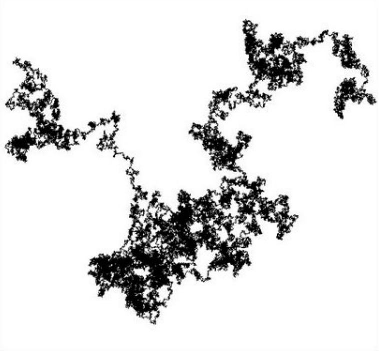
Deep factorisation of the stable process:
Part I: Lagrange lecture, Torino University
Part II: Seminar, Collegio Carlo Alberto.

Andreas E. Kyprianou, University of Bath, UK.

Your favourite Markov process

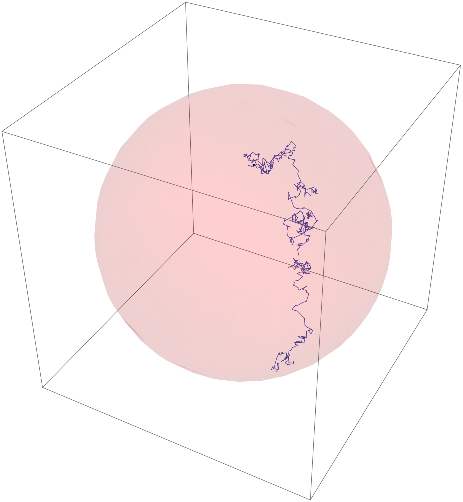
Brownian motion in $\mathbb{R}^d$, $B := \{B_t : t \geq 0\}$, has the defining property that:

- For $t > s > 0$, $B_t - B_s \stackrel{d}{=} B_{t-s} \sim \mathcal{N}_d(0, \mathbf{I}(t-s))$
- For $t > s > 0$, $B_t - B_s$ is independent of $\{B_u : u \leq s\}$
- B has continuous paths



With Brownian motion, you can.....

Solve a Dirichlet boundary value problem....



With Brownian motion, you can.....

Try to model the stock market........

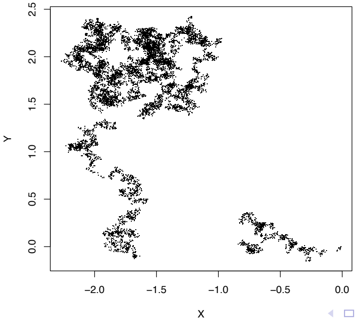
With Brownian motion, you can.....

Try to model the stock market.......and fail....

Take it to the next level: Lévy processes

The last 20 years has seen interest in bigger class of Lévy processes (that contains Brownian motion). An $\mathbb{R}^d$ valued Lévy process, $X := \{X_t : t \geq 0\}$ has almost the same properties as Brownian motion:

- For $t > s > 0$, $X_t - X_s \stackrel{d}{=} X_{t-s}$
- For $t > s > 0$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$
- X has paths that are right-continuous with left limits



- **Roughly speaking:** A Lévy process is made up of a linear Brownian motion plus a process of (up to a countable infinity) of jumps (over any finite time horizon), e.g. in one-dimension

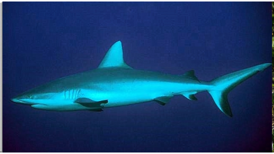
$$X_t = at + \sigma B_t + J_t.$$

- The process is entirely characterised by: a, σ and Π, the latter is a measure on $\mathbb{R}\backslash\{0\}$.
- The measure Π can be thought as a rate measure:

$$P(\text{Jump of size } x \text{ arrives at time } t) = \Pi(\mathrm{d}x)\mathrm{d}t + o(\mathrm{d}t).$$

With Lévy processes you can try.....

Try to model the foraging/flight/feeding patterns of various animals......

With Lévy processes you can.....

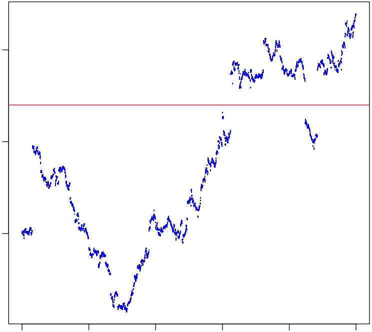
Try to model the foraging/flight/feeding patterns of various animals......

