Exploration of $\mathbb{R}^d$ by the isotropic α-stable process

Andreas Kyprianou
Based on joint work with V. Rivero and W. Satitkanitkul

A more thorough set of lecture notes can be found here:
`https://arxiv.org/abs/1707.04343`
Other related material found here
`https://arxiv.org/abs/1511.06356`
`https://arxiv.org/abs/1511.06356`
`https://arxiv.org/abs/1706.09924`

Main objectives of mini-course

To review the theory $\mathbb{R}^d$-valued stable processes in light of a number of recent developments

- Theory of self-similar Markov processes
- Radial fluctuation theory
- Space-time transformations (Riesz–Bogdan–Żak transform)
- Connections with classical potential analysis

§1. Quick review of Lévy processes

(KILLED) LÉVY PROCESS

Fundamentally we are going to spend a lot of time talking about Lévy processes in one and higher dimensions. But it is worth us briefly reminding ourselves about a few facts:

- $(\xi_t, t \geq 0)$ is a (killed) Lévy process if it has stationary and independents with RCLL paths (and is sent to a cemetery state after and independent and exponentially distributed time).
- Process is entirely characterised by its one-dimensional transitions, which are coded by the Lévy–Khinchine formula:

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\theta\cdot\xi_t}] = \mathrm{e}^{-\Psi(\theta)t}, \qquad \theta \in \mathbb{R}^d,$$

  where,

$$\Psi(\theta) = q + \mathrm{i}\mathrm{a}\cdot\theta + \frac{1}{2}\theta\cdot\mathbf{A}\theta + \int_{\mathbb{R}^d}(1 - \mathrm{e}^{\mathrm{i}\theta\cdot x} + \mathrm{i}(\theta\cdot x)\mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x),$$

  where $\mathrm{a} \in \mathbb{R}$, $\mathbf{A}$ is a $d \times d$ Gaussian covariance matrix and Π is a measure satisfying $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\Pi(\mathrm{d}x) < \infty$. Think of Π as the intensity of jumps in the sense of

$$\mathbb{P}(X \text{ has jump at time } t \text{ of size } \mathrm{d}x) = \Pi(\mathrm{d}x)\mathrm{d}t + o(\mathrm{d}t).$$

- In one dimension the path of a Lévy process can be monotone, in which case it is called a *subordinator* and we work with the Laplace exponent

$$\mathbb{E}[\mathrm{e}^{-\lambda\xi_t}] = \mathrm{e}^{-\Phi(\lambda)t}, \qquad t \geq 0$$

  where

$$\Phi(\lambda) = q + \delta\lambda + \int_{(0,\infty)}(1 - \mathrm{e}^{-\lambda x})\Upsilon(\mathrm{d}x), \qquad \lambda \geq 0.$$

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Lévy process: One dimension

Two examples in one dimension:

- **Stable subordinator** $(\xi_t, t \geq 0)$ is a subordinator which satisfies the additional scaling property: For $c > 0$

$$\text{under } \mathbb{P}, \text{ the law of } (c\xi_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P},$$

where $\alpha \in (0,1)$. We have

$$\Phi(\lambda) = \lambda^\alpha, \qquad \lambda \geq 0, \qquad \text{and} \qquad \Pi(\mathrm{d}x) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{1}{x^{1+\alpha}} \mathrm{d}x, \qquad x > 0.$$

- **Hypgergeometric Lévy process:** For $\beta \leq 1, \gamma \in (0,1), \hat{\beta} \geq 0, \hat{\gamma} \in (0,1)$

$$\Psi(\theta) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}\theta)}{\Gamma(1-\beta-\mathrm{i}\theta)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}\theta)}{\Gamma(\hat{\beta}+\mathrm{i}\theta)} \qquad \theta \in \mathbb{R}.$$

The Lévy measure has a density with respect to Lebesgue measure which is given by

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\hat{\gamma})\Gamma(-\gamma)} \mathrm{e}^{-(1-\beta+\gamma)x} {}_2F_1\left(1+\gamma, \eta; \eta-\hat{\gamma}; \mathrm{e}^{-x}\right), & \text{if } x > 0, \\ -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\gamma)\Gamma(-\hat{\gamma})} \mathrm{e}^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1\left(1+\hat{\gamma}, \eta; \eta-\gamma; \mathrm{e}^{x}\right), & \text{if } x < 0, \end{cases}$$

where $\eta := 1 - \beta + \gamma + \hat{\beta} + \hat{\gamma}$.

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Lévy process: One dimension

- If ξ has a characteristic exponent Ψ then necessarily

$$\Psi(\theta) = \kappa(-\mathrm{i}\theta)\hat{\kappa}(\mathrm{i}\theta), \qquad \theta \in \mathbb{R}.$$

where κ and $\hat{\kappa}$ are Bernstein functions, e.g.

$$\kappa(\lambda) = q + \delta\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\Upsilon(\mathrm{d}x), \qquad \lambda \geq 0.$$

- The factorisation has a physical interpretation:
  - range of the κ-subordinator agrees with the range of $\sup_{s\leq t} \xi_s$, $t \geq 0$
  - range $\hat{\kappa}$-subordinator agrees with the range of $-\inf_{s\leq t} \xi_s$, $t \geq 0$.
- Note if $\delta > 0$, then $\mathbb{P}(\xi_{\tau_x^+} = x) > 0$, where $\tau_x^+ = \inf\{t > 0 : \xi_t = x\}$, $x > 0$.
- We have already seen the hypergeometric example

$$\Psi(\theta) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}\theta)}{\Gamma(1-\beta-\mathrm{i}\theta)} \qquad \times \qquad \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}\theta)}{\Gamma(\hat{\beta}+\mathrm{i}\theta)} \qquad \theta \in \mathbb{R}.$$

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Hitting points

- We say that ξ *can hit a point* $x \in \mathbb{R}$ if

$$\mathbb{P}(\xi_t = x \text{ for at least one } t > 0) > 0.$$

- Creeping is one way to hit a point, but not the only way

Theorem (Kesten (1969)/Bretagnolle (1971))

Suppose that ξ is not a compound Poisson process. Then ξ can hit points if and only if

$$\int_{\mathbb{R}} \mathrm{Re}\left(\frac{1}{1+\Psi(z)}\right) \mathrm{d}z < \infty.$$

If the Kesten-Bretagnolle integral test is satisfied, then

$$\mathbb{P}(\tau^{\{x\}} < \infty) = \frac{u(x)}{u(0)},$$

where $\tau^{\{x\}} = \inf\{t > 0 : \xi_t = x\}$, providing we can compute the inversion

$$u(x) = \int_{c+\mathrm{i}\mathbb{R}} \frac{\mathrm{e}^{-zx}}{\Psi(-\mathrm{i}z)} \mathrm{d}z$$

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§2. Stable processes seen as Lévy processes

Isotropic α-stable process in dimension $d \geq 2$

For $d \geq 2$, let $X := (X_t : t \geq 0)$ be a d-dimensional isotropic stable process.

- X has stationary and independent increments (it is a Lévy process)
- Characteristic exponent $\Psi(\theta) = -\log \mathbb{E}_0(e^{i\theta \cdot X_1})$ satisfies

$$\Psi(\theta) = |\theta|^\alpha, \qquad \theta \in \mathbb{R}.$$

- Necessarily, $\alpha \in (0, 2]$, we **exclude** 2 as it pertains to the setting of a Brownian motion.
- Associated Lévy measure satisfies, for $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\Pi(B) = \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|} \int_B \frac{1}{|y|^{\alpha+d}} dy$$
$$= \frac{2^{\alpha-1}\Gamma((d+\alpha)/2)\Gamma(d/2)}{\pi^d |\Gamma(-\alpha/2)|} \int_{\mathbb{S}_d} r^{d-1}\sigma_1(d\theta) \int_0^\infty \mathbf{1}_B(r\theta) \frac{1}{r^{\alpha+d}} dr,$$

where $\sigma_1(d\theta)$ is the surface measure on $\mathbb{S}_d$ normalised to have unit mass.
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Isotropic α-stable process in dimension $d \geq 2$

- Stable processes are also self-similar. For $c > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$,

$$\text{under } \mathbb{P}_x, \text{ the law of } (cX_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P}_{cx}.$$

- Isotropy means, for all rotations $U : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $x \in \mathbb{R}^d$,

$$\text{under } \mathbb{P}_x, \text{ the law of } (UX_t, t \geq 0) \text{ is equal to } \mathbb{P}_{Ux}.$$

- If $(S_t, t \geq 0)$ is a stable subordinator with index $\alpha/2$ (a Lévy process with Laplace exponent $-t^{-1} \log \mathbb{E}[e^{-\lambda S_t}] = \lambda^\alpha$) and $(B_t, t \geq 0)$ for a standard d-dimensional Brownian motion, then it is known that $X_t := \sqrt{2} B_{S_t}$, $t \geq 0$, is a stable process with index α.

$$\mathbb{E}[e^{i\theta X_t}] = \mathbb{E}\left[e^{-\theta^2 S_t}\right] = e^{-|\theta|^\alpha t}, \qquad \theta \in \mathbb{R}.$$

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- Isotropy means, for all rotations $U : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $x \in \mathbb{R}^d$,

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- If $(S_t, t \geq 0)$ is a stable subordinator with index $\alpha/2$ (a Lévy process with Laplace exponent $-t^{-1} \log \mathbb{E}[e^{-\lambda S_t}] = \lambda^\alpha$) and $(B_t, t \geq 0)$ for a standard d-dimensional Brownian motion, then it is known that $X_t := \sqrt{2} B_{S_t}$, $t \geq 0$, is a stable process with index α.

$$\mathbb{E}[e^{i\theta X_t}] = \mathbb{E}\left[e^{-\theta^2 S_t}\right] = e^{-|\theta|^\alpha t}, \qquad \theta \in \mathbb{R}.$$

