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Topics in Stochastic Geometry

Lecture 5
Models of continuum percolation

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1. Percolation in the Boolean model

Definition

A random closed subset of $\mathbb{R}^d$ (or random graph) percolates if it has an unbounded component.

Theorem

Consider a Boolean model Z in $\mathbb{R}^d$ (with $d \geq 2$) where the typical grain is a deterministic ball with radius $R_0 > 0$. Then there exists a critical intensity $\lambda_c > 0$ such such Z percolates for $\lambda > \lambda_c$ and does not percolate for $\lambda < \lambda_c$.

Remark

Let Z be a Boolean model as above. The critical percolation threshold and the critical volume fraction $p_c := 1 - e^{-\lambda_c \kappa_d R_0^d}$ are „known“ from simulation:

$$p_c \approx \begin{cases} 0.6763475, & \text{if } d = 2, \\ 0.289573, & \text{if } d = 3. \end{cases}$$

Theorem (Penrose ’96)

As $d \to \infty$, the critical degree $\lambda_c(d)\kappa_d(2R_0)^d$ tends to 1.

Theorem (Meester and Roy ’94)

A Boolean model Z as above has (almost surely) at most one unbounded component. Also the complement has at most one unbounded component. In particular, for $\lambda > \lambda_c$ there is exactly one unbounded component.

Remark

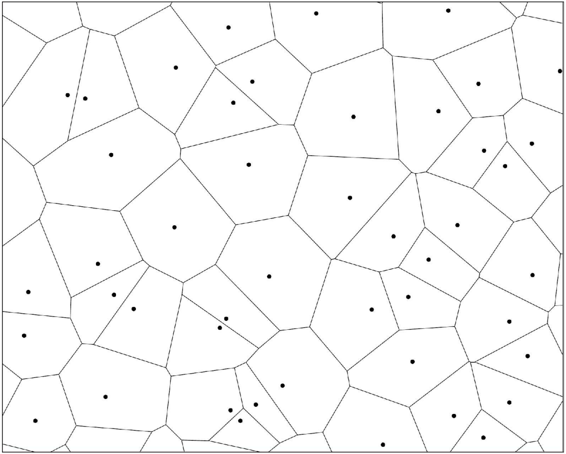
It is believed that Z does not percolate at the critical intensity. This was proved for $d = 2$ (Alexander ’96) and for large d (Tanemura ’96).

Remark

The above concepts can also be discussed for more general Boolean models. The critical threshold λ_c depends on the distribution (and the geometry) of the typical grain in a non-trivial way.

2. Poisson Voronoi percolation

Let X be a Poisson Voronoi tessellation. Declare the cells in X independently open with probability p and let Z be the union of all open cells.



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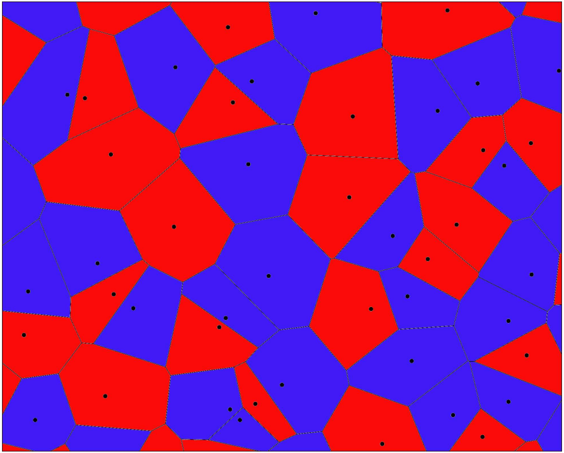
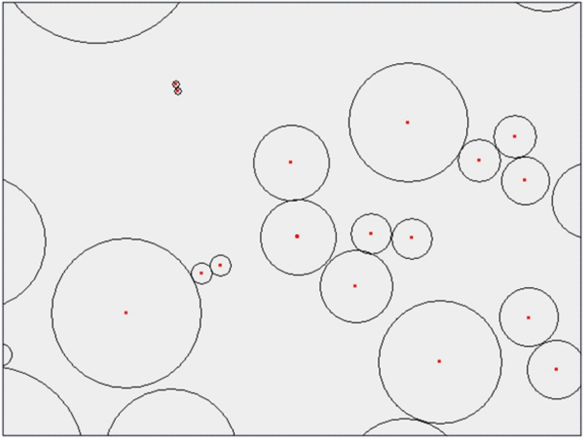


