Spines, backbones and orthopedic surgery.

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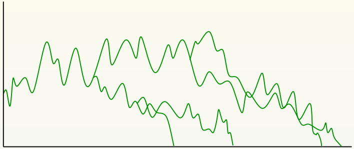

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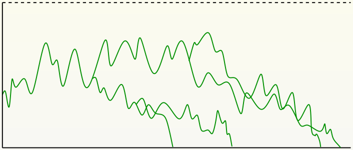

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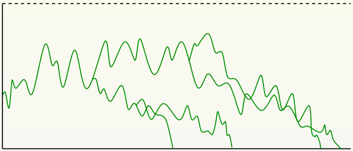

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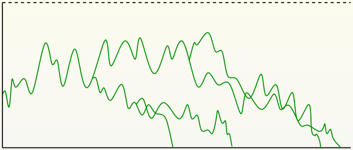
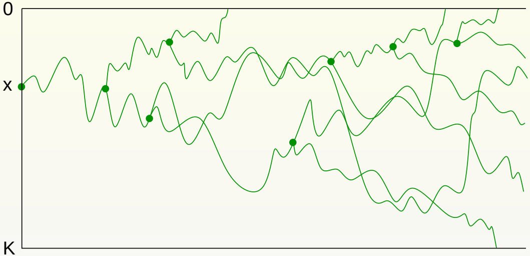


- Their analysis revolves around the behaviour of branching Brownian motion conditioned to stay in a strip next to the origin.
- It is also a natural question to ask how such a process behaves as the strip becomes thinner.
- Specifically, is there a critical width below which there is no possibility of surviving and how does the process behave at criticality?

Branching Brownian motion in a strip $(0, K)$



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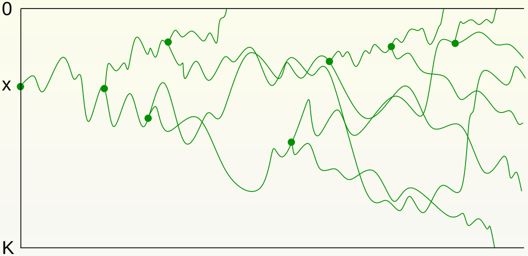

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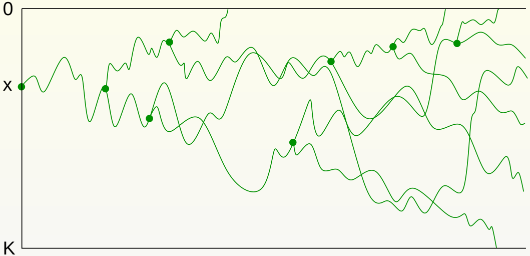

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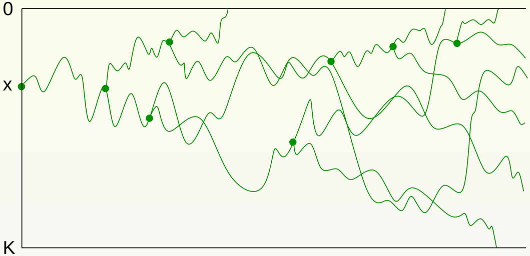

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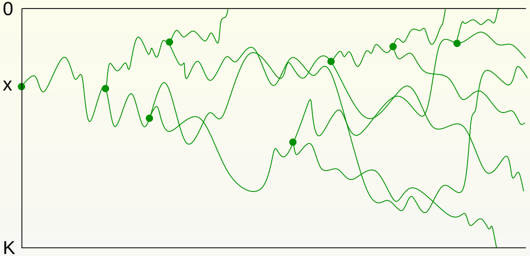

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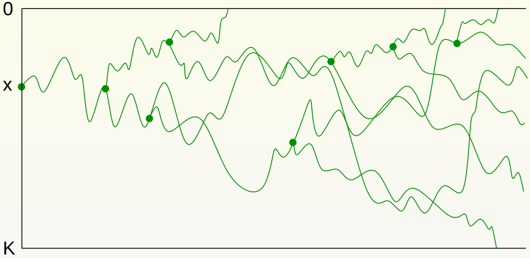

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- $Z = \{Z_t(\cdot) : t \geq 0\}$, where $Z_t(\cdot) = \sum_{i=1}^{N_t} \delta_{x_i(t)}(\cdot)$, is the sequence of random measures which describes the evolution of particles.
- The process becomes extinct at time $\zeta^K := \inf\{t > 0 : Z_t(0, K) = 0\}$.