ISOTROPIC α-STABLE PROCESS IN DIMENSION $d \geq 2$

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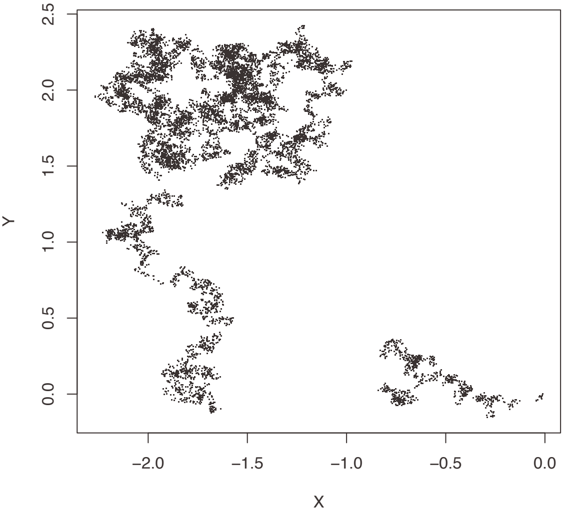
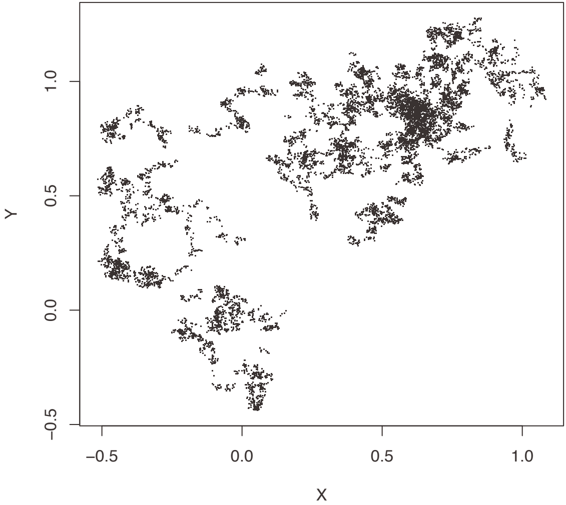
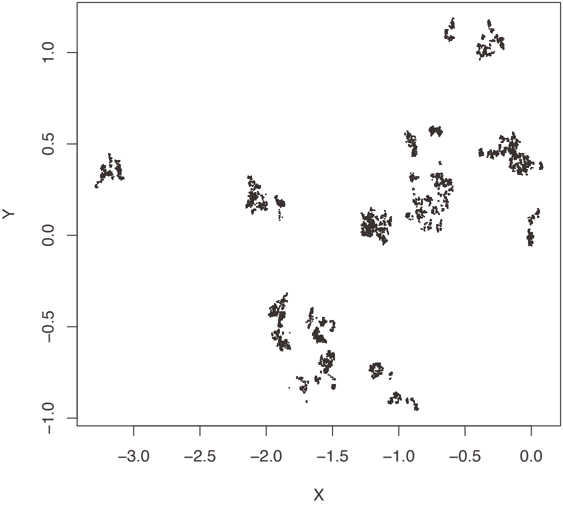
SAMPLE PATH, $\alpha = 1.9$



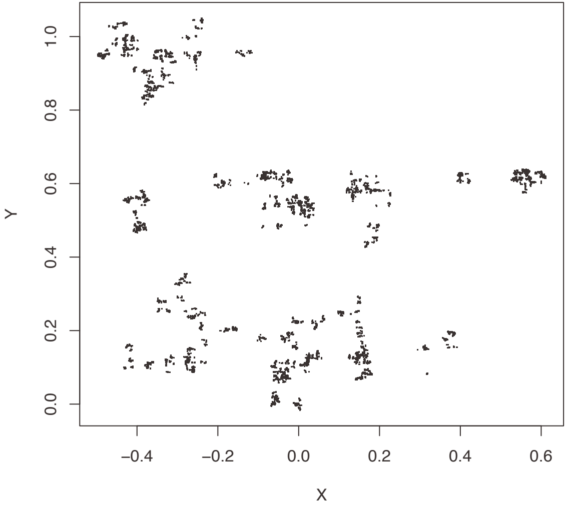
SAMPLE PATH, $\alpha = 1.7$



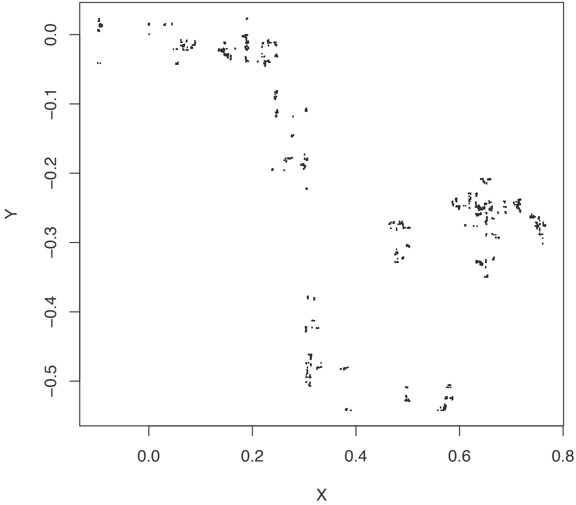
SAMPLE PATH, $\alpha = 1.5$



SAMPLE PATH, $\alpha = 1.2$



SAMPLE PATH, $\alpha = 0.9$



SOME CLASSICAL PROPERTIES: TRANSIENCE

We are interested in the potential measure

$$U(x, \mathrm{d}y) = \int_0^\infty \mathbb{P}_x(X_t \in \mathrm{d}y)\mathrm{d}t = \left(\int_0^\infty \mathrm{p}_t(y-x)\mathrm{d}t\right)\mathrm{d}y, \qquad x, y \in \mathbb{R}.$$

Note: stationary and independent increments means that it suffices to consider $U(0, \mathrm{d}y)$.

Theorem

The potential of X is absolutely continuous with respect to Lebesgue measure, in which case, its density in collaboration with spatial homogeneity satisfies $U(x, \mathrm{d}y) = u(y-x)\mathrm{d}y$, $x, y \in \mathbb{R}^d$, where

$$u(z) = 2^{-\alpha}\pi^{-d/2}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}|z|^{\alpha-d}, \qquad z \in \mathbb{R}^d.$$

In this respect X is transient. It can be shown moreover that

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Proof of Theorem

Now note that, for bounded and measurable $f : \mathbb{R}^d \mapsto \mathbb{R}^d$,

$$
\begin{aligned}
\mathbb{E}\left[\int_0^\infty f(X_t)\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^\infty f(\sqrt{2}B_{S_t})\mathrm{d}t\right] \\
&= \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}t\, \mathbb{P}(S_t \in \mathrm{d}s) \int_{\mathbb{R}} \mathbb{P}(B_s \in \mathrm{d}x) f(\sqrt{2}x) \\
&= \frac{1}{\Gamma(\alpha/2)\pi^{d/2}2^d} \int_{\mathbb{R}} \mathrm{d}y \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-|y|^2/4s} s^{-1+(\alpha-d)/2} f(y) \\
&= \frac{1}{2^\alpha \Gamma(\alpha/2)\pi^{d/2}} \int_{\mathbb{R}} \mathrm{d}y\, |y|^{(\alpha-d)} \int_0^\infty \mathrm{d}u\, e^{-u} u^{-1+(d-\alpha/2)} f(y) \\
&= \frac{\Gamma((d-\alpha)/2)}{2^\alpha \Gamma(\alpha/2)\pi^{d/2}} \int_{\mathbb{R}} \mathrm{d}y\, |y|^{(\alpha-d)} f(y).
\end{aligned}
$$

Some classical properties: Polarity

- Kesten-Bretagnolle integral test, in dimension $d \geq 2$,

$$\int_{\mathbb{R}} \mathrm{Re}\left(\frac{1}{1+\Psi(z)}\right) \mathrm{d}z = \int_{\mathbb{R}} \frac{1}{1+|z|^{\alpha}} \mathrm{d}z \propto \int_{\mathbb{R}} \frac{1}{1+r^{\alpha}} r^{d-1} \mathrm{d}r\, \sigma_1(\mathrm{d}\theta) = \infty.$$

- $\mathbb{P}_x(\tau^{\{y\}} < \infty) = 0$, for $x, y \in \mathbb{R}^d$.
- i.e. the stable process cannot hit individual points almost surely.

§3. Stable processes seen as a self-similar Markov process

The radial part of a stable process

Lemma

The process $(|X_t|, t \geq 0)$ is strong Markov and self-similar.

- Temporarily write $(X_t^{(x)}, t \geq 0)$ in place of $(X, \mathbb{P}_x)$
- Markov property of X tells us that, for $s, t \geq 0$,

$$X_{t+s}^{(x)} = \tilde{X}_s^{(X_t^{(x)})},$$

where $\tilde{X}^{(x)}$ is an independent copy of $X^{(x)}$.

- Isotropy implies that

$$|X_{t+s}^{(x)}| = |\tilde{X}_s^{(y)}|\Big|_{y=X_t^{(x)}} \stackrel{d}{=} |\tilde{X}_s^{(z)}|\Big|_{z=(|X_t^{(x)}|,0,0\cdots,0)}$$

- Hence Markov property holds, strong Markov property (and Feller property) can be developed from this argument
- Self-similarity of $|X|$ follows directly from the self-similarity of X.

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Positive self-similar Markov processes

The process $|X|$ is an example of a positive self-similar Markov process.

Definition

A $[0, \infty)$-valued regular Feller process $Z = (Z_t, t \geq 0)$ is called a *positive self-similar Markov process* if there exists a constant $\alpha > 0$ such that, for any $x > 0$ and $c > 0$,

$$\text{the law of } (cZ_{c^{-\alpha}t}, t \geq 0) \text{ under } P_x \text{ is } P_{cx},$$

where P_x is the law of Z when issued from x. In that case, we refer to α as the *index of self-similarity*.

LAMPERTI TRANSFORM

Theorem (Lamperti 1972)

Fix $\alpha > 0$.

(i) *If* (Z, P_x), $x > 0$, *is a positive self-similar Markov process with index of self-similarity* α, *then up to absorption at the origin, it can be represented as follows:*

$$Z_t \mathbf{1}_{(t<\zeta)} = \exp\{\xi_{\varphi(t)}\}, \qquad t \geq 0,$$

where

$$\varphi(t) = \inf\{s > 0 : \int_0^s \exp(\alpha \xi_u) \mathrm{d}u > s\},$$

$\xi_0 = \log x$ *and either*

(1) $P_x(\zeta = \infty) = 1$ *for all* $x > 0$, *in which case,* ξ *is a Lévy process satisfying* $\limsup_{t\uparrow\infty} \xi_t = \infty$,

(2) $P_x(\zeta < \infty$ *and* $Z_{\zeta-} = 0) = 1$ *for all* $x > 0$, *in which case* ξ *is a Lévy process satisfying* $\lim_{t\uparrow\infty} \xi_t = -\infty$, *or*

(3) $P_x(\zeta < \infty$ *and* $Z_{\zeta-} > 0) = 1$ *for all* $x > 0$, *in which case* ξ *is a Lévy process killed at an independent and exponentially distributed random time.*

In all cases, we may identify $\zeta = I_\infty := \int_0^\infty \mathrm{e}^{\alpha \xi_t} \mathrm{d}t$.

