Backbone decomposition for superprocesses and applications.

A. E. Kyprianou[1]

Department of Mathematical Sciences, University of Bath

[1] Julien Berestycki, Rongli Liu, Antonio Murillo-Salas, Yanxia Ren

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- **Notation:** $\langle f, Z_t\rangle = \int_E f(x) Z_t(\mathrm{d}x) = \sum_{i=1}^{N_t} f(z_i(t))$ when $Z_t(\cdot) = \sum_{i=1}^{N_t} \delta_{z_i(t)}(\cdot)$.

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- **Total mass:** The process $\{\langle 1, Z_t \rangle : t \geq 0\}$ is a continuous time Galton-Watson process.

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- **Markov property:** For all $f \in C_c^+(E)$,

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- **Semigroup:** For positive, bounded measurable f,

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$$u_f(x,t) = \mathcal{P}_t[f](x) - \int_0^t \mathrm{d}s \cdot \mathcal{P}_s[\psi(u_f(\cdot, t-s))](x).$$

and

$$\psi(\lambda) = -\alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x)\Pi(\mathrm{d}x),$$

such that $\int_{(0,\infty)}(x \wedge x^2)\Pi(\mathrm{d}x) < \infty$.

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- **Total mass:** $||X_t|| := \{\langle 1, X_t\rangle : t \geq 0\}$ is a continuous state branching process (CSBP) with branching mechanism ψ.

Long-term behaviour of CSBP (total mass process)

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- **Finite expected growth (assumed):** General theory for superprocesses generally excludes the case that $-\psi'(0+) < \infty$ for $||X_t||$.
- **No explosion (assumed):** As a process, we also want $||X_t||$ to be conservative

$$\int_{0+} |\psi(\lambda)|^{-1} \mathrm{d}\lambda = \infty.$$

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- It turns out that $\mathbb{P}_{y\delta_x}(\lim_{t\uparrow\infty} ||X_t|| = 0) = e^{-\lambda_* y}$ for all $y > 0$ and $x \in E$ where $\psi(\lambda_*) = 0$.
- Straightforward computation:

$$\begin{aligned}\mathbb{E}^*_{\delta_x}(e^{-\langle f, X_t\rangle}) &:= \mathbb{E}_{\delta_x}(e^{-\langle f, X_t\rangle} | \lim_{s\uparrow\infty} ||X_{t+s}|| = 0) \\ &= e^{-u^*_f(x,t)}\end{aligned}$$

$$u^*_f(x,t) = \mathcal{P}_t[f](x) - \int_0^t \mathrm{d}s \cdot \mathcal{P}_s[\psi^*(u^*_f(\cdot, t-s))](x)$$

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$$\mathbb{E}_\mu(e^{-\langle f, X_t\rangle}) = e^{-\langle u_f^*(\cdot,t),\mu\rangle}\mathbb{E}_{\mathcal{P}(\frac{\mathtt{a}}{\mathtt{b}}\mu)}\left[\exp\left\{-\int_0^t 2\beta\langle u_f^*(\cdot,t-s), Z_s\rangle \mathrm{d}s\right\}\right]$$

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  - In each $\mathrm{d}s$ "immigrate" independent copies of $(X, \mathbb{P}^*_\cdot)$, having initial mass $2\beta\mathrm{d}s$ at the spatial positions of particles in Z.

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  - In each $\mathrm{d}s$ "immigrate" independent copies of $(X, \mathbb{P}^*_\cdot)$, having initial mass $2\beta\mathrm{d}s$ at the spatial positions of particles in Z.
  - **Alternative:** "immigrate at rate 2β" independent copies of $(X, \mathbb{P}^*)$ along the path of Z.

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$$\mathbb{P}^*_{\delta_x} = \mathbb{P}^*_{\frac{1}{n}\delta_x} \star \cdots \star \mathbb{P}^*_{\frac{1}{n}\delta_x}.$$

Dynkin and Kuznetsov (2004) describe the "Lévy measure" of $\mathbb{P}^*_{\delta_x}$ and call it $\mathbb{N}^*_x$ and can be thought of an “excursion measure" on path space of the superprocess. We have

$$e^{-u^*_f(x,t)} = \mathbb{E}^*_{\delta_x}(e^{-\langle f, X_t\rangle}) = \exp\left\{-\int (1 - e^{-\langle f, X_t\rangle}) \mathrm{d}\mathbb{N}^*_x\right\}$$

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- **Key ingredient 2:** A measure on $\{2, 3, \dots\} \times (0, \infty)$ in the form $\eta_n(\mathrm{d}x) = p_n(\mathrm{d}x)/p_n$ with $p_n = p_n(0, \infty)$ and

$$p_n(\mathrm{d}x) = \frac{1}{\lambda_* \psi'(\lambda_*)}\left\{\beta(\lambda_*)^2\delta_0(\mathrm{d}x)\mathbf{1}_{\{n=2\}} + (\lambda_*)^n \frac{x^n}{n!} e^{-\lambda^* x}\Pi(\mathrm{d}x)\right\}.$$

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- Run an independent copy of $(X, \mathbb{P}^*_\mu)$
- Independently, run a copy of a $(\mathcal{P}, F; E)$ branching Markov process, Z with branching generator $F(s) = \psi(\lambda_*(1-s))/\lambda_*$ and with initial configuration independently determined by $\mathcal{P}(\lambda_*\mu)$, a Poisson random field with intensity $\lambda_*\mu$. What happens when this number is zero?

