Topics in Stochastic Geometry

Lecture 3
Random partitions
and balanced invariant transports

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1. The Monge-Kantorovich problem

Setting

Let ξ and η be measures on $\mathbb{R}^d$ such that

$$0 < \xi(\mathbb{R}^d) = \eta(\mathbb{R}^d) < \infty.$$

Let $c(x, y)$ be the cost of transporting one unit of mass from $x \in \mathbb{R}^d$ to $y \in \mathbb{R}^d$.

Problem (Monge 1781)

Minimize

$$\int c(x, \tau(x))\xi(dx)$$

among all transport maps $\tau : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $\tau^*(\xi) = \eta$, that is

$$\int \mathbf{1}\{\tau(x) \in B\}\xi(dx) = \eta(B), \quad B \in \mathcal{B}^d.$$

Such a τ is called admissable.

Remark

If ξ and η have the same number of atoms of equal size, the Monge Problem corresponds to optimal matching.

Remark

Admissable transports need not exist, for instance if ξ and η have atoms of different sizes.

Remark

If ξ and η are absolutely continuous and $c(x, y) = \|x - y\|^p$ for some $p > 1$ then (under moment assumptions on ξ and η) there is a unique solution of the Monge problem.

Definition (Coupling)

Let $\Pi(\xi, \eta)$ denote the set of all (finite) measures π on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\pi(\cdot \times \mathbb{R}^d) = \xi$ and $\pi(\mathbb{R}^d \times \cdot) = \eta$. Any such π is called a coupling of ξ and η.

Problem (Kantorovich 1940)

Minimize

$$\int c(x, y)\pi(d(x, y))$$

among all $\pi \in \Pi(\xi, \eta)$.

Remark

Any $\pi \in \Pi(\xi, \eta)$ can be identified with a stochastic kernel $T(x, dy)$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that

$$\int T(x, B)\xi(dx) = \eta(B), \quad B \in \mathcal{B}^d.$$

Such a T is called transport kernel.

Remark

If the costs are finite for some transport kernel, then there exists a solution to the Monge-Kantorovich problem.

2. Invariant transports

Setting

Consider measurable mappings $\theta_x : \Omega \to \Omega$, $x \in \mathbb{R}^d$, satisfying $\theta_0 = \mathrm{id}_\Omega$ and the flow property

$$\theta_x \circ \theta_y = \theta_{x+y}, \quad x, y \in \mathbb{R}^d.$$

The mapping $(\omega, x) \mapsto \theta_x\omega$ is assumed measurable. The probability measure $\mathbb{P}$ is assumed stationary under the flow, that is

$$\mathbb{P} \circ \theta_x = \mathbb{P}, \quad x \in \mathbb{R}^d.$$

Definition

A random measure ξ is invariant if

$$\xi(\theta_x\omega, B - x) = \xi(\omega, B), \quad \omega \in \Omega, x \in \mathbb{R}^d, B \in \mathcal{B}^d.$$

Remark

By invariance of $\mathbb{P}$, any invariant random measure is stationary. Any pair of invariant random measures is jointly stationary. If nothing else is said, random measures will always assumed to be invariant.

Definition (Transport kernels and allocation rules)

(i) A weighted transport-kernel is a kernel T from $\Omega \times \mathbb{R}^d$ to $\mathbb{R}^d$ such that $T(\omega, x, \cdot)$ is a locally finite measure for all $(\omega, x) \in \Omega \times \mathbb{R}^d$.

(ii) If T is Markovian, i.e. $T(\omega, x, \cdot)$ is a probability measure for all $(\omega, x) \in \Omega \times G$, then T is called transport kernel.

(iii) A weighted transport-kernel T is invariant, if

$$T(\theta_y \omega, x - y, B - y) = T(\omega, x, B), \quad x, y \in \mathbb{R}^d, \omega \in \Omega, \\ B \in \mathcal{B}^d.$$

If T is of the form $T(\omega, x, \cdot) = \delta_{\tau(\omega,x)}$, then $\tau : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is called allocation rule.

Definition

Let ξ and η be random measures. A weighted transport kernel is balancing ξ and η if

$$\int T(\omega, x, \cdot)\, \xi(\omega, dx) = \eta(\omega, \cdot)$$

$\mathbb{P}$-a.e. $\omega \in \Omega$.

Theorem (L. and Thorisson '09)

Let ξ, η be two (invariant) and ergodic random measures. Then there exists an invariant transport-kernel T balancing ξ and η iff ξ and η have equal intensities, that is

$$\mathbb{E}[\xi([0,1]^d)] = \mathbb{E}[\eta([0,1]^d)].$$

Definition

The cost of an invariant transport kernel T balancing ξ and η is defined as

$$c_T := \mathbb{E}\left[\iint \mathbf{1}\{x \in [0,1]^d\} c(x,y) T(x,dy) \xi(dx)\right].$$

In the case of an allocation rule τ, this simplifies to

$$c_\tau := \mathbb{E}\left[\int \mathbf{1}\{x \in [0,1]^d\} c(x,\tau(x)) \xi(dx)\right].$$

3. Balancing transports and Palm measures

Definition

Let ξ be an invariant random measure of intensity 1. The probability measure

$$\mathbb{P}_\xi(A) := \iint \mathbf{1}\{\theta_x\omega \in A, x \in [0,1]^d\}\,\xi(\omega, dx)\,\mathbb{P}(d\omega), \quad A \in \mathcal{F},$$

is called the Palm measure of ξ.