(ii) *Conversely, for each* $x > 0$, *suppose that* ξ *is a given (killed) Lévy process, issued from* $\log x$. *Define*

$$Z_t = \exp\{\xi_{\varphi(t)}\} \mathbf{1}_{(t<I_\infty)}, \qquad t \geq 0.$$

Then Z *defines a positive self-similar Markov process up to its absorption time* $\zeta = I_\infty$, *which satisfies* $Z_0 = x$ *and which has index* α.

LAMPERTI-TRANSFORM OF $|X|$

Theorem (Caballero-Pardo-Perez (2011))

For the pssMp constructed using the radial part of an isotropic d-dimensional stable process, the underlying Lévy process, ξ that appears through the Lamperti has characteristic exponent given by

$$\Psi(z) = 2^{\alpha} \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z+\alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z+d))}{\Gamma(\frac{1}{2}(\mathrm{i}z+d-\alpha))}, \qquad z \in \mathbb{R}.$$

- The fact that $\lim_{t\to\infty} |X_t| = \infty$ implies that $\lim_{t\to\infty} \xi_t = \infty$
- If we write $\psi(\lambda) = -\Psi(-\mathrm{i}\lambda) = \log \mathbb{E}[\mathrm{e}^{\lambda X_1}]$ for the Laplace exponent of ξ, then it is well defined for $\lambda \in (-d, \alpha)$ with roots at $\lambda = 0$ and $\lambda = \alpha - d$.
- Note that

  $$\exp((\alpha - d)\xi_t), \qquad t \geq 0,$$

  is a martingale
- Recalling that $|X_t| = \exp(\xi_{\varphi_t})$ and that φ_t is an almost surely finite stopping time (because $\lim_{t\to\infty} \xi_t = \infty$) we can deduce that

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CONDITIONED STABLE PROCESS

- We can define the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}_x^\circ}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \qquad t \geq 0, x \neq 0$$

- Suppose that f is a bounded measurable function then, for all $c > 0$,

$$\begin{aligned}\mathbb{E}_x^\circ[f(cX_{c^{-\alpha}s}, s \leq t)] &= \mathbb{E}_x\left[\frac{|cX_{c^{-\alpha}t}|^{\alpha-d}}{|cx|^{d-\alpha}} f(cX_{c^{-\alpha}s}, s \leq t)\right] \\ &= \mathbb{E}_{cx}\left[\frac{|X_t|^{\alpha-d}}{|cx|^{d-\alpha}} f(X_s, s \leq t)\right] = \mathbb{E}_{cx}^\circ[f(X_s, , s \leq t)]\end{aligned}$$

- Markovian, isotropy and self-similarity properties pass through to $(X, \mathbb{P}_x^\circ)$, $x \neq 0$.
- Similarly $(|X|, \mathbb{P}_x^\circ)$, $x \neq 0$ is a positive self-similar Markov process.

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$$= \mathbb{E}_{cx}\left[\frac{|X_t|^{\alpha-d}}{|cx|^{d-\alpha}} f(X_s, s \leq t)\right] = \mathbb{E}^\circ_{cx}[f(X_s, , s \leq t)]$$

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Conditioned stable process

- It turns out that $(X, \mathbb{P}^\circ_x)$, $x \neq 0$, corresponds to the stable process conditioned to be continuously absorbed at the origin.
- More precisely, for $A \in \sigma(X_s, s \leq t)$, if we set $\{0\}$ to be 'cemetery' state and $\Bbbk = \inf\{t > 0 : X_t = 0\}$, then

$$\mathbb{P}^\circ_x(A, t < \Bbbk) = \lim_{a \downarrow 0} \mathbb{P}_x(A,\ t < \Bbbk | \tau_a^\oplus < \infty),$$

  where $\tau_a^\oplus = \inf\{t > 0 : |X_t| < a\}$.
- In light of the associated Esscher transform on ξ, we note that the Lamperti transform of $(|X|, \mathbb{P}^\circ_x)$, $x \neq 0$, corresponds to the Lévy process with characteristic exponent

$$\Psi^\circ(z) = 2^\alpha \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z + d))}{\Gamma(-\frac{1}{2}(\mathrm{i}z + \alpha - d))} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z + \alpha))}{\Gamma(\frac{1}{2}\mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

- Given the pathwise interpretation of $(X, \mathbb{P}^\circ_x)$, $x \neq 0$, it follows immediately that $\lim_{t\to\infty} \xi_t = -\infty$, $\mathbb{P}^\circ_x$ almost surely, for any $x \neq 0$.

CONDITIONED STABLE PROCESS

- It turns out that $(X, \mathbb{P}^\circ_x)$, $x \neq 0$, corresponds to the stable process conditioned to be continuously absorbed at the origin.
- More precisely, for $A \in \sigma(X_s, s \leq t)$, if we set $\{0\}$ to be 'cemetery' state and $\mathtt{k} = \inf\{t > 0 : X_t = 0\}$, then

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$\mathbb{R}^d$-SELF-SIMILAR MARKOV PROCESSES

Definition

A $\mathbb{R}^d$-valued regular Feller process $Z = (Z_t, t \geq 0)$ is called a $\mathbb{R}^d$*-valued self-similar Markov process* if there exists a constant $\alpha > 0$ such that, for any $x > 0$ and $c > 0$,

$$\text{the law of } (cZ_{c^{-\alpha}t}, t \geq 0) \text{ under } P_x \text{ is } P_{cx},$$

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- Same definition as before except process now lives on $\mathbb{R}^d$.
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Lamperti–Kiu transform

In order to introduce the analogue of the Lamperti transform in d-dimensions, we need to introduce the notion of a Markov additive process.

Definition

An $\mathbb{R} \times E$ valued regular Feller process $(\xi, \Theta) = ((\xi_t, \Theta_t) : t \geq 0)$ with probabilities $\mathbf{P}_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in E$, and cemetery state $(-\infty, \dagger)$ is called a *Markov additive process* (MAP) if Θ is a regular Feller process on E with cemetery state $\dagger$ such that, for every bounded measurable function $f : (\mathbb{R} \cup \{-\infty\}) \times (E \cup \{\dagger\}) \to \mathbb{R}$, $t, s \geq 0$ and $(x, \theta) \in \mathbb{R} \times E$, on $\{t < \varsigma\}$,

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- Roughly speaking, one thinks of a MAP as a ‘Markov modulated’ Lévy process
- It has ‘conditional stationary and independent increments’
- Think of the E-valued Markov process Θ as modulating the characteristics of ξ (which would otherwise be a Lévy processes).

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LAMPERTI–KIU TRANSFORM

Theorem

Fix $\alpha > 0$. The process Z is a ssMp with index α if and only if there exists a (killed) MAP, (ξ, Θ) on $\mathbb{R} \times \mathbb{S}_d$ such that

$$Z_t := \mathrm{e}^{\xi_{\varphi(t)}}\Theta_{\varphi(t)} \qquad , \qquad t \leq I_\varsigma, \tag{1}$$

where

$$\varphi(t) = \inf\left\{ s > 0 : \int_0^s \mathrm{e}^{\alpha \xi_u}\,\mathrm{d}u > t \right\}, \qquad t \leq I_\varsigma,$$

and $I_\varsigma = \int_0^\varsigma \mathrm{e}^{\alpha\xi_s}\mathrm{d}s$ is the lifetime of Z until absorption at the origin. Here, we interpret $\exp\{-\infty\} \times \dagger := 0$ and $\inf \emptyset := \infty$.

- In the representation (1), the time to absorption in the origin,

$$\zeta = \inf\{t > 0 : Z_t = 0\},$$

  satisfies $\zeta = I_\varsigma$.
- Note $x \in \mathbb{R}^d$ if and only if

$$x = (|x|, \mathrm{Arg}(x)),$$

  where $\mathrm{Arg}(x) = x/|x| \in \mathbb{S}_d$. The Lamperti–Kiu decomposition therefore gives us a d-dimensional skew product decomposition of self-similar Markov processes.

LAMPERTI-STABLE MAP

- The stable process X is an $\mathbb{R}^d$-valued self-similar Markov process and therefore fits the description above
- **How do we characterise its underlying MAP** (ξ, Θ)?
- We already know that $|X|$ is a positive similar Markov process and hence ξ is a Lévy process, albeit corollated to Θ
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MAP ISOTROPY

Theorem

Suppose (ξ, Θ) is the MAP underlying the stable process. Then $((\xi, U^{-1}\Theta), \mathbf{P}_{x,\theta})$ is equal in law to $((\xi, \Theta), \mathbf{P}_{x,U^{-1}\theta})$, for every orthogonal d-dimensional matrix U and $x \in \mathbb{R}^d$, $\theta \in \mathbb{S}_d$.

Proof.

First note that $\varphi(t) = \int_0^t |X_u|^{-\alpha} \mathrm{d}u$. It follows that

$$(\xi_t, \Theta_t) = (\log |X_{A(t)}|, \operatorname{Arg}(X_{A(t)})), \qquad t \geq 0,$$

where the random times $A(t) = \inf \{s > 0 : \int_0^s |X_u|^{-\alpha} \mathrm{d}u > t\}$ are stopping times in the natural filtration of X.

Now suppose that U is any orthogonal d-dimensional matrix and let $X' = U^{-1}X$. Since X is isotropic and since $|X'| = |X|$, and $\operatorname{Arg}(X') = U^{-1}\operatorname{Arg}(X)$, we see from (**??**) that, for $x \in \mathbb{R}$ and $\theta \in \mathbb{S}_d$

$$\begin{aligned}
((\xi, U^{-1}\Theta), \mathbf{P}_{\log|x|,\theta}) &= ((\log |X_{A(t)}|,\, U^{-1}\operatorname{Arg}(X_{A(t)})), \mathbb{P}_x) \\
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- We will work with the increments $\Delta\xi_t = \xi_t - \xi_{t-} \in \mathbb{R}$, $t \geq 0$,

Theorem (Bo Li, Victor Rivero, Bertoin-Werner (1996))

Suppose that f is a bounded measurable function on $[0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}^d \times \mathbb{S}^d$ such that $f(\cdot,\cdot,0,\cdot,\cdot) = 0$, then, for all $\theta \in \mathbb{S}_d$,

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MAP OF $(X, \mathbb{P}^\circ_\cdot)$

- Recall that $(|X_t|^{\alpha-d}, t \geq 0)$, is a martingale.
- Informally, we should expect $\mathcal{L}h = 0$, where $h(x) = |x|^{\alpha-d}$ and $\mathcal{L}$ is the infinitesimal generator of the stable process, which has action

$$\mathcal{L}f(x) = \mathrm{a} \cdot \nabla f(x) + \int_{\mathbb{R}^d} [f(x+y) - f(x) - \mathbf{1}_{(|y|\leq 1)} y \cdot \nabla f(x)] \Pi(\mathrm{d}y), \qquad |x| > 0,$$

for appropriately smooth functions.

