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Topics in Stochastic Geometry

Lecture 2
Random tessellations

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1. Definition of a tessellation

Definition

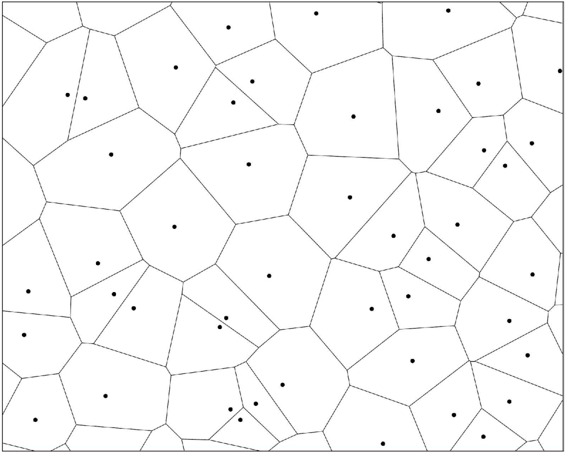
A tessellation (mosaic) in $\mathbb{R}^d$ is a countable system of compact subsets of $\mathbb{R}^d$ (cells) with the following properties.

1. Any bounded set is intersected by only a finite number of the cells.
2. All cells are convex and have a non-empty interior.
3. The union of the cells is all of $\mathbb{R}^d$.
4. The interiors of the cells are mutually disjoint.

Remark

The cells of a mosaic are convex polytopes, that is finite intersections of half-spaces.

A Voronoi tessellation:



Definition

Let φ denote a locally finite subset of $\mathbb{R}^d$, The Voronoi cell $C(\varphi, x)$ of $x \in \varphi$ is the set of all sites $y \in \mathbb{R}^d$ whose distance from x is smaller or equal than the distances to all other points of φ. The point x is called generator of $C(\varphi, x)$. The Voronoi tessellation based on φ is the system

$$\{C(\varphi, x) : x \in \varphi\}.$$

Remark

If the convex hull of φ equals $\mathbb{R}^d$ then the Voronoi cells are bounded. Moreover, if the points of φ are in general quadratic position, then the Voronoi tessellation is normal in the sense that any k-face is contained in exactly $d - k + 1$ cells.

Definition

Let φ denote a locally finite systems of hyperplanes in $\mathbb{R}^d$. The closures of the connected components of $\mathbb{R}^d \setminus \cup_{H \in \varphi} H$ is called hyperplane tessellation generated by φ.

Remark

Voronoi and hyperplane tessellations are face to face, that is faces of different cells do not overlap. In the following all tessellations will be assumed face to face.

Definition

Let C be a convex polytope. Then

$$C = \bigcup_{k \in \{0,\dots,d\}} \bigcup_{C \in \mathcal{S}_k(C)} \operatorname{relint} F,$$

where $\mathcal{S}_k(C)$ is a finite set of k-dimensional polytopes whose affine hulls are pairwise not equal. A polytope $F \in \mathcal{S}_k(C)$ is called a k-face of C.

2. Particle processes and random tessellations

Definition

Let $\mathcal{C}^d$ denote the space of all compact non-empty subsets (particles) of $\mathbb{R}^d$. The Hausdorff distance between two compact sets K, L is given by

$$d(K, L) := \sup_{x \in L} d(x, K) \vee \sup_{x \in K} d(x, L),$$

where $d(x, K) := \inf\{\|x - y\| : y \in K\}$ is the (Euclidean) distance between x and K. This metric makes $\mathcal{C}^d$ a complete, separable and locally compact metric space. We equip $\mathcal{C}^d$ with the associated Borel σ-field.

Definition

A particle process X is a simple point process on $\mathcal{C}^d$.

Definition

A random tessellation X is a particle process, whose particles almost surely form a (face to face) tessellation of $\mathbb{R}^d$.

2. Stationary tessellations

Definition

A random tessellation X is called stationary, if

$$X \overset{d}{=} X + y, \quad y \in \mathbb{R}^d.$$

Example

Let η be a stationary Poisson process on $\mathbb{R}^d$. Then the Poisson Voronoi tessellation $\{C(x, \eta) : x \in \eta\}$ is a stationary (and normal) tessellation. Other stationary point processes can be considered as well.

