^- < \infty)$, for $a < 0$.

Method for $\alpha \in (0,1]$

Use the ladder process:

$$\begin{aligned}\mathbb{P}(\xi_{S_a^-} \in \mathrm{d}w,\ S_a^- < \infty) &= \mathbb{P}(\underline{\xi}_{S_a^-} \in \mathrm{d}w,\ S_a^- < \infty) \\ &= \mathbb{P}(-\hat{H}_{S_{-a}^+} \in \mathrm{d}w) \\ &= \int_{[0,-a]} \hat{U}(\mathrm{d}z)\Pi_{\hat{H}}(-\mathrm{d}w - z),\end{aligned}$$

recalling that $-\hat{H}$ is a time-change of the running minimum $\underline{\xi}$.

Results

So now we are looking for $\mathbb{P}(\xi_{S_a^-} \in \mathrm{d}w,\ S_a^- < \infty)$, for $a < 0$.

Method for $\alpha \in (1,2)$

Use the Pecherskii-Rogozin identity:

$$\int_0^\infty \int \exp(qa - \beta(a - \xi_{S_a^-}))\,\mathrm{d}\mathbb{P}\,\mathrm{d}a = \frac{\hat{\kappa}(q) - \hat{\kappa}(\beta)}{(q-\beta)\hat{\kappa}(q)},$$

for $a < 0$, $q, \beta > 0$.

The theorem

Theorem

Let $x > 1$. Then, when $\alpha \in (0, 1]$,

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$$= \frac{\sin(\pi\alpha\hat{\rho})}{\pi}(x+1)^{\alpha\rho}(x-1)^{\alpha\hat{\rho}}(1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat{\rho}}(x-y)^{-1},$$

for $y \in (-1, 1)$.

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&\qquad - (\alpha-1)\frac{\sin(\pi\alpha\hat{\rho})}{\pi}(1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat{\rho}} \\
&\qquad\qquad \times \int_1^x (t-1)^{\alpha\hat{\rho}-1}(t+1)^{\alpha\rho-1}\,\mathrm{d}t,
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for $y \in (-1,1)$.

Robustness

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The expected occupation measure for X of $(-1,1)^c$ until hitting $(-1,1)$,

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When $\alpha \in (1,2)$, the law of first entry into $(1,\infty)$ of X on avoiding the origin,

$$\mathrm{P}_x(X_{\tau_1^+} \in \mathrm{d}u,\ \tau_1^+ < \tau_0), \qquad x \leq 1,$$

where $\tau_1^+ = \inf\{t > 0 : X_t > 1\}$.