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$$\mathcal{L}f(x) = \mathrm{a} \cdot \nabla f(x) + \int_{\mathbb{R}^d} [f(x+y) - f(x) - \mathbf{1}_{(|y|\leq 1)} y \cdot \nabla f(x)] \Pi(\mathrm{d}y), \qquad |x| > 0,$$

  for appropriately smooth functions.
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MAP OF $(X, \mathbb{P}^\circ_\cdot)$

Theorem

Suppose that f is a bounded measurable function on $[0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}^d \times \mathbb{S}^d$ such that $f(\cdot,\cdot,0,\cdot,\cdot) = 0$, then, for all $\theta \in \mathbb{S}_d$,

$$\mathbf{E}^\circ_{0,\theta}\left(\sum_{s>0} f(s, \xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s)\right)$$
$$= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{S}_d} \int_{\mathbb{S}_d} \int_{\mathbb{R}} V^\circ_\theta(\mathrm{d}s, \mathrm{d}x, \mathrm{d}\vartheta)\sigma_1(\mathrm{d}\phi)\mathrm{d}y \frac{c(\alpha)\mathrm{e}^{yd}}{|\mathrm{e}^y\phi - \vartheta|^{\alpha+d}} f(s, x, -y, \vartheta, \phi),$$

where

$$V^\circ_\theta(\mathrm{d}s, dx, \mathrm{d}\vartheta) = \mathbf{P}^\circ_{0,\theta}(\xi_s \in \mathrm{d}x, \Theta_s \in \mathrm{d}\vartheta)\mathrm{d}s, \qquad x \in \mathbb{R}, \vartheta \in \mathbb{S}_d, s \geq 0,$$

is the space-time potential of (ξ, Θ) under $\mathbf{P}^\circ_{0,\theta}$.

Comparing the right-hand side above with that of the previous Theorem, it now becomes immediately clear that the the jump structure of (ξ, Θ) under $\mathbf{P}^\circ_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in \mathbb{S}_d$, is precisely that of $(-\xi, \Theta)$ under $\mathbf{P}_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in \mathbb{S}_d$.

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§4. Riesz–Bogdan–Żak transform

Riesz–Bogdan–Żak transform

- Define the transformation $K : \mathbb{R}^d \mapsto \mathbb{R}^d$, by

$$Kx = \frac{x}{|x|^2}, \qquad x \in \mathbb{R}^d \backslash \{0\}.$$

- This transformation inverts space through the unit sphere $\{x \in \mathbb{R}^d : |x| = 1\}$.
- Write $x \in \mathbb{R}^d$ in skew product form $x = (|x|, \mathrm{Arg}(x))$, and note that

$$Kx = (|x|^{-1}, \mathrm{Arg}(x)), \qquad x \in \mathbb{R}^d \backslash \{0\},$$

showing that the K-transform 'radially inverts' elements of $\mathbb{R}^d$ through $\mathbb{S}_d$.
- In particular $K(Kx) = x$

Theorem (d-dimensional Riesz–Bogdan–Żak Transform, $d \geq 2$)

Suppose that X is a d-dimensional isotropic stable process with $d \geq 2$. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0. \tag{2}$$

Then, for all $x \in \mathbb{R}^d \backslash \{0\}$, $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^\circ_{Kx})$.

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Proof of Riesz–Bogdan–Żak transform

We give a proof, different to the original proof of Bogdan and Żak (2010).

- Recall that $X_t = e^{\xi_{\varphi(t)}}\Theta_{\varphi(t)}$, where

$$\int_0^{\varphi(t)} e^{\alpha\xi_u}\,du = t, \qquad t \geq 0.$$

- Note also that, as an inverse,

$$\int_0^{\eta(t)} |X_u|^{-2\alpha} du = t, \qquad t \geq 0.$$

- Differentiating,

$$\frac{d\varphi(t)}{dt} = e^{-\alpha\xi_{\varphi(t)}} \text{ and } \frac{d\eta(t)}{dt} = e^{2\alpha\xi_{\varphi\circ\eta(t)}}, \qquad \eta(t) < \tau^{\{0\}}.$$

and chain rule now tells us that

$$\frac{d(\varphi\circ\eta)(t)}{dt} = \left.\frac{d\varphi(s)}{ds}\right|_{s=\eta(t)} \frac{d\eta(t)}{dt} = e^{\alpha\xi_{\varphi\circ\eta(t)}}.$$

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$$\int_0^{\varphi\circ\eta(t)} e^{-\alpha\xi_u} du = t, \qquad t \geq 0,$$

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§5. Hitting spheres

Port's Sphere hitting probability

- Recall that a stable process cannot hit points
- We are ultimately interested in the distribution of the position of X on first hitting of the sphere $\mathbb{S}_d = \{x \in \mathbb{R}^d : |x| = 1\}$.
- Define

$$\tau^{\odot} = \inf\{t > 0 : |X_t| = 1\}.$$

- We start with an easier result

Theorem (Port (1969))

If $\alpha \in (1,2)$, *then*

$$\mathbb{P}_x(\tau^{\odot} < \infty) = \frac{\Gamma\left(\frac{\alpha+d}{2} - 1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)} \begin{cases} {}_2F_1((d-\alpha)/2, 1-\alpha/2, d/2; |x|^2) & 1 > |x| \\ |x|^{\alpha-d}{}_2F_1((d-\alpha)/2, 1-\alpha/2, d/2; 1/|x|^2) & 1 \le |x|. \end{cases}$$

Otherwise, if $\alpha \in (0,1]$, *then* $\mathbb{P}_x(\tau^{\odot} = \infty) = 1$ *for all* $x \in \mathbb{R}^d$.

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- Recall that a stable process cannot hit points
- We are ultimately interested in the distribution of the position of X on first hitting of the sphere $\mathbb{S}_d = \{x \in \mathbb{R}^d : |x| = 1\}$.
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- Using Sterling's formula, we have, $|\Gamma(x + \mathrm{i}y)| = \sqrt{2\pi}e^{-\frac{\pi}{2}|y|}|y|^{x-\frac{1}{2}}(1 + o(1))$, for $x, y \in \mathbb{R}$, as $y \to \infty$, uniformly in any finite interval $-\infty < a \le x \le b < \infty$. Hence,

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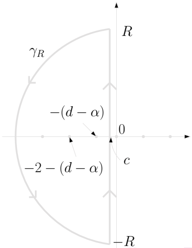

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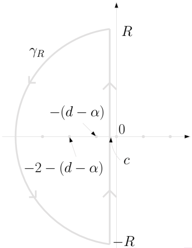

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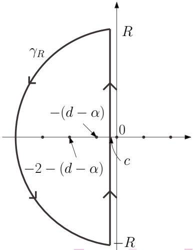

PROOF OF PORT'S HITTING PROBABILITY

- Now fix $x \leq 0$ and recall estimate $|1/\Psi(-\mathrm{i}z)| \leq |z|^{-\alpha}$. The assumption $x \leq 0$ and the fact that the arc length of $\{c + R\mathrm{e}^{\mathrm{i}\theta} : \theta \in (\pi/2, 3\pi/2)\}$ is πR, gives us

$$\left| \int_{c+R\mathrm{e}^{\mathrm{i}\theta}:\theta\in(\pi/2,3\pi/2)} \frac{\mathrm{e}^{-xz}}{\Psi(-\mathrm{i}z)} \mathrm{d}z \right| \leq CR^{-(\alpha-1)} \to 0$$

as $R \to \infty$ for some constant $C > 0$.

- Moreover,

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- To deal with the case $|x| < 1$, we can appeal to the Riesz–Bogdan–Żak transform to help us.
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Riesz representation of Port's hitting probability

Theorem

Suppose $\alpha \in (1,2)$. For all $x \in \mathbb{R}^d$,

$$\mathbb{P}_x(\tau^{\odot} < \infty) = \frac{\Gamma\left(\frac{\alpha+d}{2} - 1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)} \int_{\mathbb{S}_d} |z-x|^{\alpha-d}\sigma_1(\mathrm{d}z).$$

In particular, for $y \in \mathbb{S}_d$,

$$\int_{\mathbb{S}_d} |z-y|^{\alpha-d}\sigma_1(\mathrm{d}z) = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)}{\Gamma\left(\frac{\alpha+d}{2} - 1\right)\Gamma\left(\frac{\alpha}{2}\right)}.$$

Proof of Riesz representation of Port's hitting probability

- We know that $|X_t - z|^{\alpha-d}$, $t \geq 0$ is a martingale.
- Hence we know that

$$M_t := \int_{\mathbb{S}_d} |z - X_{t\wedge\tau^{\odot}}|^{\alpha-d}\sigma_1(\mathrm{d}z), \qquad t \geq 0,$$

  is a martingale.
- Recall that $\lim_{t\to\infty}|X_t| = 0$ and $\alpha < d$ and hence

$$M_\infty := \lim_{t\to\infty} M_t = \int_{\mathbb{S}_d} |z - X_{\tau^{\odot}}|^{\alpha-d}\sigma_1(\mathrm{d}z)\mathbf{1}_{(\tau^{\odot}<\infty)} \stackrel{d}{=} C\mathbf{1}_{(\tau^{\odot}<\infty)}.$$

  where, despite the randomness in $X_{\tau^{\odot}}$, by rotational symmetry,

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  and $1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$ is the 'North Pole' on $\mathbb{S}_d$.
- Since M is a UI martingale, taking expectations of M_∞

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PROOF OF RIESZ REPRESENTATION OF PORT'S HITTING PROBABILITY

- We know that $|X_t - z|^{\alpha-d}$, $t \geq 0$ is a martingale.
- Hence we know that

$$M_t := \int_{\mathbb{S}_d} |z - X_{t\wedge\tau^\odot}|^{\alpha-d}\sigma_1(\mathrm{d}z), \qquad t \geq 0,$$

  is a martingale.
- Recall that $\lim_{t\to\infty}|X_t| = 0$ and $\alpha < d$ and hence

$$M_\infty := \lim_{t\to\infty} M_t = \int_{\mathbb{S}_d} |z - X_{\tau^\odot}|^{\alpha-d}\sigma_1(\mathrm{d}z)\mathbf{1}_{(\tau^\odot<\infty)} \overset{d}{=} C\mathbf{1}_{(\tau^\odot<\infty)}.$$

  where, despite the randomness in $X_{\tau^\odot}$, by rotational symmetry,

$$C = \int_{\mathbb{S}_d} |z - 1|^{\alpha-d}\sigma_1(\mathrm{d}z),$$

  and $1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$ is the 'North Pole' on $\mathbb{S}_d$.
- Since M is a UI martingale, taking expectations of M_∞

$$\int_{\mathbb{S}_d} |z - x|^{\alpha-d}\sigma_1(\mathrm{d}z) = \mathbb{E}_x[M_0] = \mathbb{E}_x[M_\infty] = C\mathbb{P}_x(\tau^\odot < \infty)$$

- Taking limits as $|x| \to 0$,
  $C = 1/\mathbb{P}(\tau^\odot < \infty) = \Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)/\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)$.