.....as well as humans....

IKEA®

With Lévy processes you can.....

Try to model the stock market .....



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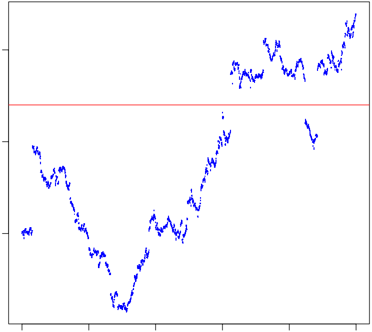

The Wiener–Hopf factorisation

For a one-dimensional Lévy process:

- To characterise the law of a Lévy process, all we need to know is $E[\mathrm{e}^{\mathrm{i}\theta X_t}]$ for all $t \geq 0$
- Stationary and independent increments imply that

$$E[\mathrm{e}^{\mathrm{i}\theta X_t}] = \mathrm{e}^{-\Psi(\theta)t}, \qquad \theta \in \mathbb{R},$$

  where

$$\Psi(\theta) = \mathrm{i}a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - \mathrm{e}^{\mathrm{i}\theta x} + \mathrm{i}\theta x \mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x), \qquad \theta \in \mathbb{R}.$$

- If X is a process with monotone paths (called a subordinator), then it is more usual to consider the Laplace exponent

$$E[\mathrm{e}^{-\lambda X_t}] = \mathrm{e}^{-\kappa(\lambda)t}, \qquad \lambda \geq 0.$$

- In that case, we call κ a **Bernstein** function.

The Wiener–Hopf factorisation

- For a given characteristic exponent of a Lévy process, Ψ, there exist unique Bernstein functions, κ and $\hat{\kappa}$ such that, up to a multiplicative constant,

$$\Psi(\theta) = \hat{\kappa}(\mathrm{i}\theta)\kappa(-\mathrm{i}\theta), \qquad \theta \in \mathbb{R}.$$

- The probabilistic significance of the subordinators corresponding to κ and $\hat{\kappa}$, is that their range corresponds precisely to the range of the running maximum of X and of $-X$ respectively.

Self-similar Markov processes (ssMp)

Definition

A regular strong Markov process $(Z_t : t \geq 0)$ on $\mathbb{R}^d$, with probabilities $\mathbb{P}_x$, $x \in \mathbb{R}^d$, is a ssMp if there exists an index $\alpha \in (0, \infty)$ such that:

for all $c > 0$ and $x \in \mathbb{R}^d$

$$(cZ_{tc^{-\alpha}} : t \geq 0) \text{ under } \mathbb{P}_x$$

is equal in law to

$$(Z_t : t \geq 0) \text{ under } \mathbb{P}_{cx}.$$

Some of your best friends are ssMp

- The moment generating function of a one-dimensional Brownian motion B, satisfies, for θR,

$$E[e^{\theta B_t}] = e^{t\theta^2/2} = e^{(c^{-2}t)(c\theta)^2/2} = E[e^{\theta(cB_{c^{-2}t})}].$$

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Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}$-Brownian motion:

- Write $\underline{X}_t := \inf_{s \leq t} X_s$. Then $(X_t, \underline{X}_t)$, $t \geq 0$ is a Markov process.
- For $c > 0$ and $\alpha = 2$,

$$\begin{pmatrix} c\underline{X}_{c^{-\alpha}t} \\ cX_{c^{-\alpha}t} \end{pmatrix} = \begin{pmatrix} c \inf_{s \leq c^{-\alpha}t} X_s \\ cX_{c^{-\alpha}t} \end{pmatrix} = \begin{pmatrix} \inf_{u \leq t} cX_{c^{-\alpha}u} \\ cX_{c^{-\alpha}t} \end{pmatrix}, \qquad t \geq 0,$$

  and the latter is equal in law to $(X, \underline{X})$, because of the scaling property of X.
- $\Rightarrow$ Markov process $Z_t := X_t - (-x \wedge \underline{X}_t)$, $t \geq 0$ is also a ssMp on $[0, \infty)$ with index 2.
- $\Rightarrow$ $Z_t := X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \geq 0$ is also a ssMp, again on $[0, \infty)$.