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- Independently, **dress**[4] each spatial branch $\{\xi_t : \tau_{\texttt{birth}} \leq t < \tau_{\texttt{death}}\}$ of Z, with an $\mathcal{M}^{[0,\infty)}$ trajectory rooted at space time point (ξ_t, t) according to an independent Poisson random field with intensity $2\beta \mathrm{d}t \times \mathrm{d}\mathbb{N}^*_{\xi_t}$.

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For finite and compactly supported μ, $(X, \mathbb{P}_\mu)$ is equal in law to the following superposition

- Run an independent copy of $(X, \mathbb{P}^*_\mu)$
- Independently, run a copy of a $(\mathcal{P}, F; E)$ branching Markov process, Z with branching generator $F(s) = \psi(\lambda_*(1-s))/\lambda_*$ and with initial configuration independently determined by $\mathcal{P}(\lambda_*\mu)$, a Poisson random field with intensity $\lambda_*\mu$. What happens when this number is zero?
- Independently, **dress**[4] each spatial branch $\{\xi_t : \tau_{\mathtt{birth}} \le t < \tau_{\mathtt{death}}\}$ of Z, with an $\mathcal{M}^{[0,\infty)}$ trajectory rooted at space time point (ξ_t, t) according to an independent Poisson random field with intensity $2\beta \mathrm{d}t \times \mathrm{d}\mathbb{N}^*_{\xi_t}$.
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- Independently, at each branch point of Z, if there are n offspring as well as a rooted an independent copy of $(X, \mathbb{P}^*_{x\delta_{\xi_{\tau_{\texttt{death}}}}})$ with random initial mass x with probability $\eta_n(\mathrm{d}x)$.

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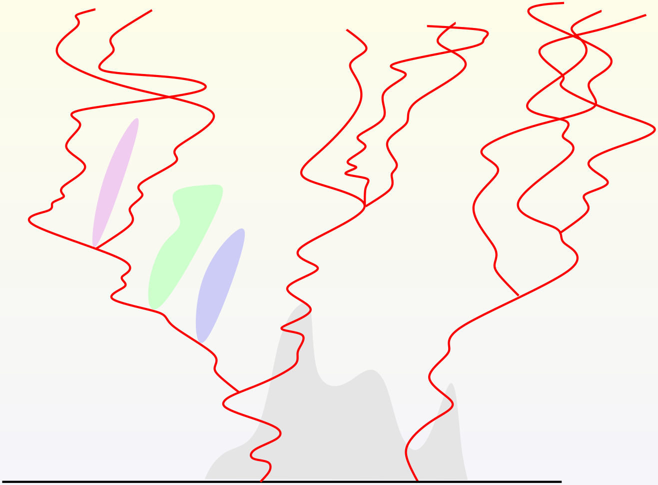
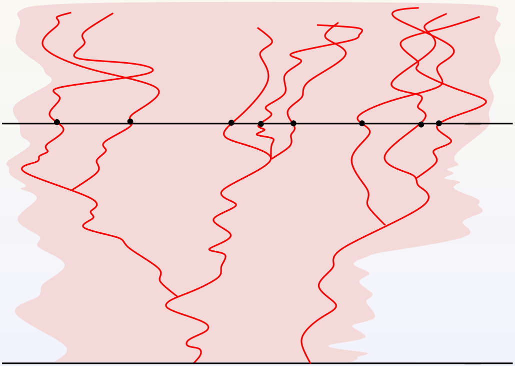


Prolific Poissonization

An important feature of the backbone decomposition. Let $((Z_t, \Lambda_t), \mathbf{P}_\mu)$ be the backbone configuration and aggregation of the dressed mass at time $t \geq 0$ so that $(\Lambda, \mathbf{P}_\mu) = (X, \mathbb{P}_\mu)$

$$\mathsf{Law}(Z_t(\cdot)|\Lambda_t(\cdot)) \sim \text{Poisson Random Field}(\lambda^* \Lambda_t(\cdot))$$



Additional remarks

This backbone decomposition is in some sense the final step following many other steps taken by others as well as concurrent work:

- Engländer and Pinsky (1999) consider a semi-group backbone decomposition for superdiffusions with spatial quadratic branching mechanism.
- Fleishmann and Swart (2002) Consider semi-pathwise decomposition for superdiffusions with spatially dependent quadratic branching mechanism.
- Dusquene and Winkel (2007) Consider pathwise decomposition for CSBPs.
- Bertoin, Fontbona & Martinez (2008) Consider semi-pathwise decomposition for CSBPs
- Abraham and Delmas (2009) Related decompositions for critical and supercritical (see previous talks!!)
- In principle the method we use should be able to handle

$$\psi(\lambda, x) = \alpha(x)\lambda + \beta(x)\lambda^2 + \int_{(0,\infty)} (e^{-\lambda y} - 1 + \lambda y)\Pi(x, \mathrm{d}y)$$

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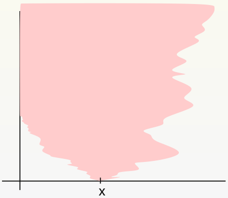
- Define Λ_{D_x} represents Dynkin's exit measure from the space-time domain $(0,\infty) \times (0,\infty)$ of $(\Lambda, \mathbb{P}_{\delta_x})$ where $x > 0$. Define Z_{D_x} similarly for the backbone.



