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Topics in Stochastic Geometry

Lecture 4
The Boolean model

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1. Definition of the Boolean model

Definition

Let $\eta = \{X_n : n \in \mathbb{N}\}$ be a Poisson process with intensity λ and $Z_0, Z_1, \ldots$ a sequence of independent random particles with common distribution $\mathbb{Q}$ satisfying the integrability assumption

$$\mathbb{E} V_d(Z_0 + K) < \infty, \quad K \in \mathcal{C}^d.$$

Assume that η and (Z_n) are independent. Then

$$Z = \bigcup_{n \in \mathbb{N}} (Z_n + \xi_n),$$

is called Boolean model with grain distribution $\mathbb{Q}$ and typical grain Z_0.

Theorem

Let $\eta = \{X_n : n \in \mathbb{N}\}$ be a Poisson process with positive intensity and $R_0, R_1, \ldots$ a sequence of independent and identically distributed non-negative random variables. Define

$$Z = \bigcup_{n \in \mathbb{N}} B(X_n, R_n),$$

where $B(x, r)$ is a ball with centre $x \in \mathbb{R}^d$ and radius $r \geq 0$. Then $\mathbb{P}(Z = \mathbb{R}^d) = 1$ iff $\mathbb{E} R_0^d = \infty$.

Remark

Let $Z_0 := (0, R_0)$, where R_0 is as above. Then Z_0 satisfies the required integrability condition iff

$$\mathbb{E} R_0^d < \infty.$$

Lemma

Let Z be a Boolean model as above. Then, almost surely, any bounded set is intersected by only finitely many of the (secondary) grains $Z_n + X_n$. In particular, Z is closed.

Convention

A Boolean model Z is considered as a random closed set, that is as a random element in the space of all closed subsets of $\mathbb{R}^d$ equipped with a suitable σ-field.

2. The capacity functional

Definition

The capacity functional of a random closed set Z is the mapping $T_Z : \mathcal{C}^d \to \mathbb{R}$ defined by

$$T_Z(C) := \mathbb{P}(Z \cap C \neq \emptyset), \quad C \in \mathcal{C}^d.$$

Theorem

The capacity model of a Boolean model Z is given by

$$T_Z(C) = 1 - \exp[-\lambda \mathbb{E} V_d(Z_0 + C^*)], \quad C \in \mathcal{C}^d,$$

where $C^* := \{-x : x \in B\}$.

Corollary

A Boolean model Z is stationary, that is

$$Z \overset{d}{=} Z + x, \quad x \in \mathbb{R}^d.$$

Corollary

Assume that the typical grain Z_0 of a Boolean model Z is isotropic, i.e.

$$Z_0 \overset{d}{=} \vartheta Z_0$$

for any rotation ϑ. Then Z is isotropic.

3. Volume fraction and covariance function

Definition

The volume fraction p of a stationary random closed set Z is defined by

$$p := \mathbb{E} V_d([0,1]^d).$$

An equivalent definition is $p := \mathbb{P}(x \in Z)$ for any $x \in \mathbb{R}^d$.

Corollary

The volume fraction of a Boolean model is given by

$$p = 1 - \exp(-\lambda \mathbb{E} V_d(Z_0))$$

Definition

The covariance function of a stationary random closed set Z is the function $C : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$C(x,y) := \mathbb{P}(\{x,y\} \subset Z) = \mathbb{E}[\mathbf{1}_Z(x)\mathbf{1}_Z(y)], \quad x,y \in \mathbb{R}^d.$$

Theorem

The covariance function of a Boolean model is given by

$$C(x) = 2p - 1 + (1-p)^2 \exp(\lambda C_0(x)), \quad x \in \mathbb{R}^d,$$

where

$$C_0(x) := \mathbb{E}[V_d(Z_0 \cap (Z_0 - x))].$$

4. Ergodicity properties

Definition

A stationary random closed set Z in $\mathbb{R}^d$ is called mixing if

$$\lim_{\|x\|\to\infty} \mathbb{P}(Z \in \mathcal{H} \cap \theta_x \mathcal{H}') = \mathbb{P}(Z \in \mathcal{H})\mathbb{P}(Z \in \mathcal{H}'),$$

for any measurable sets $\mathcal{H}, \mathcal{H}'$ of closed sets. Here

$$\theta_x \mathcal{H}' := \{A + x : A \in \mathcal{H}'\}, \quad x \in \mathbb{R}^d.$$

Remark

Any mixing Z is ergodic, that is

$$\mathbb{P}(Z \in \mathcal{H}) \in \{0, 1\}$$

for all invariant $\mathcal{H}$.