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Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}^d$-Brownian motion:

- Consider $Z_t := |X_t|$, $t \geq 0$. Because of rotational invariance, it is a Markov process. Again the self-similarity (index 2) of Brownian motion, transfers to the case of $|X|$. Note again, this is a ssMp on $[0, \infty)$
- Note that $|X_t|$, $t \geq 0$ is a Bessel-d process. It turns out that all Bessel processes, *and* all squared Bessel processes are self-similar on $[0, \infty)$. Once can check this by e.g. considering scaling properties of their transition semi-groups.

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- Note when $d = 3$, $|X_t|$, $t \geq 0$ is also equal in law to a Brownian motion conditioned to stay positive: i.e if we define, for a 1-d Brownian motion $(B_t : t \geq 0)$,

$$\mathbb{P}^{\uparrow}_x(A) = \lim_{s\to\infty} \mathbb{P}_x(A|\underline{B}_{t+s} > 0) = \mathbb{E}_x\left[\frac{B_t}{x}\mathbf{1}_{(\underline{B}_t>0)}\mathbf{1}_{(A)}\right]$$

where $A \in \sigma\{X_t : u \leq t\}$, then

$$(|X_t|, t \geq 0) \text{ with } |X_0| = x \text{ is equal in law to } (B, \mathbb{P}^{\uparrow}_x).$$

More examples?

- All of the previous examples have in common that their paths are continuous. Is this a necessary condition?
- We want to find more exotic examples as most of the previous examples have been extensively studied through existing theories (of Brownian motion and continuous semi-martingales).

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- All of the previous examples are functional transforms of Brownian motion and have made use of the scaling and Markov properties and (in some cases) isometric distributional invariance.
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α-stable process

Definition

A Lévy process X is called (strictly) α-stable if it is also a self-similar Markov process.

- Necessarily $\alpha \in (0,2]$. [$\alpha = 2 \rightarrow$ BM, exclude this.]
- The characteristic exponent $\Psi(\theta) := -t^{-1} \log \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_t})$ satisfies

$$\Psi(\theta) = |\theta|^\alpha (\mathrm{e}^{\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(\theta>0)} + \mathrm{e}^{-\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(\theta<0)}), \qquad \theta \in \mathbb{R}.$$

  where $\rho = \mathrm{P}_0(X_t \geq 0)$ will frequently appear as will $\hat{\rho} = 1 - \rho$
- Assume jumps in both directions ($0 < \alpha\rho, \alpha\hat{\rho} < 1$), so that the Lévy **density** takes the form

$$\frac{\Gamma(1+\alpha)}{\pi} \frac{1}{|x|^{1+\alpha}} \left(\sin(\pi\alpha\rho)\mathbf{1}_{\{x>0\}} + \sin(\pi\alpha\hat{\rho})\mathbf{1}_{\{x<0\}}\right)$$

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Your new friends

Suppose $X = (X_t : t \geq 0)$ is within the assumed class of α-stable processes in one-dimension and let $\underline{X}_t = \inf_{s \leq t} X_s$. Your new friends are:

- $Z = X$
- $Z = X - (-x \wedge \underline{X})$, $x > 0$.
- $Z = X\mathbf{1}_{(\underline{X} > 0)}$
- $Z = |X|$ providing $\rho = 1/2$
- What about $Z =$ "X conditioned to stay positive"?