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- **Theorem:** (i) When $K > K^*$ then $\phi_K \in (0,1)$ on $(0, K)$ and is the unique solution to the ODE

$$\frac{1}{2}f'' + \beta(f^2 - f) = 0 \text{ on } (0, K) \text{ and } f(0) = f(K) = 1. \tag{1}$$

(ii) When $K \leq K^*$ then $\phi_K \equiv 1$ and the ODE (1) has no solutions valued in $[0, 1]$ other than the trivial ones.

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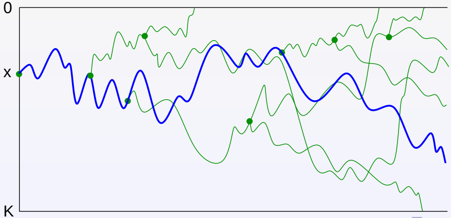
  induces a spine decomposition:
- (i) Run a Brownian motion conditioned to stay in $(0, K)$ - the spine.
  (ii) At rate 2β dress the path of the spine with independent copies of $\mathbb{P}^K_\cdot$-BBMs.



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- Both

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have the property that for $x \in (0, K)$

$$\prod_{i=1}^{N_t} \phi_K(x_i(t)) \text{ and } \prod_{i=1}^{N_t} \psi_K(x_i(t))$$

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are bounded martingales and hence both ϕ_K and ψ_K solve (1).
- Conversely, for any solution f to (1),
$$\prod_{i=1}^{N_t} f(x_i(t))$$
is a bounded martingale with limit $\mathbf{1}_{\{\zeta^K < \infty\}}$.

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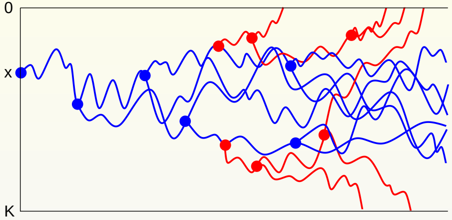
- Look instead for a quasi-stationary type result and try to understand if there is any meaning to the limit

$$\lim_{K \downarrow K^*} \mathbb{P}_x^K(\cdot | \zeta^K = \infty)$$

Blue and Red for $K > K^*$



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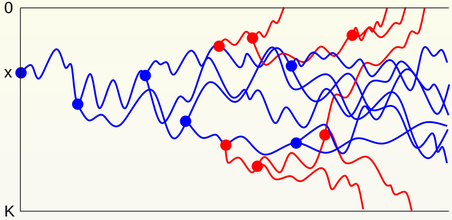

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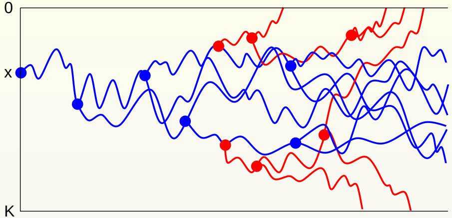

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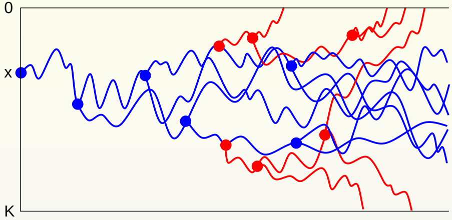

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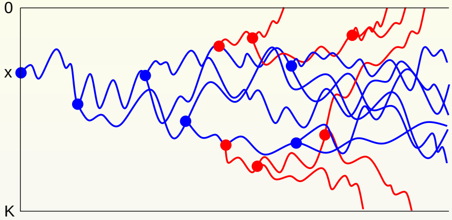

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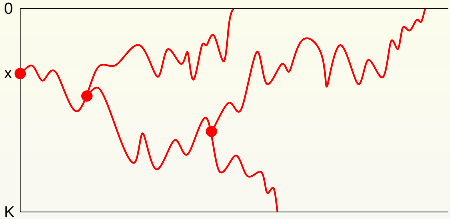

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- What happens if there is no blue tree?