Theorem

A Boolean model Z is mixing.

Remark

The assertion of the theorem can be reduced to the limit relation

$$\lim_{\|x\|\to\infty}(1 - T_Z(C \cup (C' + x)) = (1 - T_Z(C))(1 - T_Z(C')),$$

for all compact $C, C' \subset \mathbb{R}^d$. For the Boolean model this can be directly verified.

Theorem (Spatial ergodic theorem)

Assume that Z is an ergodic random closed set and let $W \subset \mathbb{R}^d$ be a convex and compact set containing the origin in its interior. Let $f(Z)$ be a measurable function of Z such that $\mathbb{E}|f(Z)| < \infty$. Then the limit

$$\lim_{r \to \infty} \frac{1}{V_d(rW)} \int_{rW} f(\theta_x Z) dx$$

exists almost surely and in $L^1(\mathbb{P})$ and is given by $\mathbb{E} f(Z)$.

Example

For a Boolean model Z the limit

$$\lim_{r \to \infty} \frac{1}{V_d(rW)} \int_{rW} \mathbf{1}\{x \in Z\} dx$$

exists and is given by the volume fraction of Z.

Definition

Let $K \subset \mathbb{R}^d$ be compact and convex. The intrinsic volumes of K are the numbers $V_0(K), \dots, V_d(K)$ uniquely determined by the Steiner formula

$$V_d(K + rB^d) = \sum_{j=0}^{d} r^j \kappa_j V_{d-j}(K), \quad r \geq 0,$$

where κ_j is the (j-dimensional) volume of the Euclidean unit ball B^j in $\mathbb{R}^j$.

Remark

$V_d(K)$ is the Lebesgue measure of K. If K has non-empty interior, then $V_{d-1}(K)$ is half the surface area of K. Moreover, $V_0(K) = \mathbf{1}\{K \neq \emptyset\}$.

Remark

Using the inclusion-exclusion formula the intrinsic volumes can be extended (uniquely!) to finite unions K of convex and compact sets. Then $V_{d-1}(K)$ is still half the surface area of K while $V_0(K)$ is the Euler characteristic of K.

Theorem

Let Z be a stationary ergodic random closed set that is locally a finite union of convex sets. Under suitable integrability assumptions the limits

$$\lambda_j := \lim_{r\to\infty} \frac{V_j(Z \cap rW)}{V_d(rW)}, \quad j = 0, \dots, d,$$

exist $\mathbb{P}$-almost surely.

Theorem (Miles)

If Z is a Boolean model with convex typical grain Z_0, then

$$\lambda_{d-1} = \lambda \mathbb{E} V_{d-1}(Z_0) e^{-\lambda \mathbb{E} V_d(Z_0)}.$$

If Z_0 is isotropic, then $\lambda_0, \dots, \lambda_{d-2}$ can be computed explicitly in terms of $e^{-\lambda \mathbb{E} V_d(Z_0)}$ and $\lambda \mathbb{E} V_i(Z_0)$, $i = 0, \dots, d-1$. For instance, we have for $d = 2$ that

$$\lambda_0 = e^{-\lambda \mathbb{E} V_d(Z_0)} \left(\lambda - \frac{\mathbb{E} V_1(Z_0)}{\pi} \right)$$

and in the case $d = 3$

$$\lambda_0 = e^{-\lambda \mathbb{E} V_d(Z_0)} \left(\lambda - \frac{\lambda^2}{4} \mathbb{E} V_1(Z_0) \mathbb{E} V_2(Z_0) + \frac{\pi \lambda^3}{48} (\mathbb{E} V_2(Z_0))^3 \right).$$

5. References

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