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Conditioned α-stable processes

- It can be shown that for $A \in \sigma(\xi_u : u \leq t)$,

$$\mathbb{P}^{\uparrow}_x(A) = \lim_{s\to\infty} \mathbb{P}_x(A|\underline{X}_{t+s} > 0) = \mathbb{E}_x\left[\frac{X_t^{\alpha\hat{\rho}}}{x^{\alpha\hat{\rho}}}\mathbf{1}_{(\underline{X}_t>0)}\mathbf{1}_{(A)}\right]$$

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Notation

- Use $\xi := \{\xi_t : t \geq 0\}$ to denote a Lévy process which is killed and sent to the cemetery state $-\infty$ at an independent and exponentially distributed random time, $\mathbf{e}_q$, with rate in $q \in [0, \infty)$. The characteristic exponent of ξ is thus written

$$-\log E(\mathrm{e}^{\mathrm{i}\theta\xi_1}) = \Psi(\theta) = q + \text{Lévy–Khintchine}$$

- Define the associated integrated exponential Lévy process

$$I_t = \int_0^t \mathrm{e}^{\alpha\xi_s} \mathrm{d}s, \qquad t \geq 0. \tag{1}$$

and its limit, $I_\infty := \lim_{t\uparrow\infty} I_t$.

- Also interested in the inverse process of I:

$$\varphi(t) = \inf\{s > 0 : I_s > t\}, \qquad t \geq 0. \tag{2}$$

As usual, we work with the convention $\inf \emptyset = \infty$.

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Lamperti transform for POSITIVE ssMp

Theorem (Part (i))

Fix $\alpha > 0$. If $Z^{(x)}$, $x > 0$, is a positive self-similar Markov process with index of self-similarity α, then up to absorption at the origin, it can be represented as follows. For $x > 0$,

$$Z_t^{(x)} \mathbf{1}_{(t<\zeta^{(x)})} = x \exp\{\xi_{\varphi(x^{-\alpha} t)}\}, \qquad t \geq 0,$$

where $\zeta^{(x)} = \inf\{t > 0 : Z_t^{(x)} = 0\}$ and either

(1) $\zeta^{(x)} = \infty$ *almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying* $\limsup_{t\uparrow\infty} \xi_t = \infty$,

(2) $\zeta^{(x)} < \infty$ *and $Z^{(x)}_{\zeta^{(x)}-} = 0$ almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying* $\lim_{t\uparrow\infty} \xi_t = -\infty$, *or*

(3) $\zeta^{(x)} < \infty$ *and $Z^{(x)}_{\zeta^{(x)}-} > 0$ almost surely for all $x > 0$, in which case ξ is a Lévy process killed at an independent and exponentially distributed random time.*

In all cases, we may identify $\zeta^{(x)} = x^\alpha I_\infty$.

Lamperti transform for POSITIVE ssMp

Theorem (Part (ii))

Conversely, suppose that ξ is a given (killed) Lévy process. For each $x > 0$, define

$$Z_t^{(x)} = x \exp\{\xi_{\varphi(x^{-\alpha} t)}\}\mathbf{1}_{(t < x^\alpha I_\infty)}, \qquad t \geq 0.$$

Then $Z^{(x)}$ defines a positive self-similar Markov process, up to its absorption time $\zeta^{(x)} = x^\alpha I_\infty$, with index α.

Lamperti transform for POSITIVE ssMp

$(Z, \mathrm{P}_x)_{x>0}$ pssMp	$\leftrightarrow$	$(\xi, \mathbb{P}_y)_{y\in\mathbb{R}}$ killed Lévy
$Z_t = \exp(\xi_{S(t)})$,		$\xi_s = \log(Z_{T(s)})$,
S a random time-change		T a random time-change

$$\left.\begin{array}{c} Z \text{ never hits zero} \\ Z \text{ hits zero continuously} \\ Z \text{ hits zero by a jump} \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \xi \to \infty \text{ or } \xi \text{ oscillates} \\ \xi \to -\infty \\ \xi \text{ is killed} \end{array}\right.$$

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Stable process killed on entry to $(-\infty, 0)$

- The stable process cannot 'creep' downwards across the threshold 0 and so must do so with a jump.
- This puts $Z^*_t := X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \geq 0$, in the class of pssMp for which the underlying Lévy process experiences exponential killing.
- Write $\xi^* = \{\xi^*_t : t \geq 0\}$ for the underlying (killed) Lévy process.