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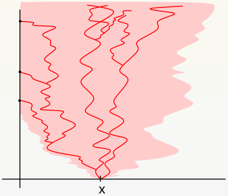

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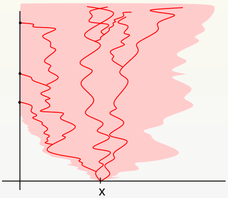

- The processes $(\langle 1, Z_{D_x}\rangle, x \geq 0)$ is a cts time Galton-Watson process (observed by Neveu) and $(\langle 1, X_{D_x}\rangle, x \geq 0)$ is a CSBP:
  - supercritical for $c \leq -\sqrt{2\mathtt{a}}$
  - subcritical for $c \geq \sqrt{2\mathtt{a}}$ $\Rightarrow$ max left (right) most speed $-\sqrt{2\mathtt{a}}$ $(\sqrt{2\mathtt{a}})$.

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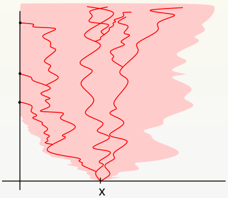

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- Moreover Law$(Z_{D_x}(\cdot)|\Lambda_{D_x}(\cdot)) \sim$ Poisson Random Field$(\lambda^*\Lambda_{D_x}(\cdot))$.

Growth of mass on the exit boundary

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- Maillard (2010) (see also Addario-Berry and Broutin (2009) for BRW) show that when $c = -\sqrt{2\mathrm{a}}$

$$\mathbf{P}_{\delta_x}(Z_{D_x} > n) \sim \frac{\sqrt{2\mathrm{a}}xe^{\sqrt{2\mathrm{a}}x}}{n(\log n)^2}, \; n \to \infty$$

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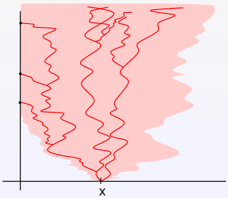
- Through the Poissonization of Z_{D_x} by Λ_{D_x} it is easy to show that the above asymptotic transfers through the above FKPP equation, into the FKPP equation for Λ

$$\frac{1}{2}\Phi''(x) - \sqrt{2\mathrm{a}}\Phi'(x) - \psi(\Phi(x)) = 0,\ \ \Phi(0) = 0,\ \Phi(+\infty) = \lambda^*.$$

to give

$$\mathbf{P}_{\delta_x}(\Lambda_{D_x} > t) \sim \frac{\sqrt{2\mathrm{a}}xe^{\sqrt{2\mathrm{a}}x}}{t(\log t)^2},\ t \to \infty.$$

Right most speed of the support for killed superBM



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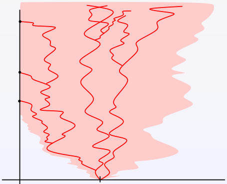
- Consider the case that we kill superBM with drift $c > \sqrt{2\mathrm{a}}$ at the origin.



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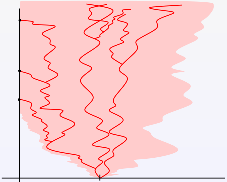
- Consider the case that we kill superBM with drift $c > \sqrt{2a}$ at the origin.
- $R_t^Z = \inf\{y > 0 : Z_t(y, \infty) = 0\}$ and $R_t^\Lambda = \inf\{y > 0 : \Lambda_t(y, \infty) = 0\}$

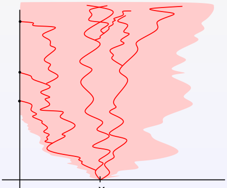


Right most speed of the support for killed superBM

- Consider the case that we kill superBM with drift $c > \sqrt{2\mathsf{a}}$ at the origin.
- $R^Z_t = \inf\{y > 0 : Z_t(y,\infty) = 0\}$ and $R^\Lambda_t = \inf\{y > 0 : \Lambda_t(y,\infty) = 0\}$
- We already know that

$$\limsup_{t\to\infty} \frac{R^\Lambda_t}{t} \le \sqrt{2\mathsf{a}}$$

on survival.



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- We already know that

$$\limsup_{t\to\infty} \frac{R_t^\Lambda}{t} \leq \sqrt{2\mathsf{a}}$$

  on survival.
- From the backbone embedding and known results on killed branching Brownian motion,

$$\liminf_{t\to\infty} \frac{R_t^\Lambda}{t} \geq \lim_{t\to\infty} \frac{R_t^Z}{t} = \sqrt{2\mathsf{a}}$$

  on survival.