Proof of Riesz representation of Port's hitting probability

- We know that $|X_t - z|^{\alpha-d}$, $t \geq 0$ is a martingale.
- Hence we know that

$$M_t := \int_{\mathbb{S}_d} |z - X_{t\wedge\tau^{\odot}}|^{\alpha-d}\sigma_1(\mathrm{d}z), \qquad t \geq 0,$$

  is a martingale.
- Recall that $\lim_{t\to\infty}|X_t| = 0$ and $\alpha < d$ and hence

$$M_\infty := \lim_{t\to\infty} M_t = \int_{\mathbb{S}_d} |z - X_{\tau^{\odot}}|^{\alpha-d}\sigma_1(\mathrm{d}z)\mathbf{1}_{(\tau^{\odot}<\infty)} \stackrel{d}{=} C\mathbf{1}_{(\tau^{\odot}<\infty)}.$$

  where, despite the randomness in $X_{\tau^{\odot}}$, by rotational symmetry,

$$C = \int_{\mathbb{S}_d} |z - 1|^{\alpha-d}\sigma_1(\mathrm{d}z),$$

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- Taking limits as $|x| \to 0$,
  $C = 1/\mathbb{P}(\tau^{\odot} < \infty) = \Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)/\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)$.

Sphere inversions

Sphere inversions

- Fix a point $b \in \mathbb{R}^d$ and a value $r > 0$.
- The spatial transformation $x^* : \mathbb{R}^d \backslash \{b\} \mapsto \mathbb{R}^d \backslash \{b\}$

$$x^* = b + \frac{r^2}{|x-b|^2}(x-b),$$

is called an *inversion through the sphere* $\mathbb{S}_d(b, r) := \{x \in \mathbb{R}^d : |x - b| = r\}$.

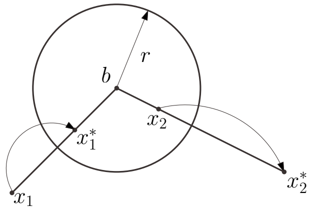


Figure: Inversion relative to the sphere $\mathbb{S}_d(b, r)$.

INVERSION THROUGH $\mathbb{S}_d(b, r)$: KEY PROPERTIES

Inversion through $\mathbb{S}_d(b, r)$

$$x^* = b + \frac{r^2}{|x-b|^2}(x-b),$$

The following can be deduced by straightforward algebra

- Self inverse

$$x = b + r^2 \frac{(x^*-b)}{|x^*-b|^2}$$

- Symmetry

$$r^2 = |x^*-b||x-b|$$

- Difference

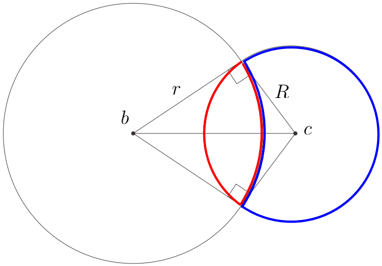
$$|x^*-y^*| = \frac{r^2|x-y|}{|x-b||y-b|}$$

- Differential

$$\mathrm{d}x^* = \frac{r^{2d}}{|x-b|^{2d}}\mathrm{d}x$$

INVERSION THROUGH $\mathbb{S}_d(b, r)$: KEY PROPERTIES

- The sphere $\mathbb{S}_d(c, R)$ maps to itself under inversion through $\mathbb{S}_d(b, r)$ provided the former is orthogonal to the latter, which is equivalent to $r^2 + R^2 = |c - b|^2$.



- In particular, the area contained in the blue segment is mapped to the area in the red segment and vice versa.

Sphere inversion with reflection

A variant of the sphere inversion transform takes the form

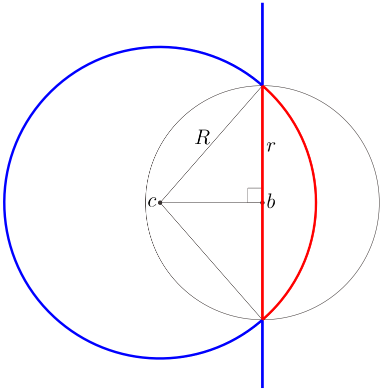
$$x^{\diamond} = b - \frac{r^2}{|x-b|^2}(x-b),$$

and has properties

- Self inverse
$$x = b - \frac{r^2}{|x^{\diamond}-b|^2}(x^{\diamond}-b),$$
- Symmetry
$$r^2 = |x^{\diamond}-b||x-b|,$$
- Difference
$$|x^{\diamond}-y^{\diamond}| = \frac{r^2|x-y|}{|x-b||y-b|}.$$
- Differential
$$\mathrm{d}x^{\diamond} = \frac{r^{2d}}{|x-b|^{2d}}\mathrm{d}x$$

SPHERE INVERSION WITH REFLECTION

- Fix $b \in \mathbb{R}^d$ and $r > 0$. The sphere $\mathbb{S}_d(c, R)$ maps to itself through $\mathbb{S}_d(b, r)$ providing $|c - b|^2 + r^2 = R^2$.



- However, this time, the exterior of the sphere $\mathbb{S}_d(c, R)$ maps to the interior of the sphere $\mathbb{S}_d(c, R)$ and vice versa. For example, the region in the exterior of $\mathbb{S}_d(c, R)$ contained by blue boundary maps to the portion of the interior of $\mathbb{S}_d(c, R)$ contained by the red boundary.

§6. Spherical hitting distribution

Port's Sphere hitting distribution

A richer version of the previous theorem:

Theorem (Port (1969))

Define the function

$$h^{\odot}(x,y) = \frac{\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)} \frac{||x|^2-1|^{\alpha-1}}{|x-y|^{\alpha+d-2}}$$

for $|x| \neq 1$, $|y| = 1$. *Then, if* $\alpha \in (1,2)$,

$$\mathbb{P}_x(X_{\tau^{\odot}} \in \mathrm{d}y) = h^{\odot}(x,y)\sigma_1(\mathrm{d}y)\mathbf{1}_{(|x|\neq 1)} + \delta_x(\mathrm{d}y)\mathbf{1}_{(|x|=1)}, \qquad |y| = 1,$$

where $\sigma_1(\mathrm{d}y)$ *is the surface measure on* $\mathbb{S}_d$, *normalised to have unit total mass.*

Otherwise, if $\alpha \in (0,1]$, $\mathbb{P}_x(\tau^{\odot} = \infty) = 1$, *for all* $|x| \neq 1$.

PROOF OF PORT'S SPHERE HITTING DISTRIBUTION

- Write $\mu_x^{\odot}(\mathrm{d}z) = \mathbb{P}_x(X_{\tau^{\odot}} \in \mathrm{d}z)$ on $\mathbb{S}_d$ where $x \in \mathbb{R}^d \backslash \mathbb{S}_d$.
- Recall the expression for the resolvent of the stable process in Theorem 2 which states that, due to transience,

$$\int_0^\infty \mathbb{P}_x(X_t \in \mathrm{d}y)\mathrm{d}t = C(\alpha)|x-y|^{\alpha-d}\mathrm{d}y, \qquad x, y \in \mathbb{R}^d,$$

where $C(\alpha)$ is an unimportant constant in the following discussion.

- The measure $\mu_x^{\odot}$ is the solution to the 'functional fixed point equation'

$$|x-y|^{\alpha-d} = \int_{\mathbb{S}_d} |z-y|^{\alpha-d}\mu(\mathrm{d}z), \qquad y \in \mathbb{S}_d.$$

- With a little work, we can show it is the unique solution in the class of probability measures.

Proof of Port's sphere hitting distribution

- Write $\mu_x^{\odot}(\mathrm{d}z) = \mathbb{P}_x(X_{\tau^{\odot}} \in \mathrm{d}z)$ on $\mathbb{S}_d$ where $x \in \mathbb{R}^d \backslash \mathbb{S}_d$.
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PROOF OF PORT'S SPHERE HITTING DISTRIBUTION

- Write $\mu_x^{\odot}(\mathrm{d}z) = \mathbb{P}_x(X_{\tau^\odot} \in \mathrm{d}z)$ on $\mathbb{S}_d$ where $x \in \mathbb{R}^d \backslash \mathbb{S}_d$.
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$$|x-y|^{\alpha-d} = \int_{\mathbb{S}_d} |z-y|^{\alpha-d}\mu(\mathrm{d}z), \qquad y \in \mathbb{S}_d.$$

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Proof of Port's sphere hitting distribution

Recall, for $y^* \in \mathbb{S}_d$, from the Riesz representation of the sphere hitting probability,

$$\frac{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)}{\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)} = \int_{\mathbb{S}_d} |z^*-y^*|^{\alpha-d}\sigma_1(\mathrm{d}z^*).$$

we are going to manipulate this identity using sphere inversion to solve the fixed point equation **first assuming that** $|x| > 1$

- Apply the sphere inversion with respect to the sphere $\mathbb{S}_d(x, (|x|^2-1)^{1/2})$ remembering that this transformation maps $\mathbb{S}_d$ to itself and using

$$\frac{1}{|z^*-x|^{d-1}}\sigma_1(\mathrm{d}z^*) = \frac{1}{|z-x|^{d-1}}\sigma_1(\mathrm{d}z)$$

$$(|x|^2-1) = |z^*-x||z-x| \quad \text{and} \quad |z^*-y^*| = \frac{(|x|^2-1)|z-y|}{|z-x||y-x|}$$