Stable process killed on entry to $(-\infty, 0)$

- The stable process cannot 'creep' downwards across the threshold 0 and so must do so with a jump.
- This puts $Z^*_t := X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \geq 0$, in the class of pssMp for which the underlying Lévy process experiences exponential killing.
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Stable process killed on entry to $(-\infty, 0)$

Theorem

For the pssMp constructed by killing a stable process on first entry to $(-\infty, 0)$, the underlying Lévy process, ξ^*, that appears through the Lamperti transform has characteristic exponent given by

$$\Psi^*(z) = \frac{\Gamma(\alpha - \mathrm{i}z)}{\Gamma(\alpha\hat{\rho} - \mathrm{i}z)} \frac{\Gamma(1 + \mathrm{i}z)}{\Gamma(1 - \alpha\hat{\rho} + \mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

Stable processes conditioned to stay positive

- Use the Lamperti representation of the α-stable process X to write, for $A \in \sigma(X_u : u \leq t)$,

$$\mathbb{P}_x^{\uparrow}(A) = \mathbb{E}_x\left[\frac{X_t^{\alpha\hat{\rho}}}{x^{\alpha\hat{\rho}}}\mathbf{1}_{(\underline{X}_t>0)}\mathbf{1}_{(A)}\right] = E\left[\mathrm{e}^{\alpha\hat{\rho}\xi^*_\tau}\mathbf{1}_{(\tau<\mathbf{e}_{q^*})}\mathbf{1}_{(A)}\right],$$

where $\tau = \varphi(x^{-\alpha}t)$ is a stopping time in the natural filtration of ξ^*.

- Noting that $\Psi^*(-\mathrm{i}\alpha\hat{\rho}) = 0$, the change of measure constitutes an Esscher transform at the level of ξ^*.

Theorem

The underlying Lévy process, $\xi^{\uparrow}$, that appears through the Lamperti transform applied to $(X, \mathbb{P}_x^{\uparrow})$, $x > 0$, has characteristic exponent given by

$$\Psi^{\uparrow}(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(-\mathrm{i}z)}\frac{\Gamma(1+\alpha\hat{\rho}+\mathrm{i}z)}{\Gamma(1+\mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

- In particular $\Psi^{\uparrow}(z) = \Psi^*(z - \mathrm{i}\alpha\hat{\rho})$, $z \in \mathbb{R}$ so that $\Psi^{\uparrow}(0) = 0$ (i.e. no killing!)
- One can also check by hand that $\Psi^{\uparrow\prime}(0+) = E[\xi_1^{\uparrow}] > 0$ so that $\lim_{t\to\infty}\xi_t^{\uparrow} = \infty$.

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Did you spot the other root?

- In essence, the case of the stable process conditioned to stay positive boils down to an Esscher transform in the underlying (Lamperti-transformed) Lévy process.
- It was important that we identified a root of $\Psi^*(z) = 0$ in order to avoid involving a 'time component' of the Esscher transform.
- However, there is another root of the equation

$$\Psi^*(z) = \frac{\Gamma(\alpha - \mathrm{i}z)}{\Gamma(\alpha\hat{\rho} - \mathrm{i}z)} \frac{\Gamma(1 + \mathrm{i}z)}{\Gamma(1 - \alpha\hat{\rho} + \mathrm{i}z)} = 0,$$

namely $z = -\mathrm{i}(1 - \alpha\hat{\rho})$.

- And this means that

$$\mathrm{e}^{(1-\alpha\hat{\rho})\xi^*}, \qquad t \geq 0,$$

is a unit-mean Martingale, which can also be used to construct an Esscher transform:

$$\Psi^{\downarrow}(z) = \Psi^*(z - \mathrm{i}(1 - \alpha\hat{\rho})) = \Psi^{\downarrow}(z) = \frac{\Gamma(1 + \alpha\rho - \mathrm{i}z)}{\Gamma(1 - \mathrm{i}z)} \frac{\Gamma(\mathrm{i}z + \alpha\hat{\rho})}{\Gamma(\mathrm{i}z)}.$$

- The choice of notation is pre-emptive since we can also check that $\Psi^{\downarrow}(0) = 0$ and $\Psi^{\downarrow\prime}(0) < 0$ so that if $\xi^{\downarrow}$ is a Lévy process with characteristic exponent $\Psi^{\downarrow}$, then $\lim_{t\to\infty} \xi^{\downarrow}_t = -\infty$.