Definition

A hyperplane process is a simple point process η on the space $A(d-1)$ of all hyperplanes in $\mathbb{R}^d$ such that

$$\eta(\{H \in A(d-1) : H \cap K \neq \emptyset\}) < \infty \quad \mathbb{P}\text{-a.s.}, K \in \mathcal{C}^d.$$

It is stationary if $\eta \overset{d}{=} \eta + y$ for all $y \in \mathbb{R}^d$.

Theorem

Under an integrability assumption, the intensity measure Λ of a stationary hyperplane process η is of the form

$$\Lambda(\cdot) = \lambda \int_{G(d-1)} \int_{H^\perp} \mathbf{1}\{H + x \in \cdot\} dx \mathbb{Q}(dH),$$

where $\lambda \geq 0$ (the intensity of η), and the directional distribution $\mathbb{Q}$ is a probability measure on the space $G(d-1)$ of all $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$.

Example

Consider a stationary Poisson process η of hyperplanes and assume that its directional distribution is non-degenerate, in the sense that it is not concentrated on a great subsphere. Then the hyperplanes of η are a.s. in general position, that is every k-dimensional plane is contained in at most $d - k$ hyperplanes of the system. The generated hyperplane tessellation is a stationary tessellation.

3. Intensities of faces

Setting

X is a stationary face to face tessellation. The system of all k-faces of the cells of X is denoted by $\mathcal{S}_k(X)$.

Definition

For a k-dimensional convex set C let $\pi_k(C)$ denote the centre of the k-dimensional circumball of C. Define the stationary point process of centres of k-faces by

$$N_k := \{\pi_k(F) : F \in \mathcal{S}_k(X)\}$$

and denote its intensity by

$$\gamma_k := \mathbb{E}[N_k[0,1]^d].$$

Theorem

We have

$$\sum_{i=0}^{d} (-1)^i \gamma_i = 0.$$

If X is normal, then moreover, for any $k \in \{0, \dots, d\}$,

$$(1 - (-1)^k)\gamma_k = \sum_{j=0}^{k-1} (-1)^j \binom{d+1-j}{k-j} \gamma_j = 0,$$

and in particular $2\gamma_1 = (d+1)\gamma_0$. In two dimensions this implies $\gamma_0 = 2\gamma_2$ and $\gamma_1 = 3\gamma_2$.

Theorem (Miles ’71, Møller ’81)

Assume that X is the Voronoi tessellation generated by a stationary Poisson process of intensity λ. Then $\gamma_0 = c_d\lambda$ for an explicitly known constant $c_d > 0$. In particular, we have in case $d = 3$ that

$$\gamma_0 = \frac{24\pi^2}{35}\lambda, \quad \gamma_1 = \frac{48\pi^2}{35}\lambda, \quad \gamma_2 = \Big(\frac{24\pi^2}{35} + 1\Big)\lambda.$$

Theorem (Mecke '83)

Assume that X is a stationary hyperplane tessellation in general position. Then

$$\gamma_k = \binom{d}{k}\gamma_0, \quad k = 0, \ldots, d.$$

Theorem

Assume that X is a hyperplane tessellation generated by a Poisson hyperplane process of intensity

$$\lambda = \frac{1}{2}\mathbb{E}[\mathrm{card}\{H \in \eta : H \cap B^d \neq \emptyset\}].$$

Then γ_0 is determined by the associated zonoid of X. In the isotropic case

$$\gamma_0 = \frac{\kappa_{d-1}^d}{d^d \kappa_d^{d-1}} \lambda^d,$$

where κ_d is the volume of the unit ball in $\mathbb{R}^d$.

4. Typical faces

Definition

- A random set Z with distribution

$$\mathbb{P}(Z \in \cdot) := \frac{1}{\gamma_d V_d(B)} \mathbb{E}\left[\sum_{C \in X, \pi_d(C) \in B} \mathbf{1}\{C - \pi_d(C) \in \cdot\}\right]$$

  is called typical cell of X. Here the volume $V_d(B)$ is assumed to be positive and finite.