- We have

$$\frac{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)}{\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)} = \int_{\mathbb{S}_d} |z^*-x|^{d-1}|z^*-y^*|^{\alpha-d}\frac{\sigma_1(\mathrm{d}z^*)}{|z^*-x|^{d-1}}$$

$$= \frac{(|x|^2-1)^{\alpha-1}}{|y-x|^{\alpha-d}}\int_{\mathbb{S}_d}\frac{|z-y|^{\alpha-d}}{|z-x|^{\alpha+d-2}}\sigma_1(\mathrm{d}z).$$

- For the case $|x| < 1$, calculate similarly by replacing x^* by $x^\diamond$ i.e. inverting and reflecting in the sphere $\mathbb{S}_d(x, (1-|x|^2)^{1/2})$

Proof of Port's sphere hitting distribution

Recall, for $y^* \in \mathbb{S}_d$, from the Riesz representation of the sphere hitting probability,

$$\frac{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)}{\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)} = \int_{\mathbb{S}_d} |z^*-y^*|^{\alpha-d}\sigma_1(\mathrm{d}z^*).$$

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$$= \frac{(|x|^2-1)^{\alpha-1}}{|y-x|^{\alpha-d}}\int_{\mathbb{S}_d}\frac{|z-y|^{\alpha-d}}{|z-x|^{\alpha+d-2}}\sigma_1(\mathrm{d}z).$$

- For the case $|x|<1$, calculate similarly by replacing x^* by $x^\diamond$ i.e. inverting and reflecting in the sphere $\mathbb{S}_d(x,(1-|x|^2)^{1/2})$

Proof of Port's sphere hitting distribution

Recall, for $y^* \in \mathbb{S}_d$, from the Riesz representation of the sphere hitting probability,

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$$\frac{\Gamma\left(\frac{d}{2}\right)\Gamma(\alpha-1)}{\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)} = \int_{\mathbb{S}_d} |z^*-x|^{d-1}|z^*-y^*|^{\alpha-d}\frac{\sigma_1(\mathrm{d}z^*)}{|z^*-x|^{d-1}}$$

$$= \frac{(|x|^2-1)^{\alpha-1}}{|y-x|^{\alpha-d}}\int_{\mathbb{S}_d}\frac{|z-y|^{\alpha-d}}{|z-x|^{\alpha+d-2}}\sigma_1(\mathrm{d}z).$$

- For the case $|x|<1$, calculate similarly by replacing x^* by $x^\diamond$ i.e. inverting and reflecting in the sphere $\mathbb{S}_d(x, (1-|x|^2)^{1/2})$

§7. Spherical entrance/exit distribution

BLUMENTHAL–GETOOR–RAY EXIT / ENTRANCE DISTRIBUTION

Theorem

Define the function

$$g(x,y) = \pi^{-(d/2+1)}\,\Gamma(d/2)\,\sin(\pi\alpha/2)\frac{|1-|x|^2|^{\alpha/2}}{|1-|y|^2|^{\alpha/2}}|x-y|^{-d}$$

for $x, y \in \mathbb{R}^d \backslash \mathbb{S}_d$. *Let*

$$\tau^{\oplus} := \inf\{t > 0 : |X_t| < 1\} \text{ and } \tau_a^{\ominus} := \inf\{t > 0 : |X_t| > 1\}.$$

(i) *Suppose that* $|x| < 1$, *then*

$$\mathbb{P}_x(X_{\tau^\ominus} \in \mathrm{d}y) = g(x,y)\mathrm{d}y, \qquad |y| \geq 1.$$

(ii) *Suppose that* $|x| > 1$, *then*

$$\mathbb{P}_x(X_{\tau^\oplus} \in \mathrm{d}y,\, \tau^\oplus < \infty) = g(x,y)\mathrm{d}y, \qquad |y| \leq 1.$$

PROOF OF B–G–R ENTRANCE/EXIT DISTRIBUTION (I)

- Appealing again to the potential density and the strong Markov property, it suffices to find a solution to

$$|x-y|^{\alpha-d} = \int_{|z|\geq 1} |z-y|^{\alpha-d}\mu(\mathrm{d}z), \qquad |y|>1,$$

with a straightforward argument providing uniqueness.

- The proof is complete as soon as we can verify that

$$|x-y|^{\alpha-d} = c_{\alpha,d}\int_{|z|\geq 1} |z-y|^{\alpha-d}\frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}}|x-z|^{-d}\mathrm{d}z$$

for $|y|>1>|x|$, where

$$c_{\alpha,d} = \pi^{-(1+d/2)}\,\Gamma(d/2)\,\sin(\pi\alpha/2).$$

Proof of B–G–R entrance/exit distribution (i)

- Appealing again to the potential density and the strong Markov property, it suffices to find a solution to

$$|x-y|^{\alpha-d} = \int_{|z|\geq 1} |z-y|^{\alpha-d}\mu(\mathrm{d}z), \qquad |y|>1,$$

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- The proof is complete as soon as we can verify that

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for $|y|>1>|x|$, where

$$c_{\alpha,d} = \pi^{-(1+d/2)}\,\Gamma(d/2)\,\sin(\pi\alpha/2).$$

PROOF OF B–G–R ENTRANCE / EXIT DISTRIBUTION (I)

- Transform $z \mapsto z^\diamond$ (sphere inversion with reflection) through the sphere $\mathbb{S}_d(x, (1-|x|^2)^{1/2})$, noting in particular that

$$|z^\diamond - y^\diamond| = (1-|x|^2)\frac{|z-y|}{|z-x||y-x|} \quad \text{and} \quad |z|^2 - 1 = \frac{|z-x|^2}{1-|x|^2}(1-|z^\diamond|^2)$$

and

$$\mathrm{d}z^\diamond = (1-|x|^2)^d |z-x|^{-2d}\mathrm{d}z, \qquad z \in \mathbb{R}^d.$$

- For $|x| < 1 < |y|$,

$$\int_{|z|\geq 1} |z-y|^{\alpha-d}\frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}}|x-z|^{-d}\mathrm{d}z = |y-x|^{\alpha-d}\int_{|z^\diamond|\leq 1}\frac{|z^\diamond-y^\diamond|^{\alpha-d}}{|1-|z^\diamond|^2|^{\alpha/2}}\mathrm{d}z^\diamond.$$

- Now perform similar transformation $z^\diamond \mapsto w$ (inversion with reflection), albeit through the sphere $\mathbb{S}_d(y^\diamond, (1-|y^\diamond|^2)^{1/2})$.

$$|y-x|^{\alpha-d}\int_{|z^\diamond|\leq 1}\frac{|z^\diamond-y^\diamond|^{\alpha-d}}{|1-|z^\diamond|^2|^{\alpha/2}}\mathrm{d}z^\diamond = |y-x|^{\alpha-d}\int_{|w|\geq 1}\frac{|1-|y^\diamond|^2|^{\alpha/2}}{|1-|w|^2|^{\alpha/2}}|w-y^\diamond|^{-d}\mathrm{d}w.$$

Proof of B–G–R entrance/exit distribution (I)

- Transform $z \mapsto z^\diamond$ (sphere inversion with reflection) through the sphere $\mathbb{S}_d(x, (1-|x|^2)^{1/2})$, noting in particular that

$$|z^\diamond - y^\diamond| = (1-|x|^2)\frac{|z-y|}{|z-x||y-x|} \quad \text{and} \quad |z|^2 - 1 = \frac{|z-x|^2}{1-|x|^2}(1-|z^\diamond|^2)$$

and

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- For $|x| < 1 < |y|$,

$$\int_{|z|\geq 1} |z-y|^{\alpha-d}\frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}}|x-z|^{-d}\mathrm{d}z = |y-x|^{\alpha-d}\int_{|z^\diamond|\leq 1}\frac{|z^\diamond - y^\diamond|^{\alpha-d}}{|1-|z^\diamond|^2|^{\alpha/2}}\mathrm{d}z^\diamond.$$

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Proof of B–G–R entrance/exit distribution (I)

- Transform $z \mapsto z^\diamond$ (sphere inversion with reflection) through the sphere $\mathbb{S}_d(x, (1-|x|^2)^{1/2})$, noting in particular that

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and

$$\mathrm{d}z^\diamond = (1-|x|^2)^d |z-x|^{-2d}\mathrm{d}z, \qquad z \in \mathbb{R}^d.$$

- For $|x| < 1 < |y|$,

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Proof of B–G–R entrance/exit distribution (I)

Thus far:

$$\int_{|z|\geq 1} |z-y|^{\alpha-d} \frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}} |x-z|^{-d} \mathrm{d}z = |y-x|^{\alpha-d} \int_{|w|\geq 1} \frac{|1-|y^\diamond|^2|^{\alpha/2}}{|1-|w|^2|^{\alpha/2}} |w-y^\diamond|^{-d} \mathrm{d}w.$$

- Taking the integral in red and decomposition into generalised spherical polar coordinates

$$\int_{|v|\geq 1} \frac{1}{|1-|w|^2|^{\alpha/2}} |w-y^\diamond|^{-d} \mathrm{d}w = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_1^\infty \frac{r^{d-1}\mathrm{d}r}{|1-r^2|^{\alpha/2}} \int_{\mathbb{S}_d(0,r)} |z-y^\diamond|^{-d} \sigma_r(\mathrm{d}z)$$

- Poisson's formula (the probability that a Brownian motion hits a sphere of radius $r > 0$) states that

$$\int_{\mathbb{S}_d(0,r)} \frac{r^{d-2}(r^2-|y^\diamond|^2)}{|z-y^\diamond|^d} \sigma_r(\mathrm{d}z) = 1, \qquad |y^\diamond| < 1 < r.$$

gives us

$$\int_{|v|\geq 1} \frac{1}{|1-|w|^2|^{\alpha/2}} |w-y^\diamond|^{-d} \mathrm{d}w = \frac{\pi^{d/2}}{\Gamma(d/2)} \int_1^\infty \frac{2r}{(r^2-1)^{\alpha/2}(r^2-|y^\diamond|^2)} \mathrm{d}r$$
$$= \frac{\pi}{\sin(\alpha\pi/2)} \frac{1}{(1-|y^\diamond|^2)^{\alpha/2}}$$

- Plugging everything back in gives the result for $|x| < 1$.