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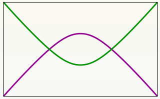

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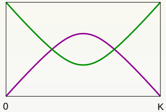

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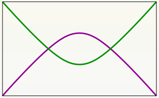

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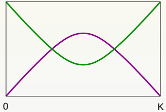

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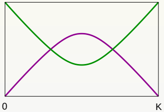

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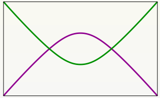

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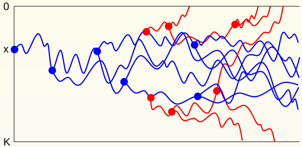
  - Can be shown that Red describes $\mathbb{P}_\cdot^K(\cdot|\zeta^K < \infty)$.

Backbone decomposition for $K > K^*$



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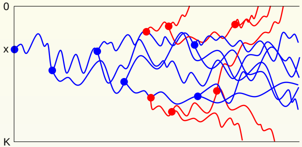

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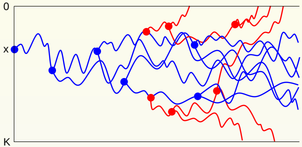

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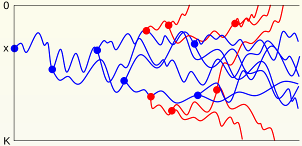

Theorem. For $x \in (0, K)$, $\mathbb{P}_x^K$ has the same law as a colour blind view of:

- Flip a coin with probability $\phi_K(x)$ of 'heads'.
- If 'heads' then grow a Red tree.
- If 'tails' then grow a Blue tree and with rate $2\beta\phi_K(\cdot)$ 'dress' the spatial paths of the Blue tree with independent Red trees.

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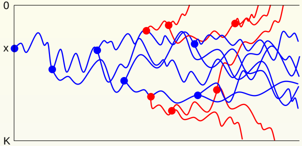

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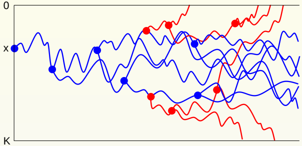
A significance convenience from this construction:

Backbone decomposition for $K > K^*$



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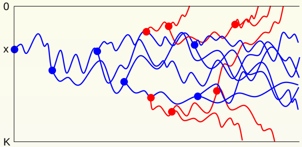

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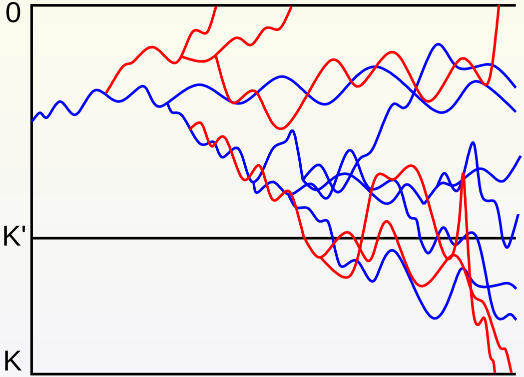
- Equivalently $\mathbb{P}_x^K(\cdot|\zeta^K = \infty)$ has the same law as the backbone construction conditioned on throwing a 'tail'.

Orthopedic surgery ($K > K' > K^*$)



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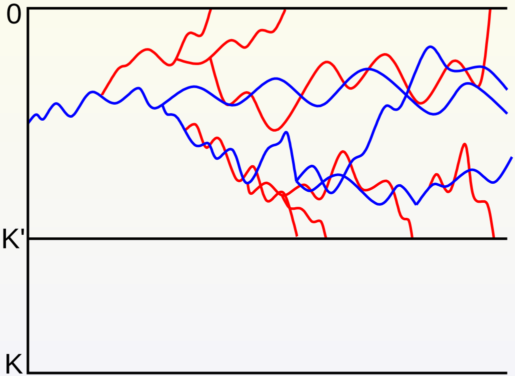

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- **Theorem.** The backbone becomes a spine through orthopedic surgery and gives the quasi-stationary result:

$$\lim_{K\downarrow K^*} \mathbb{P}^K_x(\cdot|\zeta^K = \infty) = P^*_x(\cdot),$$

  for $x \in (0, K^*)$ where P^*_x is the law of a particle system consisting of
  - a spine behaving as a Brownian motion conditioned to stay in the interval $(0, K^*)$,
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Quasi-stationary limit as $K \downarrow K^*$

- $\phi_K(\cdot) \uparrow 1$: $1 - \phi_K(x) \sim c_K \sin(\pi x/K)$ as $K \downarrow K^*$.
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