- Let $j \in \{0, \dots, d\}$. A random set $Z^{(j)}$ with distribution

$$\mathbb{P}(Z^{(j)} \in \cdot) := \frac{1}{\gamma_j V_d(B)} \mathbb{E}\left[\sum_{C \in \mathcal{S}_j(X), \pi_j(C) \in B} \mathbf{1}\{C - \pi_j(C) \in \cdot\}\right]$$

  is called typical j-cell of X.

Theorem (Miles, Matheron, Mecke)

Consider the typical cell Z and the cell Z_0 containing the origin. Then, for any $\alpha \in \mathbb{R}$,

$$\gamma_d \mathbb{E}[V_d(Z)^{\alpha+1}] = \mathbb{E}[V_d(Z_0)^{\alpha}].$$

In particular,

$$\mathbb{E}[V_d(Z)] = \gamma_d^{-1},$$
$$\mathbb{E}[V_d(Z_0)^{-1}] = \gamma_d.$$

Theorem

Let $0 \leq j \leq k \leq d$ and consider the typical k-face $Z^{(k)}$ of a normal tessellation. Then the mean number n_{kj} of j-faces of $Z^{(k)}$ satisfies

$$\gamma_j \binom{d+1-j}{k-j} = \gamma_k n_{kj}.$$

In particular, the typical cell of a stationary planar Voronoi tessellation has 6 vertices on the average.

Theorem (Mecke '83)

Let $0 \leq j \leq k \leq d$ and consider the typical k-face $Z^{(k)}$ of a stationary hyperplane tessellation in general position. Then the mean number of j-faces of $Z^{(k)}$ is given by

$$2^{k-j}\binom{k}{j}.$$

In particular, the typical cell of a stationary planar line tessellation has 4 vertices on the average.

5. Gamma distributions in Poisson-Voronoi tessellation

Setting

We consider the Voronoi tessellation generated by a stationary Poisson process of intensity λ.

Theorem (Muche '05, Baumstark and L. '07)

Let $j \in \{0, \dots, d\}$. Pick a point x on the j-faces at random (according to j-dimensional Lebesgue measure). Consider the ball S_j centred at x that has $d - j + 1$ Poisson points in its boundary and no point in its interior. Then

$$V_d(S_j) \sim \Gamma(d - j + j/d, \lambda).$$

Theorem (Cowan, Quine and Zuyev '03)

Let $d = 2$. Pick an edge $Z^{(1)}$ of the Voronoi tessellation at random. Then there are a.s. exactly two different Poisson points X_1, X_2, the neighbours of the edge $Z^{(1)}$, such that $Z^{(1)}$ is the intersection of the Voronoi cells centred at those points. The fundamental region T_1 of $Z^{(1)}$ is defined by

$$T_1 := \bigcup_{y \text{ vertex of } Z^{(1)}} B(x, \|x - X_1\|).$$

Then the area of T_1 has a $\Gamma(3, \lambda)$-distribution.

Theorem (Baumstark and L. ’09)

Let $j \in \{0, \dots, d\}$. Pick a j-face $Z^{(j)}$ of the Voronoi tessellation at random. Then there are a.s. exactly $d - j + 1$ different Poisson points $X_1, \dots, X_{d-j+1}$, the neighbours of the j-face $Z^{(j)}$, such that $Z^{(j)}$ is the intersection of the Voronoi cells centred at those points. The fundamental region T_j of $Z^{(j)}$ is defined by

$$T_j := \bigcup_{y \text{ vertex of } Z^{(j)}} B(x, \|x - X_1\|)$$

Consider T_j under the condition that $\eta(T_j) = m + d - j + 1$, where $m = 0$ in case $j = 0$, $m = 2$ in case $j = 1$ and $m \geq j + 1$ otherwise. Then the volume of the fundamental region T_j has a $\Gamma(m + d - j, \lambda)$-distribution.