PROOF OF B–G–R ENTRANCE/EXIT DISTRIBUTION (I)

Thus far:

$$\int_{|z|\geq 1}|z-y|^{\alpha-d}\frac{|1-|x|^2|^{\alpha/2}}{|1-|z|^2|^{\alpha/2}}|x-z|^{-d}\mathrm{d}z=|y-x|^{\alpha-d}\int_{|w|\geq 1}\frac{|1-|y^\diamond|^2|^{\alpha/2}}{|1-|w|^2|^{\alpha/2}}|w-y^\diamond|^{-d}\mathrm{d}w.$$

- Taking the integral in red and decomposition into generalised spherical polar coordinates

$$\int_{|v|\geq 1}\frac{1}{|1-|w|^2|^{\alpha/2}}|w-y^\diamond|^{-d}\mathrm{d}w=\frac{2\pi^{d/2}}{\Gamma(d/2)}\int_1^\infty\frac{r^{d-1}\mathrm{d}r}{|1-r^2|^{\alpha/2}}\int_{\mathbb{S}_d(0,r)}|z-y^\diamond|^{-d}\sigma_r(\mathrm{d}z)$$

- Poisson's formula (the probability that a Brownian motion hits a sphere of radius $r>0$) states that

$$\int_{\mathbb{S}_d(0,r)}\frac{r^{d-2}(r^2-|y^\diamond|^2)}{|z-y^\diamond|^d}\sigma_r(\mathrm{d}z)=1,\qquad |y^\diamond|<1<r.$$

gives us

$$\int_{|v|\geq 1}\frac{1}{|1-|w|^2|^{\alpha/2}}|w-y^\diamond|^{-d}\mathrm{d}w=\frac{\pi^{d/2}}{\Gamma(d/2)}\int_1^\infty\frac{2r}{(r^2-1)^{\alpha/2}(r^2-|y^\diamond|^2)}\mathrm{d}r$$
$$=\frac{\pi}{\sin(\alpha\pi/2)}\frac{1}{(1-|y^\diamond|^2)^{\alpha/2}}$$

- Plugging everything back in gives the result for $|x|<1$.

Proof of B–G–R entrance/exit distribution (ii)

The interesting part of the proof is the derivation of the the identity in (ii) (i.e. $|x| > 1$) from the identity in (i) (i.e. $|x| < 1$).

- Start by noting from the Riesz–Bogdan–Żak transform that, for $|x| > 1$,

$$\mathbb{P}_x(X_{\tau^\oplus} \in D) = \mathbb{P}^\circ_{Kx}(KX_{\tau^\ominus} \in D),$$

  where $Kx = x/|x|^2$, $|Kx - Kz| = |x - z|/|x||z|$ and $KD = \{Kx : x \in D\}$.

- Noting that $\mathrm{d}(Kz) = |z|^{-2d}\mathrm{d}z$, we have

$$\begin{aligned}
&\mathbb{P}_x(X_{\tau^\oplus} \in D)\\
&= \int_{KD} \frac{|y|^{\alpha-d}}{|Kx|^{\alpha-d}} g(Kx, y)\mathrm{d}y\\
&= c_{\alpha,d} \int_{KD} |z|^{d-\alpha}|Kx|^{d-\alpha} \frac{|1 - |Kx|^2|^{\alpha/2}}{|1 - |y|^2|^{\alpha/2}} |Kx - y|^{-d}\mathrm{d}y\\
&= c_{\alpha,d} \int_{D} |z|^{2d} \frac{|1 - |x|^2|^{\alpha/2}}{|1 - |z|^2|^{\alpha/2}} |x - z|^{-d}\mathrm{d}(Kz)\\
&= c_{\alpha,d} \int_{D} \frac{|1 - |x|^2|^{\alpha/2}}{|1 - |z|^2|^{\alpha/2}} |x - z|^{-d}\mathrm{d}z
\end{aligned}$$

PROOF OF B–G–R ENTRANCE/EXIT DISTRIBUTION (II)

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\end{aligned}$$

§8. Radial excursion theory

Excursions from the radial minimum

Recall that we can represent an isotropic Lévy process through the Lamperti transform

$$X_t := \mathrm{e}^{\xi_{\varphi(t)}}\Theta_{\varphi(t)} \qquad t \geq 0,$$

where

$$\varphi(t) = \inf\left\{ s > 0 : \int_0^s \mathrm{e}^{\alpha\xi_u}\,\mathrm{d}u > t \right\}$$

and (ξ, Θ) with probabilities $\mathbf{P}_{x,\theta}$, $x \neq 0$, $\theta \in \mathbb{S}_d$, is a MAP. Recall also that, although corollated to Θ, ξ alone is a Lévy process.

- Let $\ell = (\ell_t, t \geq 0)$, the local time at 0 of the reflected Lévy process $\xi_t - \underline{\xi}_t$, $t \geq 0$, where $\underline{\xi}_t := \inf_{s \leq t} \xi_s$, $t \geq 0$.
- The process ℓ serves as an adequate choice for the local time of the Markov process $(\xi - \underline{\xi}, \Theta)$ on the set $\{0\} \times \mathbb{S}_d$.
- Define

$$\mathtt{g}_t = \sup\{s < t : \xi_s = \underline{\xi}_s\} \text{ and } \mathtt{d}_t = \inf\{s > t : \xi_s = \underline{\xi}_s\}.$$

- For all $t > 0$ such that $\mathtt{d}_t > \mathtt{g}_t$ the process

$$(\epsilon_{\mathtt{g}_t}(s), \Theta^{\epsilon}_{\mathtt{g}_t}(s)) := (\xi_{\mathtt{g}_t+s} - \xi_{\mathtt{g}_t}, \Theta_{\mathtt{g}_t+s}), \qquad s \leq \zeta_{\mathtt{g}_t} := \mathtt{d}_t - \mathtt{g}_t,$$

  codes the excursions of $(\xi - \underline{\xi}, \Theta)$ from the set $(0, \mathbb{S}_d)$ or equivalently, excursions of $(X_t / \inf_{s \leq t} |X_s|, t \geq 0)$, from $\mathbb{S}_d$, or equivalently an excursion of X from its running radial infimum.
- Moreover, we see that, for all $t > 0$ such that $\mathtt{d}_t > \mathtt{g}_t$,

$$X_{\mathtt{g}_t+s} = \mathrm{e}^{\xi_{\mathtt{g}_t}}\mathrm{e}^{\epsilon_{\mathtt{g}_t}(s)}\Theta^{\epsilon}_{\mathtt{g}_t}(s) = |X_{\mathtt{g}_t}|\mathrm{e}^{\epsilon_{\mathtt{g}_t}(s)}\Theta^{\epsilon}_{\mathtt{g}_t}(s), \qquad s \leq \zeta_{\mathtt{g}_t}.$$

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Excursions from the radial minimum

- The classical theory of exit systems in Maisonneuve (1975) now implies that there exists a family of *excursion measures*, $\mathbb{N}_\theta$, $\theta \in \mathbb{S}_d$, such that:
- the map $\theta \mapsto \mathbb{N}_\theta$ is a kernel from $\mathbb{S}_d$ to $\mathbb{R} \times \mathbb{S}_d$, such that $\mathbb{N}_\theta(1 - e^{-\zeta}) < \infty$ and $\mathbb{N}_\theta$ is carried by the set $\{(\epsilon(0), \Theta^\epsilon(0) = (0, \theta)\}$ and $\{\zeta > 0\}$;
- we have the *exit formula*

$$\mathbf{E}_{x,\theta}\left[\sum_{\mathtt{g}\in G} F((\xi_s, \Theta_s) : s < \mathtt{g}) H((\epsilon_{\mathtt{g}}, \Theta^\epsilon_{\mathtt{g}}))\right]$$
$$= \mathbf{E}_{x,\theta}\left[\int_0^\infty F((\xi_s, \Theta_s) : s < t) \mathbb{N}_{\Theta_t}(H(\epsilon, \Theta^\epsilon)) \mathrm{d}\ell_t\right],$$

  for $x \neq 0$, where F and H are continuous on the space of càdlàg paths on $\mathbb{R} \times \mathbb{S}_d$) and $G = \{\mathtt{g}_s : s \geq 0\}$
- under any measure $\mathbb{N}_\theta$ the process $(\epsilon, \Theta^\epsilon)$ is Markovian with the same *transition* semigroup as (ξ, Θ) stopped at its first hitting time of $(-\infty, 0] \times \mathbb{S}_d$.
- The couple $(\ell, \mathbb{N}_\cdot)$ is called an exit system. The pair ℓ and the kernels $\mathbb{N}_\theta$, $\theta \in \mathbb{S}_d$, are not unique, but once ℓ is chosen the measures $\mathbb{N}_\theta$ are determined but for a ℓ-neglectable set.

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Radial ladder MAP

- For bounded measurable f on $\mathbb{R}^d$ and $G(\infty) := \sup\{s \geq 0 : |X_s| = \inf_{u \leq s} |X_u|\}$,

$$\mathbb{E}_x[f(X_{G(\infty)})] = \mathbf{E}_{\log|x|,\arg(x)}\left[\sum_{t \in G} f(\mathrm{e}^{\xi_t}\Theta_t)\mathbf{1}(\zeta_t = \infty)\right]$$

$$= \mathbf{E}_{\log|x|,\arg(x)}\left[\int_0^\infty f(\mathrm{e}^{\xi_t}\Theta_t)\mathbb{N}_{\Theta_t}(\zeta = \infty)\mathrm{d}\ell_t\right]$$

$$= \mathbf{E}_{\log|x|,\arg(x)}\left[\int_0^{\ell_\infty} f(\mathrm{e}^{-H_t^-}\Theta_t^-)\mathbb{N}_{\Theta_t^-}(\zeta = \infty)\mathrm{d}t\right]$$

  where $(H_t^-, \Theta_t^-) = (-\xi_{\ell_t^{-1}}, \Theta_{\ell_t^{-1}})$, $t < \ell_\infty$.