Theorem (Bollobás & Riordan '06)

Consider planar Poisson Voronoi percolation. Then $p_c = 1/2$*. At this critical density there is no percolation, while above there is exactly one unbounded component.*

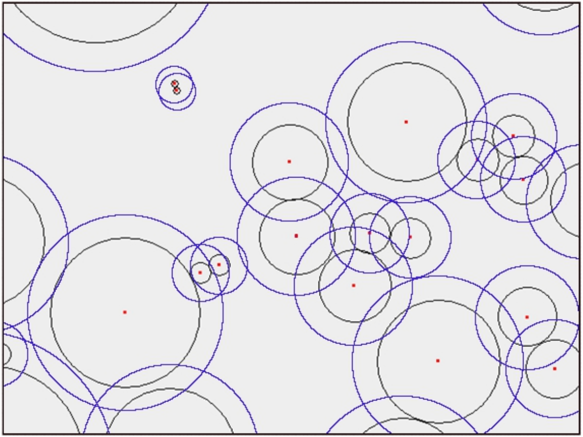
3. Poisson lilypond model

Poisson points grow at uniform speed in all normal directions until they hit another ball.



Theorem (Häggström & Meester '96, L. & Penrose '10)

There is no percolation in the Poisson lilypond model Z. This remains even true for the parallel set Z^{δ} of Z for sufficiently small but positive δ.



4. Continuum percolation on planar tessellations

Idea

Introduce a hierachy of percolation models on a planar stationary tessellation X. Study the model via the geometric properties of a suitably defined random set Z having the same connectivity properties.

Definition

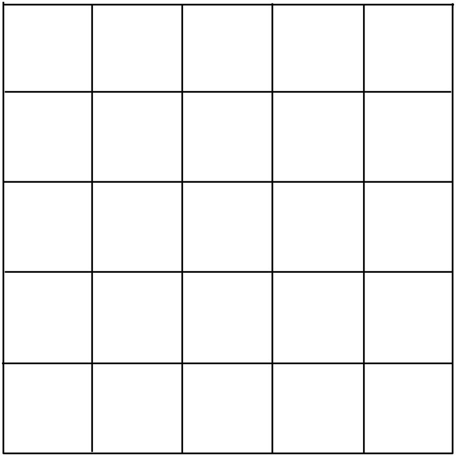
The mean Euler characteristic of Z is defined as

$$\bar{\chi} := \lim_{r\to\infty} \frac{\mathbb{E}[\chi(Z \cap rW)]}{V_d(rW)},$$

where $\chi(\cdot)$ denotes the Euler characteristic.

Vertex percolation

Given X, the vertices are independently declared open with probability p. An edge is declared open if its endpoints are open. A cell is open if all its vertices are open.



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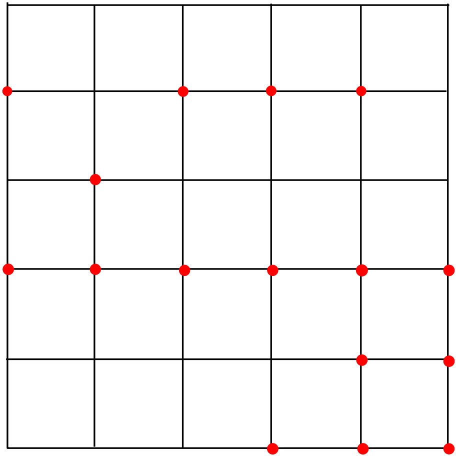

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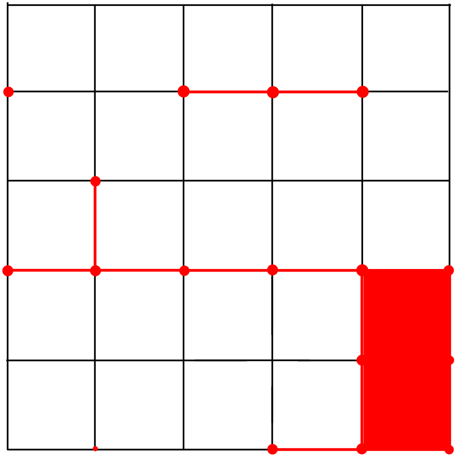
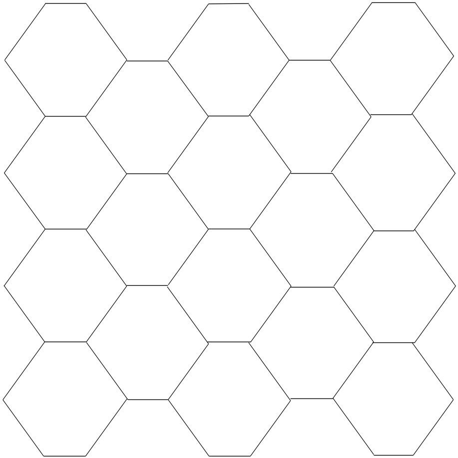