Remark

The definition of the Palm measure is written as

$$\mathbb{P}_\xi(A) := \mathbb{E}_\mathbb{P}\left[\int \mathbf{1}\{\theta_x \in A, x \in B\}\,\xi(dx)\right], \quad A \in \mathcal{F}.$$

Theorem (L. and Thorisson '09)

Consider two invariant random measures ξ and η of intensity 1 and let T be an invariant weighted transport-kernel. Then T is balancing ξ and η iff

$$\mathbb{E}_{\mathbb{P}_\xi}\left[\int \mathbf{1}\{\theta_y \in \cdot\}\, T(0, dy)\right] = \mathbb{P}_\eta.$$

4. Marriage of Lebesgue and Poisson

Setting

Let η be a stationary Poisson process of intensity 1.

Theorem (Holroyd and Peres '05)

There is an allocation rule balancing Lebesgue measure and η.

Theorem (Holroyd and Peres '05)

Assume that $d \in \{1, 2\}$. Any allocation rule balancing Lebesgue measure and η satisfies

$$\mathbb{E}[|\tau(0)|^{d/2}] = \infty.$$

Theorem (Holroyd and Peres '05)

There exists an invariant transport kernel T balancing Lebesgue measure and η such that

$$\mathbb{E}\left[\exp[c|y|^d]T(0,dy)\right] < \infty$$

for some $c > 0$.

Theorem (Chatterjee, Peled, Peres and Romik '10)

Assume that $d \geq 3$. There exists an allocation rule τ balancing Lebesgue measure and η such that

$$\mathbb{E}\left[\exp[c|\tau(0)|^{1-\varepsilon}]\right] < \infty$$

for some $c > 0$ and all $\varepsilon > 0$.

Theorem (Huesmann and Sturm '10)

Assume that the costs are of the form $c(x, y) = g(|x - y|)$ *for some strictly increasing, continuous function with* $g(0) = 0$ *and* $\lim_{r \to \infty} g(r) = \infty$. *Let* $\Pi(\eta)$ *be the set of all invariant transport kernels balancing Lebesgue measure and* η. *Assume that* c_T *is finite for some* $T \in \Pi(\eta)$. *Then there exist a unique* $T^* \in \Pi(\eta)$ *minimizing the costs. In fact,* T^* *reduces to an allocation rule.*

5. Stationary partitions

Setting

$\eta \neq 0$ is an invariant point process on $\mathbb{R}^d$ with finite intensity λ and $\mathbb{P}$ is a stationary probability measure on $(\Omega, \mathcal{F})$.

Definition

A stationary partition (based on η) is an allocation rule $\tau : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\tau(\omega, x) \in \eta(\omega), \quad \omega \in \Omega.$$

Definition

Let τ be a stationary partition based on η.

(i) The cell with centre $x \in \eta$ is the Borel set

$$C(x) = \{y \in \mathbb{R}^d : \tau(y) = x\}.$$

(ii) The cell containing the site $y \in \mathbb{R}^d$ is the Borel set

$$V(y) := \{z \in \mathbb{R}^d : \tau(z) = \tau(y)\}.$$

Remark

The system $\{C(x) : x \in \eta\}$ forms a partition of $\mathbb{R}^d$ into measurable sets.

Theorem

We have for all measurable $f, g : \Omega \to [0, \infty)$ *that*

$$\mathbb{E}[f \cdot g(\theta_\tau)] = \mathbb{E}_{\mathbb{P}_\eta}\left[g \cdot \int_{C(0)} f(\theta_x)\, dx\right],$$

where $\theta_\tau(\omega) := \theta_{\tau(0,\omega)}(\omega)$. *In particular*

$$\mathbb{E}_{\mathbb{P}_\eta}[V_d(C(0)) \cdot g] = \mathbb{E}[g(\theta_\tau)].$$

Definition

A stationary partition τ (based on η) is balanced, if τ is balancing a multiple of Lebesgue measure and η, that is

$$\mathbb{P}(V_d(C(x)) = \lambda^{-1} \text{ for all } x \in \eta) = 1,$$

or, equivalently,

$$\mathbb{P}^0_\eta(V_d(C(0)) = \lambda^{-1}) = 1.$$

Theorem (Holroyd and Peres '05)

Let τ be a stationary partition. Then τ is balanced if and only if

$$\mathbb{P}(\theta_\tau \in \cdot) = \mathbb{P}^0_\eta.$$

Definition

Let τ be a stationary partition of $\mathbb{R}^d$. A pair $(x, y) \in \mathbb{R}^d \times \eta$ is called unstable (with respect to Euclidean distance $d(\cdot, \cdot)$) if it has the following two properties.

(i) $d(x, y) < d(x, \tau(x))$.

(ii) $d(x, y) < d(z, y)$ for some $z \in C(y)$.

Property (i) means that x prefers y to its actual centre $\tau(x)$. The second property means that y would like to have x in its territory.

Theorem (Hoffman, Holroyd and Peres '06)

Assume that η is ergodic. There is a stationary η-measurable and balanced partition of $\mathbb{R}^d$ that a.s. does not contain unstable pairs. In fact, this partition is uniquely determined.

6. References

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