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Reverse engineering

- What now happens if we define for $A \in \sigma(X_u : u \le t)$,

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- In the same way we checked that $(X, \mathbb{P}^{\uparrow}_x)$, $x > 0$, is a pssMp, we can also check that $(X, \mathbb{P}^{\downarrow}_x)$, $x > 0$ is a pssMp.
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ξ^*, $\xi^\uparrow$ and $\xi^\downarrow$

- The three examples of pssMp offer quite striking underlying Lévy processes.
- In particular, each of them have characteristic exponent written as the ratio of two pairs of gamma functions.
- Is this exceptional?

Censored stable processes

- Start with X, the stable process.
- Let $A_t = \int_0^t \mathbf{1}_{(X_t>0)}\,\mathrm{d}t$.
- Let γ be the right-inverse of A, and put $\check{Z}_t := X_{\gamma(t)}$.
- Finally, make zero an absorbing state: $Z_t = \check{Z}_t \mathbb{1}_{(t<T_0)}$ where

$$T_0 = \inf\{t > 0 : X_t = 0\}.$$

  Note $T_0 < \infty$ a.s. if and only if $\alpha \in (1,2)$ and otherwise $T_0 = \infty$ a.s.
- This is the censored stable process.

Censored stable processes

Theorem

Suppose that the underlying Lévy process for the censored stable process is denoted by $\overset{\rightsquigarrow}{\xi}$. *Then its characteristic exponent is given by*

$$\overset{\rightsquigarrow}{\Psi}(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(-\mathrm{i}z)} \frac{\Gamma(1 - \alpha\rho + \mathrm{i}z)}{\Gamma(1 - \alpha + \mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

The radial part of a stable process

- Suppose that X is a symmetric stable process, i.e $\rho = 1/2$.
- We know that $|X|$ is a pssMp.

Theorem

Suppose that the underlying Lévy process for $|X|$ is written $\xi^{\odot}$, then it characteristic exponent is given by

$$\Psi^{\odot}(z) = 2^{\alpha}\frac{\Gamma(\frac{1}{2}(-\mathrm{i}z+\alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)}\frac{\Gamma(\frac{1}{2}(\mathrm{i}z+1))}{\Gamma(\frac{1}{2}(\mathrm{i}z+1-\alpha))}, \qquad z \in \mathbb{R}.$$

Hypergeometric Lévy processes

Definition (and Theorem)

For $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ in

$$\left\{ \ \beta \leq 2,\ \gamma, \hat{\gamma} \in (0,1)\ \hat{\beta} \geq -1,\ \text{and } 1 - \beta + \hat{\beta} + \gamma \wedge \hat{\gamma} \geq 0 \ \right\}$$

there exists a (killed) Lévy process, henceforth refered to as a hypergeometric Lévy process, having the characteristic function

$$\Psi(z) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}z)}{\Gamma(1-\beta-\mathrm{i}z)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}z)}{\Gamma(\hat{\beta}+\mathrm{i}z)} \qquad z \in \mathbb{R}.$$

The Lévy measure of Y has a density with respect to Lebesgue measure is given by

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\hat{\gamma})\Gamma(-\gamma)} \mathrm{e}^{-(1-\beta+\gamma)x} {}_2F_1\left(1+\gamma, \eta; \eta-\hat{\gamma}; \mathrm{e}^{-x}\right), & \text{if } \ x > 0, \\ -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\gamma)\Gamma(-\hat{\gamma})} \mathrm{e}^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1\left(1+\hat{\gamma}, \eta; \eta-\gamma; \mathrm{e}^{x}\right), & \text{if } \ x < 0, \end{cases}$$

where $\eta := 1 - \beta + \gamma + \hat{\beta} + \hat{\gamma}$, for $|z| < 1$, ${}_2F_1(a, b; c; z) := \sum_{k \geq 0} \frac{(a)_k (b)_k}{(c)_k k!} z^k$.

Grazie!

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