Edge percolation

Declare the edges in X independently open with probability p. A cell is open if all its edges are open.



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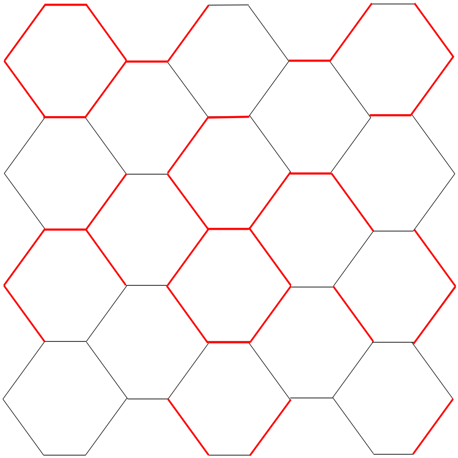

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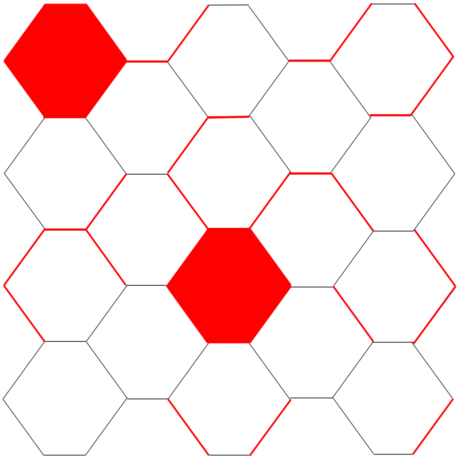

Definition

(i) Let $\gamma_{0,i}$ denote the intensity of vertices of degree $i \geq 3$.

(ii) Let $\gamma_{2,i}$ denote the intensity of cells with i vertices.

(iii) Recall that

$$\gamma_0 := \gamma_{0,3} + \gamma_{0,4} + \cdots$$

is the intensity of vertices and

$$\gamma_2 := \gamma_{2,3} + \gamma_{2,4} + \cdots$$

is the intensity of cells.

Theorem (Neher, Mecke & Wagner ’08, L. ’11)

For continuum percolation on a stationary planar tessellation the mean Euler characteristic is given as follows.
Vertex percolation:

$$\bar{\chi}(p) = \gamma_0 p - (\gamma_0 + \gamma_2)p^2 + \sum_{i=3}^{\infty} \gamma_{2,i} p^i.$$

Edge percolation:

$$\bar{\chi}(p) = \gamma_0 - (\gamma_0 + \gamma_2)p - \sum_{i=3}^{\infty} \gamma_{0,i}(1-p)^i + \sum_{i=3}^{\infty} \gamma_{2,i} p^i.$$

Cell percolation:

$$\bar{\chi}(p) = -\gamma_2(1-p) + (\gamma_0 + \gamma_2)(1-p)^2 - \sum_{i=3}^{\infty} \gamma_{0,i}(1-p)^i.$$

Observation by Neher, Mecke & Wagner '08

For percolation on lattices $\bar{\chi}(p)$ has exactly one zero on the open interval $(0, 1)$ that is pretty close to the critical probability.

Remark

The Euler characteristic of cell percolation on a planar Voronoi tessellation is given by

$$\bar{\chi}(p) = \gamma_2 p(1 - p)(1 - 2p).$$

It has a zero at $p = 1/2$ which is critical in the Poisson case (Bollobás & Riordan '06) and perhaps also for other short-range dependent point processes.

5. References

- B. Bollobás and O. Riordan (2006). The critical probability for random Voronoi percolation in the plane is 1/2. *Probab. Theor. Related Fields* **136**, 417-468.
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