- Define the potential

$$U_x^-(\mathrm{d}z) := \int_0^\infty \mathbf{P}_{\log|x|,\arg(x)}(\mathrm{e}^{-H_t^-}\Theta_t^- \in \mathrm{d}z,\ t < \ell_\infty)\mathrm{d}t, \qquad |z| \leq |x|.$$

- As X is transient, (H^-, Θ^-) experiences killing at Θ^--dependent rate $\mathbb{N}_\theta(\zeta = \infty)$, $\theta \in \mathbb{S}_d$. Isotropy implies $\mathbb{N}_\theta(\zeta = \infty)$ independent of θ. Scaling of local time ℓ chosen so that $\mathbb{N}_\theta(\zeta = \infty) = 1$.
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RADIAL LADDER MAP

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POINT OF CLOSEST REACH

Theorem (Point of Closest Reach to the origin)

The law of the point of closest reach to the origin is given by

$$\mathbb{P}_x(X_{G(\infty)} \in \mathrm{d}y) = \pi^{-d/2} \frac{\Gamma(d/2)^2}{\Gamma((d-\alpha)/2)\,\Gamma(\alpha/2)} \frac{(|x|^2 - |y|^2)^{\alpha/2}}{|x-y|^d |y|^\alpha} \mathrm{d}y, \qquad 0 < |y| < |x|.$$

Point of closest reach: sketch proof

- First define, for $x \neq 0$, $|x| > r$, $\delta > 0$ and continuous, positive and bounded f on $\mathbb{R}^d$,

$$\Delta_r^\delta f(x) := \frac{1}{\delta}\mathbb{E}_x\left[f(\arg(X_{\mathtt{G}_\infty})), |X_{\mathtt{G}_\infty}| \in [r-\delta, r]\right].$$

- Then, with the help of Blumenthal–Getoor–Ray first entry distribution,

$$\begin{aligned}
&\Delta_r^\delta f(x)\\
&= \frac{1}{\delta}\int_{|y|\in[r-\delta,r]} \mathbb{P}_x(X_{\tau_r^\oplus} \in \mathrm{d}y;\ \tau_r^\oplus < \infty)\mathbb{E}_y\left[f(\arg(X_{\mathtt{G}_\infty}));\ |X_{\mathtt{G}_\infty}| \in (r-\delta, |y|]\right]\\
&= \frac{1}{\delta}C_{\alpha,d}\int_{|y|\in[r-\delta,r]} \mathrm{d}y\left|\frac{r^2-|x|^2}{r^2-|y|^2}\right|^{\alpha/2}|y-x|^{-d}\mathbb{E}_y\left[f(\arg(X_{\mathtt{G}_\infty}));\ |X_{\mathtt{G}_\infty}| \in (r-\delta, |y|]\right]\\
&= \frac{1}{\delta}C_{\alpha,d}|r^2-|x|^2|^{\alpha/2}\int_{|y|\in(r-\delta,r]} \mathrm{d}y\frac{|y-x|^{-d}}{|r^2-|y|^2|^{\alpha/2}}\int_{r-\delta\leq|z|\leq|y|} U_y^-(\mathrm{d}z)f(\arg(z)),
\end{aligned}$$

Lemma

Suppose that f is a bounded continuous function on $\mathbb{R}^d$. Then

$$\lim_{\delta\to 0}\sup_{|y|\in(r-\delta,r]}\left|\frac{\int_{r-\delta\leq|z|\leq|y|}U_y^-(\mathrm{d}z)f(z)}{\int_{r-\delta\leq|z|\leq|y|}U_y^-(\mathrm{d}z)} - f(y)\right| = 0.$$

POINT OF CLOSEST REACH: SKETCH PROOF

- First define, for $x \neq 0$, $|x| > r$, $\delta > 0$ and continuous, positive and bounded f on $\mathbb{R}^d$,

$$\Delta_r^\delta f(x) := \frac{1}{\delta}\mathbb{E}_x\left[f(\arg(X_{\mathtt{G}_\infty})), |X_{\mathtt{G}_\infty}| \in [r-\delta, r]\right].$$

- Then, with the help of Blumenthal–Getoor–Ray first entry distribution,

$$\begin{aligned}
&\Delta_r^\delta f(x) \\
&= \frac{1}{\delta}\int_{|y|\in[r-\delta,r]} \mathbb{P}_x(X_{\tau_r^\oplus} \in \mathrm{d}y;\ \tau_r^\oplus < \infty)\mathbb{E}_y\left[f(\arg(X_{\mathtt{G}_\infty}));\ |X_{\mathtt{G}_\infty}| \in (r-\delta, |y|]\right] \\
&= \frac{1}{\delta}C_{\alpha,d}\int_{|y|\in[r-\delta,r]} \mathrm{d}y\left|\frac{r^2-|x|^2}{r^2-|y|^2}\right|^{\alpha/2}|y-x|^{-d}\mathbb{E}_y\left[f(\arg(X_{\mathtt{G}_\infty}));\ |X_{\mathtt{G}_\infty}| \in (r-\delta, |y|]\right] \\
&= \frac{1}{\delta}C_{\alpha,d}|r^2-|x|^2|^{\alpha/2}\int_{|y|\in(r-\delta,r]} \mathrm{d}y\frac{|y-x|^{-d}}{|r^2-|y|^2|^{\alpha/2}}\int_{r-\delta\le|z|\le|y|} U_y^-(\mathrm{d}z)f(\arg(z)),
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Point of closest reach: sketch proof

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Point of closest reach: sketch proof

- Hence

$$\Delta_r^\delta f(x) \overset{\delta\downarrow 0}{\sim} \frac{1}{\delta} C_{\alpha,d}|r^2-|x|^2|^{\alpha/2}\int_{|y|\in(r-\delta,r]}\mathrm{d}y\,\frac{|y-x|^{-d}}{|r^2-|y|^2|^{\alpha/2}}f(\arg(y))\int_{r-\delta\le|z|\le|y|}U_y^-(\mathrm{d}z)$$

and for $|y|\in(r-\delta,r]$,

$$\int_{r-\delta\le|z|\le|y|}U_y^-(\mathrm{d}z)=\mathbb{P}_y(\tau^{\oplus}_{r-\delta}=\infty)=\mathbf{P}(\underline{\xi}_\infty\ge\log((r-\delta)/y))$$

- The right hand side above can be determined explicitly thanks to the known Wiener–Hopf factorisation of ξ
- Note also

$$\Delta_r^\delta f(x) \overset{\delta\downarrow 0}{\sim} C_{\alpha,d}|r^2-|x|^2|^{\alpha/2}\frac{1}{\delta}\int_{r-\delta}^{r}\rho^{d-1}\mathrm{d}\rho\,\frac{\mathbf{P}(\underline{\xi}_\infty\ge\log((r-\delta)/y))}{|r^2-\rho^2|^{\alpha/2}}\int_{\rho\mathbb{S}_d}\sigma_\rho(\mathrm{d}\theta)|\rho\theta-x|^{-d}f(\theta)$$

Lemma

Let $D_{\alpha,d}=\Gamma(d/2)/\Gamma((d-\alpha)/2)\Gamma(\alpha/2)$. Then

$$\lim_{\delta\to 0}\sup_{|y|\in[r-\delta,r]}\left|(\rho^2-(r-\delta)^2)^{-\alpha/2}r^\alpha\mathbf{P}(\underline{\xi}_\infty\ge\log((r-\delta)/y))-\frac{2D_{\alpha,d}}{\alpha}\right|=0$$

Point of closest reach: sketch proof

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and for $|y| \in (r-\delta, r]$,

$$\int_{r-\delta\le|z|\le|y|} U_y^-(\mathrm{d}z) = \mathbb{P}_y(\tau_{r-\delta}^\oplus = \infty) = \mathbf{P}(\underline{\xi}_\infty \ge \log((r-\delta)/y))$$

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Point of closest reach: sketch proof

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Let $D_{\alpha,d}=\Gamma(d/2)/\Gamma((d-\alpha)/2)\Gamma(\alpha/2)$. *Then*

$$\lim_{\delta\to 0}\sup_{|y|\in[r-\delta,r]}\left|(\rho^2-(r-\delta)^2)^{-\alpha/2} r^\alpha \mathbf{P}(\underline{\xi}_\infty \geq \log((r-\delta)/y)) - \frac{2D_{\alpha,d}}{\alpha}\right| = 0$$

More excursion theory-based results

Theorem (Triple law at first entrance/exit of a ball)

Fix $r > 0$ and define, for $x, z, y, v \in \mathbb{R}^d \backslash \{0\}$,

$$\chi_x(z,y,v) := \pi^{-3d/2} \frac{\Gamma((d+\alpha)/2)}{|\Gamma(-\alpha/2)|} \frac{\Gamma(d/2)^2}{\Gamma(\alpha/2)^2} \frac{||z|^2-|x|^2|^{\alpha/2}||y|^2-|z|^2|^{\alpha/2}}{|z|^\alpha |z-x|^d |z-y|^d |v-y|^{\alpha+d}}.$$

(i) *Write*

$$G(\tau_r^\oplus) = \sup\{s < \tau_r^\oplus : |X_s| = \inf_{u \le s} |X_u|\}$$

for the instant of closest reach of the origin before first entry into $r\mathbb{S}_d$. For $|x| > |z| > r$, $|y| > |z|$ and $|v| < r$,

$$\mathbb{P}_x(X_{G(\tau_r^\oplus)} \in \mathrm{d}z,\ X_{\tau_r^\oplus -} \in \mathrm{d}y,\ X_{\tau_r^\oplus} \in \mathrm{d}v;\ \tau_r^\oplus < \infty) = \chi_x(z,y,v)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}v.$$

(ii) *Define $\mathcal{G}(t) = \sup\{s < t : |X_s| = \sup_{u \le s} |X_u|\}$, $t \ge 0$, and write*

$$\mathcal{G}(\tau_r^\ominus) = \sup\{s < \tau_r^\ominus : |X_s| = \sup_{u \le s} |X_u|\}.$$

for the instant of furtherest reach from the origin immediately before first exit from $r\mathbb{S}_d$. For $|x| < |z| < r$, $|y| < |z|$ and $|v| > r$,

$$\mathbb{P}_x(X_{\mathcal{G}(\tau_r^\ominus)} \in \mathrm{d}z,\ X_{\tau_r^\ominus -} \in \mathrm{d}y,\ X_{\tau_r^\ominus} \in \mathrm{d}v) = \chi_x(z,y,v)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}v.$$

More excursion theory-based results

Theorem

Write $M_t = \sup_{s\leq t}|X_t|$, $t \geq 0$. *For all bounded measurable* $f : \mathbb{B}_d \mapsto \mathbb{R}$ *and* $x \in \mathbb{R}\backslash\{0\}$

$$\lim_{t\to\infty} \mathbb{E}_x[f(X_t/M_t)] = \pi^{-d/2}\frac{\Gamma((d+\alpha)/2)}{\Gamma(\alpha/2)}\int_{\mathbb{S}_d}\sigma_1(\mathrm{d}\phi)\int_{|w|<1} f(w)\frac{|1-|w|^2|^{\alpha/2}}{|\phi-w|^d}\mathrm{d}w,$$

where $\sigma_1(\mathrm{d}y)$ *is the surface measure on* $\mathbb{S}_d$, *normalised to have unit mass.*

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